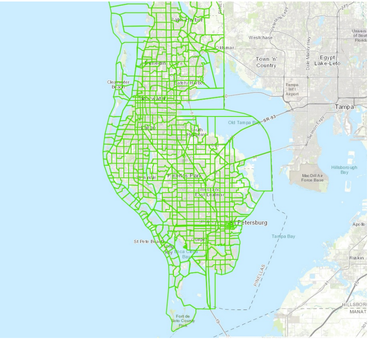
CE 272
Traffic Network Equilibrium

Lecture 11

Traffic Assignment with Elastic Demand - Part I

Previously on Traffic Network Equilibrium...

Let's extend our definition of a graph to include a subset of nodes from which trips originate or end. These nodes are called **zone centroids** and can be actual junctions or artificial nodes.



If zone centroids are artificially created, they are connected to nearby streets using artificial links called **centroid connectors**.

It is assumed that artificially created centroid connectors can be traversed instantaneously.

Previously on Traffic Network Equilibrium...

The demand information for all OD pairs is commonly referred to as **OD matrix** or **trip tables**.

The number of person trips are computed from the first two steps of the four-step process. In the third step, these trips are assigned to different modes (car, bus, two-wheeler etc.) resulting in a trip table for each mode.

But for equilibrium analysis, we assume that demand comprises of only passenger cars. The demand of other types of vehicles are adjusted by factors called **passenger car units (PCUs)** that reflect their sizes relative to that of a car.

Previously on Traffic Network Equilibrium...

Can we reverse engineer a convex function such that the KKT conditions are equivalent to the Wardrop equilibria?

Martin Beckmann, C. B. McGuire, and Christopher Winsten in 1956 discovered such a function in their seminal book *Studies in the Economics of Transportation*.

$$\sum_{(i,j)\in A} \int_0^{x_{ij}} t_{ij}(\omega)\, d\omega$$

This function is commonly referred to as the Beckmann function.

Previously on Traffic Network Equilibrium...

Suppose $\tau_p(\mathbf{y})$ denotes the travel time on path p given a path flow vector $\mathbf{y}$. From the KKT conditions, eliminating λ_p, for all $(r, s) \in Z^2$, $p \in P_{rs}$,

$$\tau_p(\mathbf{y}) \geq \mu_{rs}$$
$$y_p \left(\tau_p(\mathbf{y}) - \mu_{rs}\right) = 0$$

From the above equations, μ_{rs} is the length of the shortest path.

If $y_p > 0$, then path p must be shortest. If $y_p = 0$, the travel time of path p must be at least μ_{rs}. Voila! Wardrop Principle.

Lecture Outline

Lecture Outline

Elastic Demand

Elastic Demand

Introduction

So far, we assumed that the demand between all OD pairs is fixed. This assumption is reasonable when most travelers are regular commuters.

However, demand may sometimes depend on some supply-side conditions.

- Suppose we modify the network by building new links. This may induce new demand (in addition to route shifting).
- Individuals may have the option to select working times or to work from home. Hence, depending on the congestion levels (or tolls), one may choose to shift their departure times or to not travel.

Elastic Demand

Demand Functions

To model this phenomena, assume that the demand between an OD pair is a function of the time it takes to travel between O and D.

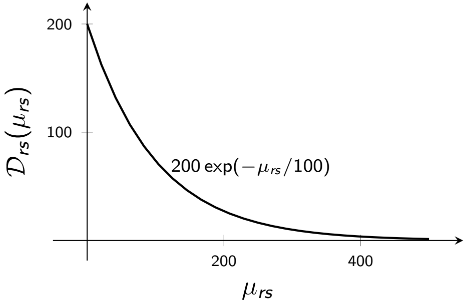
This is true in other markets as well. The demand for a good is a function of its price. Hence, as the travel time between an OD pair decreases, the consumption (demand for travel) increases.

Suppose it takes μ_{rs} min to travel between $(r, s) \in Z^2$. Let $\mathcal{D}_{rs}(\mu_{rs})$ represent the **demand function** which gives the number of users who choose to travel between (r, s).

