CE 269
Traffic Engineering

Lecture 7

Macroscopic Traffic Models

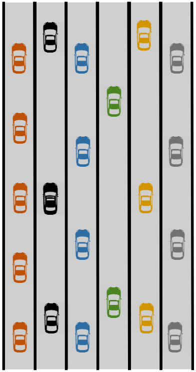
Previously on Traffic Engineering

Let us extend the earlier example to connect the time-mean and space-mean speeds.

Imagine a scenario with multiple lanes $1, \ldots, C$ each with uniform traffic with capacity q_i, density k_i, and speeds v_i.

Let $q = \sum_i q_i$ be the total flow and $k = \sum_i k_i$ be the total density.

Let $f_i = q_i/q$ and $f_i' = k_i/k$ be the proportion of observing a certain colour of vehicle across time and space.



For each lane, we can write $q_i = k_i v_i$ since the headway is q_i and spacing is v_i/q_i.

Previously on Traffic Engineering

Time-mean and space-mean speeds for this setting can be written as

$$v_t = \sum_{i=1}^{C} f_i v_i$$

$$v_s = \sum_{i=1}^{C} f_i' v_i$$

Notice from the definition of the space-mean speed that

$$v_s = \sum_{i=1}^{C} \frac{k_i}{k} v_i = \frac{1}{k} \sum_{i=1}^{C} q_i = \frac{q}{k}$$

Hence, we can write $q = kv_s$ for non-homogeneous traffic but the speed v in this expression is the space-mean speed.

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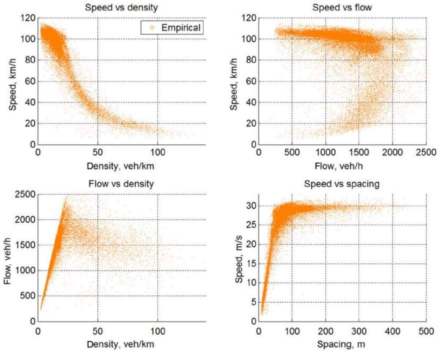
The following is a picture from Ni (2016) with one year of traffic data from a city in US aggregated into 5-minute intervals.



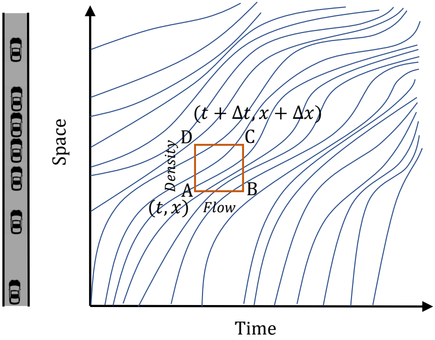
The density values are calculated from the volume and speed measurements.

Lecture Outline

Conservation Equation

Conservation Equation

Introduction



The number of vehicles crossing AB is $q\Delta t$. Likewise, the number of vehicles in AD is $k\Delta x$.

In the limiting case, these two terms must be equal. Hence, $q\Delta t = k\Delta x \Rightarrow q = kv$.

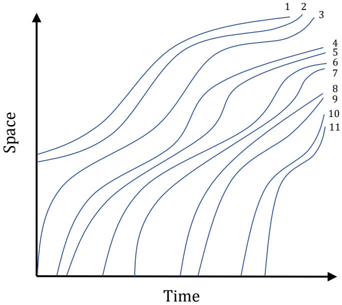
To be more precise with the notation, we can write

$$q(t, x) = k(t, x)v(t, x)$$

Conservation Equation

Cumulative Counts

There is another useful relationship between volume (v) and density (k) that can be derived using the notion of cumulative counts.

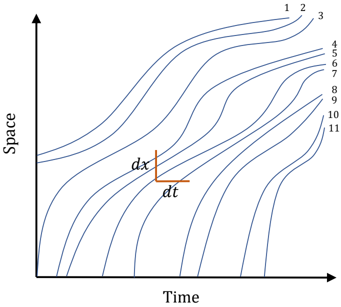


Suppose we number cars in the order in which they appear. Define $N(t, x)$ as the car number of the trajectory closest to the point (t, x). These functions are also referred to as **Moskowitz functions**.

