CE 269
Traffic Engineering

Lecture 9

Shock Waves

Previously on Traffic Engineering

In the conservation law $k_t + q_x = 0$, suppose $f'(k) = c$, a constant. Then, the PDE in terms of density can be written as

$$k_t + ck_x = 0$$
$$k(0, x) = k_0(x)$$

Suppose the density on the road at time $t = 0$ is given (*Cauchy problem*). As before, we are interested in finding the value of density at every (t, x).

Instead, suppose we try to estimate the density along a curve $x = x(t)$.

$$\frac{dk(t, x(t))}{dt} = \frac{\partial k}{\partial t}\frac{dt}{dt} + \frac{\partial k}{\partial x}\frac{dx(t)}{dt}$$
$$= k_t + \frac{dx(t)}{dt}k_x$$

If $\frac{dx(t)}{dt} = c$, what is $\frac{dk}{dt}$?

Previously on Traffic Engineering

Hence, the total time derivative of the density is constant along a curve $x = x(t)$ if $\frac{dx(t)}{dt} = c$.

In other words, the value of the density is constant along a straight line with slope c, i.e., along $x(t) = ct + x(0) = ct + x_0$.

Hence, to compute density at a point (t^*, x^*), draw the characteristic curve with slope c and look where it intersects the y axis. That is,

$$k(t^*, x^*) = k(0, x^* - ct^*) = k_0(x^* - ct^*)$$

Notice that this solution is in the form of a traveling wave. Which fundamental diagram is suited for this framework?

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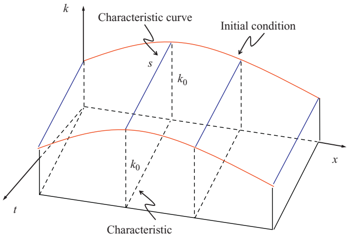
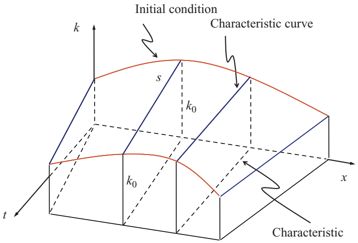
For the previous PDE, all characteristics have the same slope c and are parallel to each other. The density along these characteristics is same as the initial value.





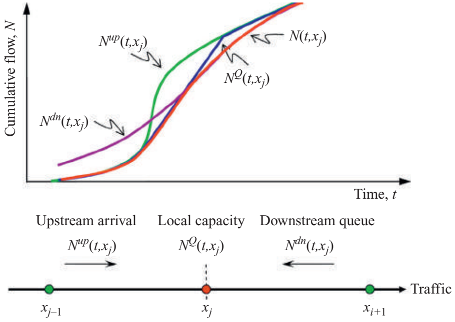
Instead, suppose c is a function of k, i.e., we have a different fundamental diagram where $c = c(k(t, x))$.

$$\frac{dx(t)}{dt} = c(k(t, x)) \Rightarrow x = c(k_0(x_0))t + x_0$$

The characteristics in this case are straight lines but need not be parallel.

Previously on Traffic Engineering

At x_j, one cannot accommodate more vehicles than what is sent from upstream, the capacity, and what can be received downstream.



Hence, the cumulative count is the lowest of all the three conditions

$$N(t, x_j) = \min\left\{N^{up}(t, x_k), N^Q(t, x_j), N^{dn}(t, x_j)\right\}$$

Lecture Outline

1. Newell's Method
2. Shock Waves
3. Queue Analysis

Lecture Outline

Newell's Method

Newell's Method

Continued

When density is less than the critical density, $f' > 0$ and traffic is in the uncongested regime. These type of states propagate downstream.

For congested regions, $f' < 0$ and these states propagate upstream.

Note that with the method of characteristics, if we knew f' to draw a characteristic and read-off the density at the initial or boundary.

But to get f' we need the density in the first place. Recall from the Green's theorem version of the conservation equation,

$$\int_C q dt - k dx = N(t_2, x_2) - N(t_1, x_1)$$

* We will use two sets of notation in this lecture since the material is from both Boyles et al. (2020) and Ni (2015).