Elastic Demand

Demand Functions

What should the shape of $\mathcal{D}_{rs}(\mu_{rs})$ be?

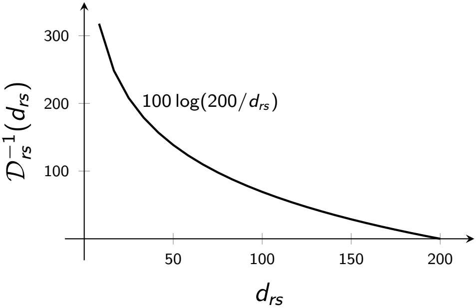


We could assume it to be non-increasing but we'll require it to be strictly decreasing so that it's inverse exists.

Elastic Demand

Demand Functions

The **inverse demand function** takes the number of travelers between the OD pair as input and provides the time to travel between O and D.

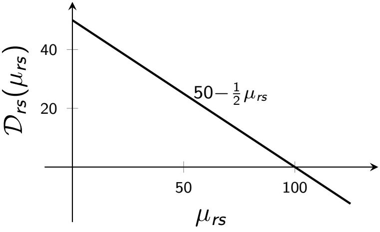


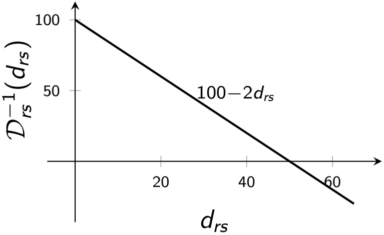
Mathematically, if d_{rs} is the demand and μ_{rs} denotes the travel time, $\mathcal{D}_{rs}^{-1}(d_{rs}) = \mu_{rs}$.

Elastic Demand

Demand Functions

Consider the demand function $\mathcal{D}_{rs}(\mu_{rs}) = 50 - \frac{1}{2}\mu_{rs}$ and its inverse $\mathcal{D}_{rs}^{-1}(d_{rs}) = 100 - 2d_{rs}$





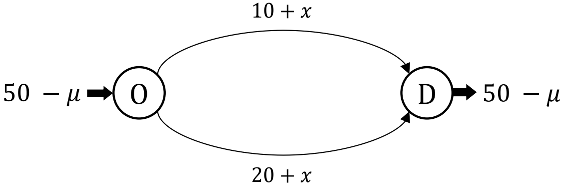
We will let the demand function $\mathcal{D}$ take negative values so that its inverse exists. Negative demand values can be avoided by setting $d_{rs} = \mathcal{D}^{+}(\mu_{rs}) = \max\{\mathcal{D}(\mu_{rs}), 0\}$

Convex Optimization Formulation

Convex Optimization Formulation

Introduction

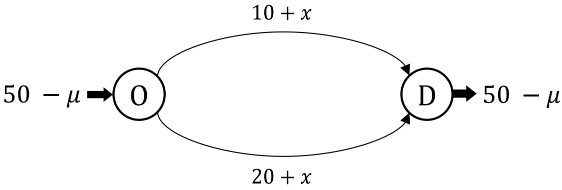
Let's analyze the following example. Assuming both paths are used, what are the equilibrium flows?



The Wardrop principle in this setting applies to all users who decide to travel.

Convex Optimization Formulation

Introduction



The equilibrium demand and link flows can be calculated using the following equations.

$$10 + x_1 = 20 + x_2 = \mu$$
$$d = 50 - \mu$$
$$x_1 + x_2 = d$$

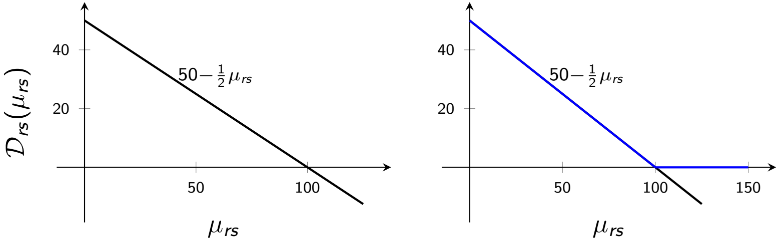
Convex Optimization Formulation

With link flows as variables, we derived the Beckmann formulation for computing an equilibrium. Can we construct a similar formulation using **link flows and demands** as variables?

