CE 272
Traffic Network Equilibrium

Lecture 1

Game Theory and Traffic Equilibria

Lecture Outline

1. Game Theory
2. Motivating Examples
3. Formulating Optimization Problems

Lecture Outline

Game Theory

Game Theory

Brainstorming

Let's revisit the primary objective of this course: **How do you choose a route when you travel?**

- You could select the fastest route.
- However, everyone else might plan to do the same thing!

Let's first take a movie break and a detour into game theory.

Game Theory

Joker's Experiment

https://www.youtube.com/watch?v=K4GAQtGtd_0

In summary, two ferries (one carrying criminals and another with civilians) are rigged with explosives.

Crew members of each ferry are given a detonator that controls the explosives on the other ferry.

Joker claims that the first to blow up the other ferry will be spared, and also threatens to blow them both up if they do not decide before midnight.

What should the members in the two ferries do?

Motivating Examples

Joker's Experiment

Let us assign some numbers to the physical outcomes and to the psychological factors associated with making a decision.

- Two possible outcomes: live $(+2)$ and die (-2).
- There's a 50-50 chance of surviving because Joker may be intercepted by Batman before he executes his plans.
- If the criminals (civilians) die because of the civilians (criminals) they'd feel revengeful/betrayed (-1). Civilians (criminals) on the other hand also experience a feeling of remorse (-1).

Hence, utilities of the criminals can be written in a matrix form as follows:

	Do Nothing	Detonate
Do Nothing	0	-3
Detonate	$+1$	-2

Motivating Examples

Joker's Experiment

The utilities of the civilians can be computed similarly. Let's write them alongside with the utilities of the criminals as follows:

	Do Nothing	Detonate
Do nothing	0, 0	-3, $+1$
Detonate	$+1$, -3	-2, -2

- If the criminals 'do nothing', civilians are better off by choosing to 'detonate'.
- If the criminals 'detonate', civilians are better off by choosing to 'detonate'.

A similar what-if analysis can be applied to the choices of the civilians. In the end, **it is optimal for both agents to 'detonate'.**

In general, what does an equilibrium look like?

Nash Equilibrium (1951)

At equilibrium, no player has an incentive to deviate.

	Do nothing	Detonate
Do nothing	0, 0	−3, +1
Detonate	+1, −3	−2, −2

In other words, to check if a particular outcome is an equilibrium solution

1. Pick a player i
2. Fix the actions of the other players that lead to the current outcome
3. Check if player i can do better by selecting a different action
4. Repeat steps 1-3 for all players

Game Theory

Prisoner's Dilemma

Joker's experiment is modeled after the **Prisoner's Dilemma**, a famous example used to introduce 2-player games in game theory.

	Do Nothing	Detonate
Do nothing	$0, 0$	$-3, +1$
Detonate	$+1, -3$	$-2, -2$

Is there a better outcome in this example?

While analyzing the Joker's experiment (Prisoner's Dilemma), we did make a few implicit assumptions.

- **Time**: Although there was a deadline in this situation, we imagined that both agents took decisions at the same instant.
- **Rationality**: Unlike in the movie, where people acted irrationally (did they?), which is very likely since their decisions had dire consequences, we supposed that both agents were fully rational.
- **Common Knowledge vs Mutual Knowledge**: It was also assumed that the criminals know the utilities of the civilians, the civilians know the utilities of the criminals, the criminals know that the civilians know that the criminals know their utilities, and so on.

Lecture Outline

Motivating Examples

Motivating Examples

Let us transfer our analysis of the Prisoner's Dilemma to a network of travelers.

We'll formally defined a network later but for starters, think of junctions as nodes and roadway segments connecting them as links/arcs. Traffic in opposite directions on a two-way street is represented using different links.

Also assume that everyone is traveling from a single origin to a single destination and that the **travel time on a link is a function of the number of travelers using it**.

- Who are the agents/players of the game?
- What are the actions available to each player?
- What are the (dis)utilities of each player for a particular outcome?

Motivating Examples

Traffic Equilibria

- **Who are the agents/players of the game?**
  Travelers
- **What are the actions available to each player?**
  Paths between the origin-destination (OD) pair
- **What are the (dis)utilities of each player for a particular outcome?**
  Each outcome determines the number of users on a path and hence the flow on every link. Travel time of a path is the sum of travel times on links belonging to the path.

What does an equilibrium look like in a traffic networks?

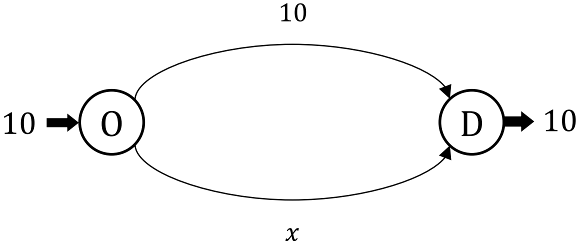
Wardrop Equilibrium (1952)

All used paths have equal and minimal travel time.