Conservation Equation

Cumulative Counts

Under the continuum approximation assumption, we treat $N(t, x)$ as a continuous function. Hence, we can define its partial derivatives.



$$\frac{\partial N(t,x)}{\partial x} = -k(t,x)$$

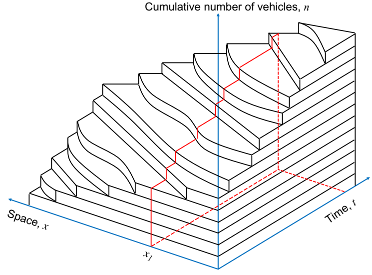
$$\frac{\partial N(t,x)}{\partial t} = q(t,x)$$

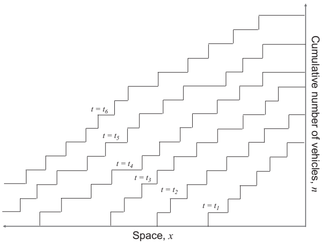
For a continuous function, we can write

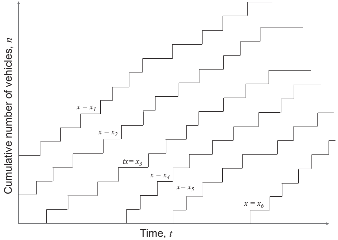
$$\frac{\partial^2 N(t,x)}{\partial t \partial x} = \frac{\partial^2 N(t,x)}{\partial x \partial t}$$

Conservation Equation

Cumulative Counts







Plugging in the expressions for the partial derivatives, we get the following PDE that must be satisfied by the flow and density functions

$$\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$$

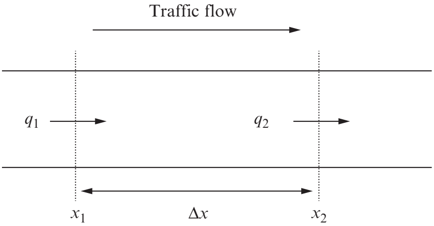
A shorthand way of writing this is $k_t + q_x = 0$

This PDE is also called the "Conservation Law" since it can be derived in a different way by assuming that vehicles do not appear or disappear inside a small infinitesimal region.

Conservation Equation

Alternate Derivation I

Suppose that from t_1 to t_2, a total of ΔN_1 and ΔN_2 vehicles cross locations x_1 and x_2. Suppose $\Delta t = t_2 - t_1$.



$$q_1 = \frac{\Delta N_1}{\Delta t} \qquad q_2 = \frac{\Delta N_2}{\Delta t}$$

Assuming that the traffic densities at t_1 and t_2 are k_1 and k_2, what is the change in the number of vehicles in terms of the flow and density variables?

Conservation Equation

Alternate Derivation I

The change in the number of vehicles in the section in terms of the flow variables are

$$\Delta N = \Delta N_2 - \Delta N_1 = q_2 \Delta t - q_1 \Delta t = \Delta q \Delta t$$

In terms of the density,

$$\Delta N = k_1 \Delta x - k_2 \Delta x = -\Delta k \Delta x$$

From the above equations,

$$\Delta q \Delta t + \Delta k \Delta x = 0$$

$$\frac{\Delta q}{\Delta x} + \frac{\Delta k}{\Delta t} = 0$$

Letting $\Delta x \to 0$ and $\Delta t \to 0$,

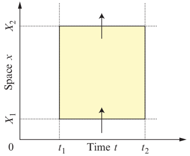
$$\frac{\partial q}{\partial x} + \frac{\partial k}{\partial t} = 0$$

Conservation Equation

Alternate Derivation II

According to Green's theorem, if L and M are functions of (t, x) and have continuous partial derivatives

$$\oint_C (Ldt + Mdx) = \int\int_A \left(\frac{\partial M}{\partial t} - \frac{\partial L}{\partial x}\right) dtdx$$



Setting $L = q$ and $M = -k$,

$$\oint_C (qdt - kdx) = -\int\int_A \left(\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x}\right) dtdx$$

Since the gradient of $N(t, x)$ is $(q, -k)$,

$$\oint_C (qdt - kdx) = N(t_2, x_2) - N(t_1, x_1)$$

which is 0 when C is closed. Since, this is true for every closed C and A, $\partial_t k + \partial_x q = 0$.

Conservation Equation
Summary

So far, we have two equations that connect traffic flow variables:

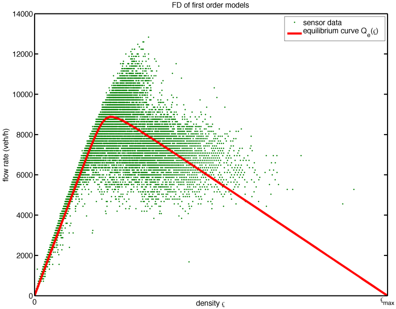
1. $q = kv$
2. $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$

To fully describe these three variables over the domain of interest, it is necessary to have a third equation.

Lecture Outline

LWR Model

The Lighthill Whitham Richards (LWR) model developed in the 50s combines the conservation equation with fundamental diagrams $q = f(k)$.



LWR Model

First-Order PDE

Having the fundamental diagram now gives us three sets of equations, which when solved will give the speed, density, and flow in the domain of interest.