Convex Optimization Formulation

Complementarity Conditions

Recall that the demand was defined as $d_{rs} = \mathcal{D}^+(\mu_{rs}) = \max\{\mathcal{D}(\mu_{rs}), 0\}$



An alternate way of expressing the above relationship is

$$d_{rs} \geq 0$$
$$d_{rs} \geq \mathcal{D}_{rs}(\mu_{rs})$$
$$d_{rs} > 0 \Rightarrow d_{rs} = \mathcal{D}_{rs}(\mu_{rs})$$

Déjà vu?

Convex Optimization Formulation

Complementarity Conditions

The conditions

$$d_{rs} \geq 0$$
$$d_{rs} \geq \mathcal{D}_{rs}(\mu_{rs})$$
$$d_{rs} > 0 \Rightarrow d_{rs} = \mathcal{D}_{rs}(\mu_{rs})$$

can be recast in terms of the inverse demand functions as follows

$$d_{rs} \geq 0$$
$$\mu_{rs} \geq \mathcal{D}_{rs}^{-1}(d_{rs})$$
$$d_{rs} > 0 \Rightarrow \mathcal{D}_{rs}^{-1}(d_{rs}) = \mu_{rs}$$

or equivalently as

$$d_{rs} \geq 0$$
$$\mu_{rs} \geq \mathcal{D}_{rs}^{-1}(d_{rs})$$
$$d_{rs}(\mathcal{D}_{rs}^{-1}(d_{rs}) - \mu_{rs}) = 0$$

Convex Optimization Formulation

Desired Conditions

The inverse demand functions have units of time which will prove useful in formulating an optimization model as we will see shortly.

The optimal solution must satisfy the following conditions:

$$
\begin{aligned}
y_p &\geq 0 && \forall\, p \in P \\
\tau_p(\mathbf{y}) &\geq \mu_{rs} && \forall\, (r,s) \in Z^2, p \in P_{rs} \\
y_p\left(\tau_p(\mathbf{y}) - \mu_{rs}\right) &= 0 && \forall\, (r,s) \in Z^2, p \in P_{rs} \\
d_{rs} &\geq 0 && \forall\, (r,s) \in Z^2 \\
\mu_{rs} &\geq \mathcal{D}_{rs}^{-1}(d_{rs}) && \forall\, (r,s) \in Z^2 \\
d_{rs}\left(\mathcal{D}_{rs}^{-1}(d_{rs}) - \mu_{rs}\right) &= 0 && \forall\, (r,s) \in Z^2
\end{aligned}
$$

Can you reverse engineer an objective from these equations?

Convex Optimization Formulation

Beckmann-like Model

The equilibrium solution for the elastic demand problem can be obtained by solving the following convex program.

$$\min_{\mathbf{y},\mathbf{d}} \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)\, d\omega$$

$$\text{s.t.} \quad \sum_{p\in P_{rs}} y_p = d_{rs} \ \forall\ (r,s) \in Z^2$$

$$y_p \geq 0 \ \forall\ p \in P$$

$$d_{rs} \geq 0 \ \forall\ (r,s) \in Z^2$$

Note that both terms in the objective have units of time. Write the KKT conditions for the above model.

Convex Optimization Formulation

Beckmann-like Model

$$\mathcal{L}(\mathbf{y}, \mathbf{d}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}) = \sum_{(i,j)\in A} \int_0^{\sum_{p\in P} \delta_{ij}^p y_p} t_{ij}(\omega)\, d\omega - \sum_{(r,s)\in Z^2} \int_0^{d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)\, d\omega$$

$$+ \sum_{p\in P} \lambda_p(-y_p) + \sum_{(r,s)\in Z^2} \nu_{rs}(-d_{rs}) + \sum_{(r,s)\in Z^2} \mu_{rs} \left(d_{rs} - \sum_{p\in P_{rs}} y_p \right)$$