Motivating Examples

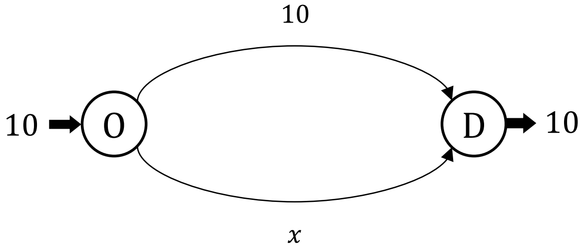
Pigou's Network

Compute the equilibrium flow in the following network.



At equilibrium, everyone takes the bottom path and experiences a travel time of 10 min. At this state, the only used path (the bottom one) is the shortest, i.e., no one has an incentive to deviate.

Is there a better way to route travelers in this network?

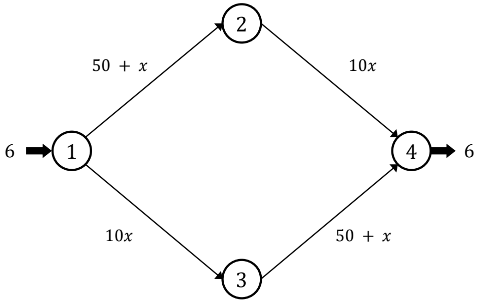


Routing 5 on the top path and 5 on the bottom reduces the overall congestion (75 minutes vs. 100 minutes).

Motivating Examples

Braess's Network

Let us repeat this exercise on this network.

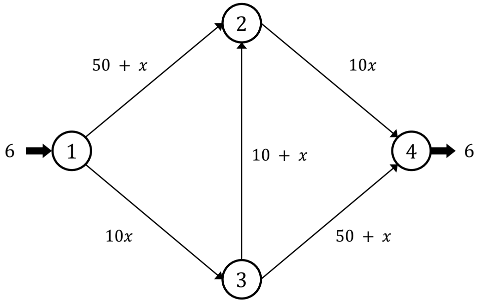


At equilibrium, 3 travelers take path 1-2-4 and 3 take path 1-3-4 and experience a travel time of 83 minutes. This flow pattern also reduces the overall congestion.

Motivating Examples

Braess's Network

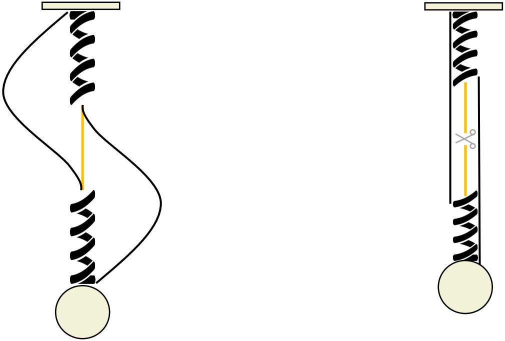
How can we improve traffic? Let's try and build a new link.



Is the old solution an equilibrium? What are the new equilibrium flows? At equilibrium, there are 2 travelers on each path and the equilibrium travel time is 92 minutes. We have made the network worse by building a new road! **A Paradox**!

Motivating Examples

Braess's Network



https://www.youtube.com/watch?v=ekd2MeDBV8s

Motivating Examples

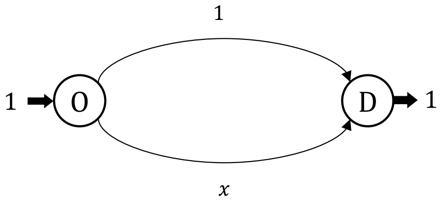
Assumptions

As before, we assumed that

- All travelers select paths at the same instant and we do not explicitly model the departure times and times at which travelers enter a link. Hence, this process of modeling route choice is also called **static traffic assignment**.
- In other words, we only analyze the spatial distribution of congestion and ignore the temporal dimension. There are advanced models which capture both aspects (**dynamic traffic assignment**).
- Travelers are fully rational and selfish and seek routes that minimize their travel times.
- Travelers are fully aware of the network topology and its response to congestion (perhaps from experience).

This simplified setting is representative of a "steady state" and allows us to study equilibrium analytically.

For most part, we also suppose that **travelers are infinitely divisible**, i.e., the flow on links is not constrained to take integer values and this assumption is not restrictive as long as a large number of travelers are involved.



The terms 'Travelers' or 'Flow' are commonly interpreted as traffic volumes and are measured in vehicles/time units.

Motivating Examples

Reality Check

The Braess paradox appears to occur in carefully orchestrated instances.
Does it happen in practice?

http://www.pbs.org/video/2192347741/

Formulating Optimization Problems

Formulating Optimization Problems

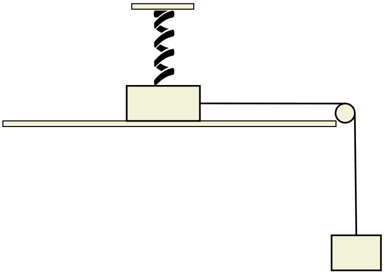
Introduction

- The logic discussed so far enabled us to check if an outcome is an equilibrium solution. This method is however computationally prohibitive when the number of players and their actions are large.
- Can we formulate an **optimization model for finding the equilibrium flows**? What would the objective and constraints be?