Suppose we draw a characteristic C from (t,x) to (t_0,x_0). Since C is a characteristic, $dx/dt = f'(k)$ and k and q are constant. Therefore,

$$
\begin{aligned}
N(t,x) &= N(t_0,x_0) + \int_C q dt - k dx \\
&= N(t_0,x_0) + \int_C (q - kf'(k)) dt \\
&= N(t_0,x_0) + (q - kf'(k))(t - t_0)
\end{aligned}
$$

But since $k(t,x)$ is not known, we do not know the second part of the above expression. The true value turns out to be the lowest possible value given by

$$N(t,x) = \inf_{k\in[0,k_j]} \left\{ N(t_k,x_k) + (q - kf'(k))(t - t_k) \right\}$$

Newell's Method

Theory

For the case of the triangular fundamental diagram, we have only two characteristics to deal with and the correct cumulative count is obtained from the most restrictive initial/boundary conditions.

Case I: If $f' = w_f$, $q = kw_f$ and hence

$$N(t, x) = N(t_U, x_U)$$

That is, we trace the same vehicle as we move along the characteristics.

Case II: If $f' = w_b$, $k + q/w_b = k_j$, and hence

$$\begin{aligned} N(t, x) &= N(t_C, x_C) + (q + kw_b)(t - t_C) \\ &= N(t_C, x_C) + k_j w(t - t_C) \\ &= N(t_C, x_C) + k_j(x_C - x) \end{aligned}$$

In this case, the cumulative count increases at the rate of the jam density.

Newell's Method

Theory

Thus, for any (t, x),

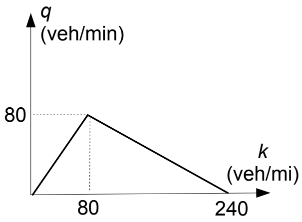
$$N(t,x) = \min\left\{N(t_U, x_U), N(t_C, x_C) + k_j(x_C - x)\right\}$$

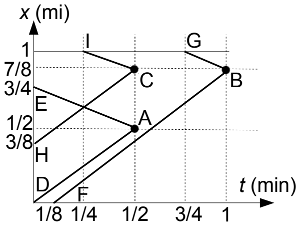
Instead of tracking the cumulative counts, we could draw characteristics and work with densities.

But as we will see shortly, shock waves can make this procedure difficult. This approach on the other hand can be applied oblivious to the existence of shock waves.

Suppose $f(k) = \min\{k, 120 - k/2\}$, and vehicles on a 1-mile stretch are initially in an uncongested state with a flow of 48 veh/min.

If at $t = 0$, the outflow was suddenly stopped because of a red light, find the cumulative count at the points A, B, and C





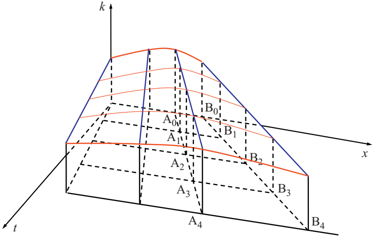
Lecture Outline

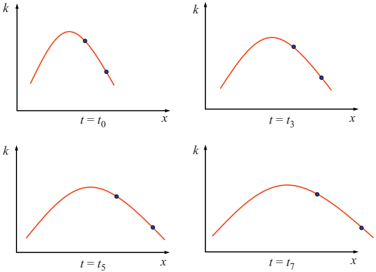
Shock Waves

Shock Waves

Expansion Waves

Suppose the slope of the characteristics is a function of density. Then we many encounter the following scenario which produces expansion waves.

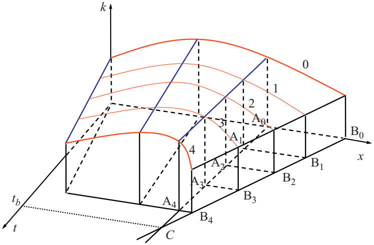


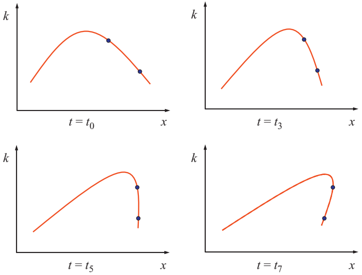


Shock Waves

Compression Waves

Likewise, if the characteristics move closer to each other over time, they produce compression waves.