Convex Optimization Formulation

KKT Conditions

Primal feasibility:

$$\sum_{p \in P_{rs}} y_p = d_{rs} \ \forall \ (r, s) \in Z^2$$

$$y_p \geq 0 \ \forall \ p \in P$$

$$d_{rs} \geq 0 \ \forall \ (r, s) \in Z^2$$

Dual feasibility:

$$\lambda_p \geq 0 \, \forall \, p \in P$$

$$\nu_{rs} \geq 0 \, \forall \, (r, s) \in Z^2$$

Complementary Slackness:

$$\lambda_p y_p = 0 \, \forall \, p \in P$$

$$\nu_{rs} d_{rs} = 0 \, \forall \, (r, s) \in Z^2$$

Gradient of the Lagrangian vanishes:

$$\sum_{(i,j) \in A} \delta_{ij}^p t_{ij}(x_{ij}) - \lambda_p - \mu_{rs} = 0 \, \forall \, (r, s) \in Z^2, p \in P_{rs}$$

$$-\mathcal{D}_{rs}^{-1}(d_{rs}) - \nu_{rs} + \mu_{rs} = 0 \, \forall \, (r, s) \in Z^2$$

Convex Optimization Formulation

KKT Conditions

As before, eliminating λ_p, for all $(r, s) \in Z^2$, $p \in P_{rs}$, we get

$$\tau_p(\mathbf{y}) \geq \mu_{rs}$$
$$y_p \left(\tau_p(\mathbf{y}) - \mu_{rs}\right) = 0$$

These two imply the Wardrop principle. Eliminating ν_{rs}, for all $(r, s) \in Z^2$

$$\mu_{rs} \geq \mathcal{D}_{rs}^{-1}(d_{rs})$$
$$d_{rs} \left(\mathcal{D}_{rs}^{-1}(d_{rs}) - \mu_{rs}\right) = 0$$

Hence, if the demand between an OD pair is strictly positive $\mathcal{D}_{rs}^{-1}(d_{rs}) = \mu_{rs}$. Else, if it is zero, the shortest path time is greater than or equal to time value at which no users are willing to travel between O and D (i.e., $\tau_p(\mathbf{y}) \geq \mu_{rs} \geq \mathcal{D}_{rs}^{-1}(0)$).

Lecture Outline

Solution Methods

Solution Methods

When the demand is fixed, the set of feasible link flows is convex. Are the link flows and demands bounded when the demand is elastic?

The optimization model discussed earlier is a convex program. (Why?) Thus, using MSA or FW we can update both the link flows and OD demands $(\mathbf{x}, \mathbf{d})$ within each iteration.

Some food for thought...

- Search direction
- Terminating criteria

Before performing an all-or-nothing assignment, we need to compute the shortest paths between each OD pair.

Suppose the shortest path labels in iteration k are μ_{rs}^k. Set the target demand to $\hat{d}_{rs}^k = \mathcal{D}_{rs}^+(\mu_{rs}^k)$ and denote the vector of demands by $\hat{\mathbf{d}}^k$.

With this demand, perform an all-or-nothing assignment and represent the resulting link flows using $\hat{\mathbf{x}}^k$.

$$\mathbf{x}^{k+1} = \eta_k \hat{\mathbf{x}}^k + (1 - \eta_k)\mathbf{x}^k$$
$$\mathbf{d}^{k+1} = \eta_k \hat{\mathbf{d}}^k + (1 - \eta_k)\mathbf{d}^k$$

Note: $\mathbf{x}^k, \hat{\mathbf{x}}^k, \mathbf{x}^{k+1}$ are the flows that correspond to the demands $\mathbf{d}^k, \hat{\mathbf{d}}^k, \mathbf{d}^{k+1}$ respectively.

Solution Methods

Terminating Criteria

We could use Relative gap and AEC as convergence measures but they do not involve the OD demands.

Hence, at each iteration k, we define the total misplaced flow (TMF) as

$$\sum_{(r,s)\in Z^2} |\mathcal{D}^{+}_{rs}(\mu^{k}_{rs}) - d^{k}_{rs}| = \sum_{(r,s)\in Z^2} |\hat{d}^{k}_{rs} - d^{k}_{rs}|$$

The algorithms are terminated when both TMF and Relative gap/AEC are less than certain thresholds.