Formulating Optimization Problems

Analogy with Statics

How can we compute the equilibrium state in the following system?

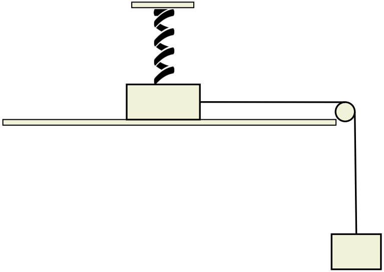


- Construct free body diagrams and solve a system of equations...
- Analyzing network equilibria looking at each player is analogous to this approach.

Formulating Optimization Problems

Analogy with Statics

Alternately, one can write an expression for the energy of the system and minimize it.



We'll follow a similar line of thought for computing traffic equilibria because it can scale up well.

Formulating Optimization Problems

General Framework

Optimization problems typically have the following three components:

1. Objective function
2. Decision variables
3. Constraints

While the first two are present in all optimization models, it is not necessary to have constraints. Problems without constraints are called *unconstrained problems*.

Formulating Optimization Problems

General Framework

Any optimization program can be written in the following standard form.

$$
\begin{aligned}
& \min_{\mathbf{x}} f(\mathbf{x}) \\
\text{s.t. } & g_i(\mathbf{x}) \leq 0 && \forall\, i = 1, 2, \ldots, l \\
& h_i(\mathbf{x}) = 0 && \forall\, i = 1, 2, \ldots, m \\
& \mathbf{x} \in X
\end{aligned}
$$

The functions g and h define the inequality and equality constraints respectively.* The set X is used to represent additional constraints (e.g., integrality), which we won't have in this course. Instead, we use X to denote implicit constraints (more on next slide).

The set of decision variables that satisfy all the constraints is called the **feasible region**.

Maximization problems can be converted into the standard form by simply changing the sign of the objective function.

* The equality constraints are superfluous (Why?) but we'll retain them for reasons that will become apparent later.

Formulating Optimization Problems

General Framework

We assume, throughout this course, that the functions f, g, and h are differentiable and have open domains.

These functions (e.g., $\log x$, $1/x$) can impose implicit constraints. We'll use X to denote the set of $\mathbf{x}$ that satisfy such implicit constraints.

$$X = \text{domain}(f) \bigcap_{i=1}^{l} \text{domain}(g_i) \bigcap_{i=1}^{m} \text{domain}(h_i)$$

It is common practice to not write the implicit constraints in the standard form.

$$\begin{aligned} & \min_{\mathbf{x}} f(\mathbf{x}) \\ \text{s.t. } & g_i(\mathbf{x}) \leq 0 \qquad && \forall\, i = 1, 2, \ldots, l \\ & h_i(\mathbf{x}) = 0 && \forall\, i = 1, 2, \ldots, m \end{aligned}$$

Formulating Optimization Problems

Examples

Formulate an optimization model in the standard form for the following problems:

- You have a 4m long wooden frame that needs to be cut to create a window frame. What would be the dimensions of the window that lets in the most light?
- You are given three fixed non-collinear points in a 2D plane. Find a fourth point that minimizes the sum of the distances from the three points. What if the fourth point must lie within the triangle formed by the three points?

Formulating Optimization Problems

Examples

Optimization models have numerous applications in the area of transportation. For example,

- What is the shortest distance/time between two points in a network?
- How to ship goods from production centers to warehouses at minimum cost while preserving capacity constraints?
- Where do you locate fire stations to maximize coverage area for a given budget?
- What is the maximum rate at which you can evacuate a city during a disaster?

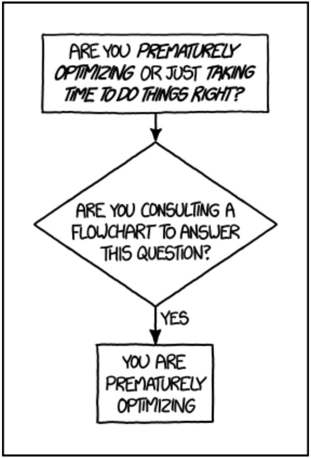
Optimization problems are commonly classified as:

- Linear Programs (LPs): Objective function and constraints are linear functions of the decision variables $\mathbf{x}$.
- Integer Programs (IPs): The decision variables take only integer values (i.e., $X \subseteq \mathbb{Z}$)
- Non-linear Programs (NLPs): The objective function and/or constraints involves a non-linear function of the decision variables $\mathbf{x}$.

Typically, LPs and convex NLPs are easy to solve when compared to IPs and non-convex NLPs.

In this course, we will mostly deal with convex NLPs in which the objective function and the feasible region are convex.

Your Moment of Zen



Source: xkcd.com