1. $q = kv$
2. $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$
3. $q = f(k)$

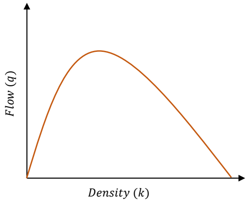
Plugging the fundamental diagram equation in the conservation law, we get a PDE purely in terms of the density

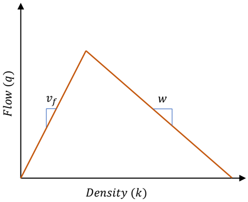
$$\frac{\partial k}{\partial t} + \frac{\partial f(k)}{\partial x} = 0$$

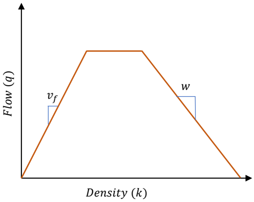
$$\frac{\partial k}{\partial t} + f'(k)\frac{\partial k}{\partial x} = 0$$

This equation is also called **first-order hyperbolic conservation law**.

Most commonly used fundamental diagrams are triangular, and trapezoidal. The parameters of these shapes have to be calibrated from data.







LWR Model

Applications

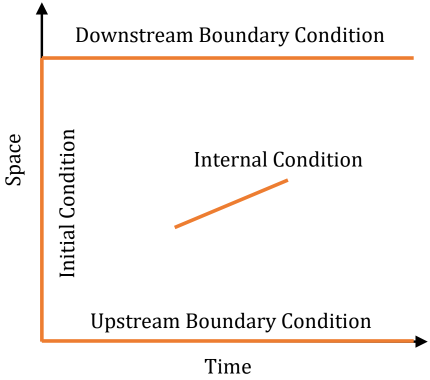
Why do we need a macroscopic model when microscopic models exist?

- Microscopic models are ideal for fine-grained traffic analysis for small networks or corridors. They scale badly for larger networks.
- Macroscopic models are faster to run and hence can be embedded within other frameworks such as dynamic traffic assignment more easily.

Some of the questions that can be addressed with macroscopic models include

- For known initial conditions and inflows, how does traffic evolve over time?
- Where do bottlenecks occur?
- How does congestion spill back, shocks propagate, and how far do queues go?

Solving the PDE requires some knowledge of the density function. This is prescribed in one or more of the following ways:

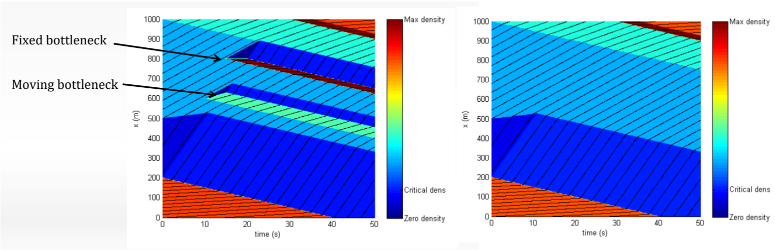


1. Initial Condition
2. Boundary Condition
3. Internal Condition

LWR Model

First-Order PDE

In many cases, given some initial conditions, one can solve the PDE exactly to get the value of density at all points in the domain.



In this course, we will restrict our attention to first-order macroscopic models. They have some limitations such as infinite accelerations, which are handled using second-order macro models.

LWR Model

Lagrangian Coordinates

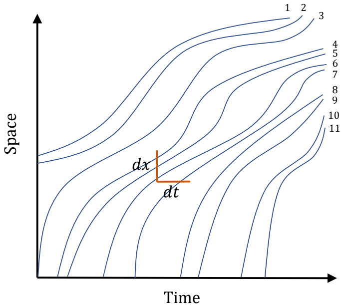
The LWR models described so far is set in Eulerian coordinates. Using a change of variables, it is possible to describe traffic in Lagrangian coordinates. This is sometimes easier to solve.

The variables of interest in Lagrangian coordinates are spacing s and velocity v instead of density k and flow q.

The independent variables are (t, N). That is, we track the individual vehicles over time instead of (t, x).

LWR Model

Lagrangian Coordinates



From the space-time trajectories, we can write

$$v(t, N) = \frac{\partial x(t, N)}{\partial t}$$

$$s(t, N) = -\frac{\partial x(t, N)}{\partial N}$$

Assuming $x(t, N)$ is continuous,

$$\frac{\partial^2 x(t, N)}{\partial t \partial N} = \frac{\partial^2 x(t, N)}{\partial N \partial t}$$

which implies

$$\frac{\partial s(t, N)}{\partial t} + \frac{\partial v(t, N)}{\partial N} = 0$$

LWR Model

Lagrangian Coordinates

We still need the fundamental diagram to write this as an equation in one variable.

To this end, the spacing-speed relationship is used, i.e., $v = f(s)$.

$$\frac{\partial s}{\partial t} + f'(s)\frac{\partial s}{\partial N} = 0$$

Your Moment of Zen