MSA(G)

```
k ← 1
Find a feasible x̂, d̂

while Relative Gap > 10^{-4} or TMF > ε do
    x ← (1/k) x̂ + (1 − 1/k) x
    d ← (1/k) d̂ + (1 − 1/k) d
    Update t(x)
    x̂ ← 0
    for r ∈ Z do
        DIJKSTRA (G, r)
        for s ∈ Z, (i, j) ∈ p*_rs do
            d̂_rs ← D+_rs(μ*_rs)
            x̂_ij ← x̂_ij + d̂_rs
        end for
    end for
    TMF ← Σ_{(r,s)∈Z^2} |d̂_rs − d_rs|
    Relative Gap ← TSTT/SPTT − 1
    k ← k + 1
end while
```

Solution Methods

Frank-Wolfe

In the FW algorithm, just as with the fixed demand case, we try to find a step size that minimizes the objective along the line between $(\mathbf{x}, \mathbf{d})$ and $(\hat{\mathbf{x}}, \hat{\mathbf{d}})$.

$$f(\eta) = \sum_{(i,j)\in A} \int_0^{\eta \hat{x}_{ij} + (1-\eta) x_{ij}} t_{ij}(\omega)\, d\omega - \sum_{(r,s)\in Z^2} \int_0^{\eta \hat{d}_{rs} + (1-\eta) d_{rs}} \mathcal{D}_{rs}^{-1}(\omega)\, d\omega$$

First-order optimality conditions for interior η imply that $f'(\eta) = 0$, i.e.,

$$\sum_{(i,j)\in A} t_{ij}\Big(\eta \hat{x}_{ij} + (1-\eta) x_{ij}\Big)(\hat{x}_{ij} - x_{ij}) - \sum_{(r,s)\in Z^2} \mathcal{D}_{rs}^{-1}\left(\eta \hat{d}_{rs} + (1-\eta) d_{rs}\right)(\hat{d}_{rs} - d_{rs}) = 0$$

Bisection or Newton's method can be used to compute the solution to the above equation.

Solution Methods

Frank-Wolfe

Frank-Wolfe(G)

$k \leftarrow 1$
Find a feasible $\hat{\mathbf{x}}, \hat{\mathbf{d}}$

while Relative Gap $> 10^{-4}$ or $TMF > \epsilon$ **do**
  if $k = 1$ **then** $\eta \leftarrow 1$ **else** $\eta \leftarrow$ Bisection($G, \mathbf{x}, \mathbf{d}, \hat{\mathbf{x}}, \hat{\mathbf{d}}$)
  $\mathbf{x} \leftarrow \eta\hat{\mathbf{x}} + (1-\eta)\mathbf{x}$
  $\mathbf{d} \leftarrow \eta\hat{\mathbf{d}} + (1-\eta)\mathbf{d}$
  Update $\mathbf{t}(\mathbf{x})$
  $\hat{\mathbf{x}} \leftarrow \mathbf{0}$
  for $r \in Z$ **do**
    Dijkstra (G, r)
    for $s \in Z, (i,j) \in p^*_{rs}$ **do**
      $\hat{d}_{rs} \leftarrow \mathcal{D}^+_{rs}(\mu^*_{rs})$
      $\hat{x}_{ij} \leftarrow \hat{x}_{ij} + \hat{d}_{rs}$
    end for
  end for
  $TMF \leftarrow \sum_{(r,s)\in Z^2} |\hat{d}_{rs} - d_{rs}|$
  Relative Gap $\leftarrow TSTT/SPTT - 1$
  $k \leftarrow k + 1$
end while

Solution Methods

A Parting Note

- The elastic demand model is definitely more realistic than the fixed demand case.
- But estimating demand and inverse demand functions is a challenge and would require elaborate surveys.

Your Moment of Zen

et al.

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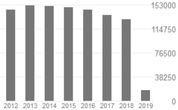

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