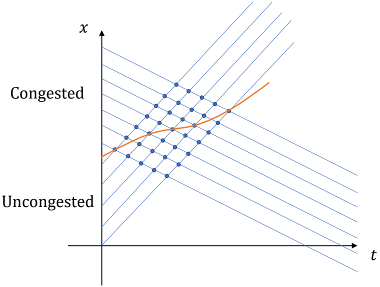


However, in this case, they may intersect and the function will become multi-valued. This is also referred to as a *gradient catastrophe*.

Shock Waves

Discontinuities

Shock waves on the other hand are said to form when densities are discontinuous. Consider the following scenario where the densities at some of the regions are unique but the characteristics meet in the range of influence.



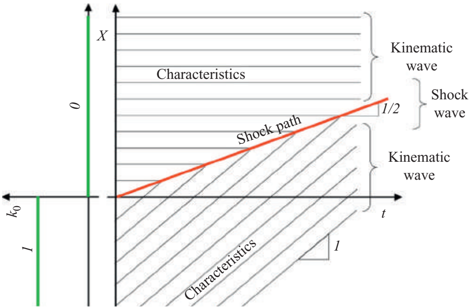
Characteristics in such cases intersect at many points. However, the locations at which the shock wave forms is unique and is dictated by *Rankine-Hugonoit Jump Condition*.

Specifically, if $x_s(t)$ is the shock path,

$$\frac{dx_s}{dt} = \frac{q(t, x_s^+) - q(t, x_s^-)}{k(t, x_s^+) - k(t, x_s^-)}$$

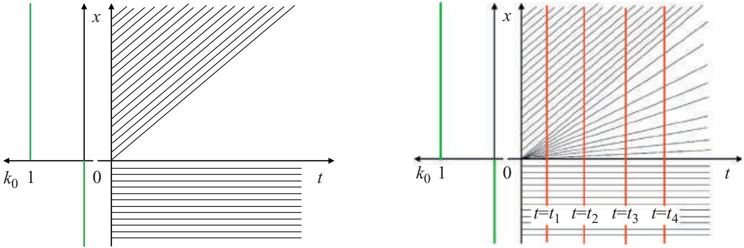
Draw the characteristics and the shock path for the following scenario

$$k_t + q_x = 0$$
$$q = k^2/2$$
$$k(0, x) = 1 \text{ if } x \leq 0 \text{ and } 0 \text{ otherwise}$$



Resolve the previous problem but let the initial conditions be $k(0, x) = 0$ if $x \leq 0$ and 1 otherwise

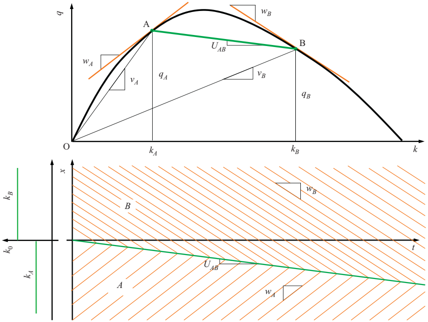
A fan on characteristics and a rarefaction wave are created in this case. The density in the wedge is 0 but there could be multiple characteristics that give the same solution.



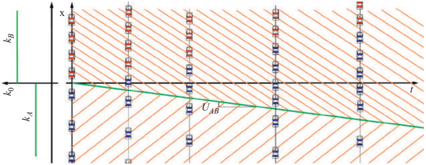
A unique set of characteristics can be generated as shown in the figure on the right using an *entropy condition*.

Are the slopes of the characteristics always constant? Are the slopes of the shock waves constant?

In case of the LWR model, the speeds of the shock waves can be related to the fundamental diagram.



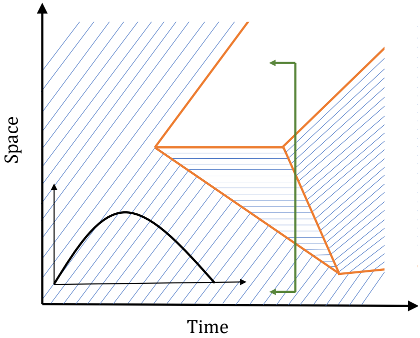
The shock path indicates the last car that enters the slow moving traffic and separates the two density regions.



Shock Waves

LWR Model

Consider a signal at junction and assume a fundamental diagram as shown.

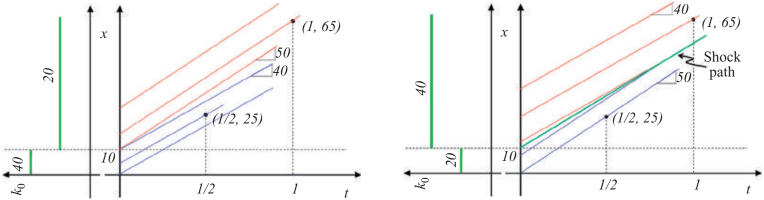


How does the density change along the highlighted cross section and where are these points on the fundamental diagram? Can we derive the RH condition using the speed of the shock wave.

Using the Greenshields fundamental diagram and $v_f = 60$ kmph and $k_j = 240$ vehicles per km, find the density at (0.5 h, 25 km) and (1 h, 65 km) for the following initial conditions

$$k(0, x) = \begin{cases} 40 \text{ vph if } 0 < x \leq 10 \\ 20 \text{ vph if } x > 10 \end{cases}$$

Repeat by switching the density values in the two regions.

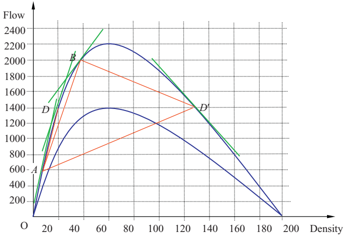


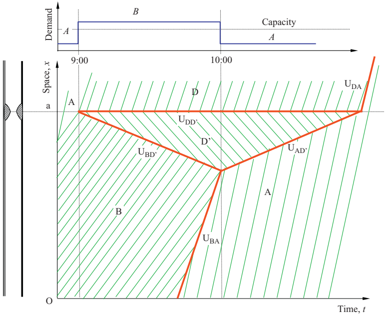
x
20
40
10
k_0
1/2
1
t
(1, 65)
50
40
(1/2, 25)
x
40
20
10
k_0
1/2
1
t
40
(1, 65)
Shock
path
50
(1/2, 25)

Lecture Outline

Queue Analysis

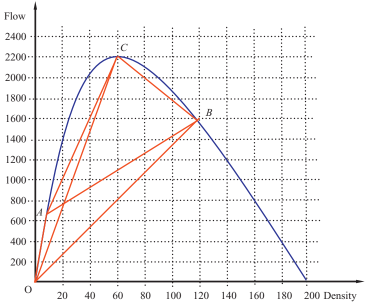
Shock waves can also merge with other shock waves. In the following example, find the length of the queue that forms due to the bottleneck.

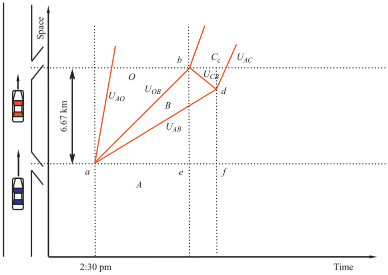




Condition	*q* (vehicles/h)	*k* (vehicles/km)	*v* (km/h)
A	600	8.57	70
B	2000	40	50
D	1400	21.5	65
D'	1400	130	10.8

A truck moving at 13.3 kmph enters a highway at 2:30 PM and leaves it 6.67 km from the entry point. Find the duration for which the truck's effect on the traffic is noticeable.





Condition	q (vehicles/h)	k (vehicles/km)	v (km/h)
A	700	10	70
B	1600	120	13.3
C	2200	60	36.7
O	0	0	75

Your Moment of Zen

Another one of those fundamental diagrams!