# CE 269
# Traffic Engineering

Lecture 4

## Car Following Models

# Previously on Traffic Engineering

**Headway**: It is the time taken between the arrivals of the front end of successive vehicles.

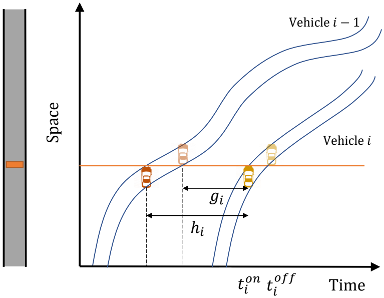
$$h_i = t_i^{on} - t_{i-1}^{on}$$

Note that a vehicle's headway is defined with respect to the vehicle in front.

**Gap**: It is the time between the arrival of the arrival of the rear end of the lead vehicle and the front end of the following vehicle.

$$g_i = t_i^{on} - t_{i-1}^{off}$$

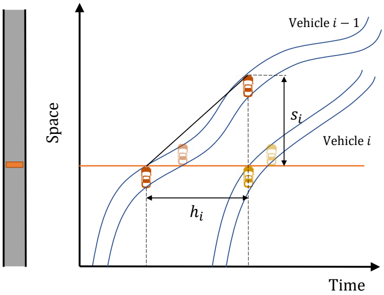
Gap may be viewed as the time to collision if the lead vehicle came to an abrupt stop.



# Previously on Traffic Engineering

The spacing between two vehicles can be approximated using the headway and the velocity of the lead vehicle.

$$s_i = v_{i-1} h_i$$




Note that this is an approximation since it implicitly assumes that the velocity of the lead vehicle remains the same during $h_i$.

# Previously on Traffic Engineering

Exact relationships

- $q = kv_s$
- $v_t = \frac{1}{\Delta N} \sum_i v_i$
- $v_s = \frac{1}{L} \sum_i v_i$
- $v_t = v_s + \frac{\sigma_s^2}{v_s}$, and $v_t \geq v_s$

Approximate relationships

- $s_i \approx v_{i-1} h_i$
- $k \approx \frac{q}{v_t} \left(1 + \frac{q}{v_t} \widehat{Cov}(\mathbf{v}, \mathbf{h})\right)^{-1} \approx \frac{q}{v_t}$
- $v_s \approx \frac{1}{\frac{1}{\Delta N} \sum_i \frac{1}{v_i}}$

# Previously on Traffic Engineering

**Speed and Density**:

$$v = v_f \left(1 - \frac{k}{k_j}\right)$$

where $v_f$ is the free flow speed and $k_j$ is the jam density.

**Flow and Density**:

$$q = v_f \left(k - \frac{k^2}{k_j}\right)$$

What is the maximum flow (capacity) according to the above equation?
$k_m = \frac{k_j}{2}$ and $q_m = \frac{v_f k_j}{4}$.

**Speed and Flow**:

$$q = k_j \left(v - \frac{v^2}{v_f}\right)$$

What is the speed at the maximum flow? $v_m = \frac{v_m}{2}$.

# Lecture Outline

1. Pre-Modern Models
2. Modern Models
3. Post-Modern Models

# Lecture Outline

## Pre-Modern Models

# Pre-Modern Models

## Introduction

Car-following models are at the heart of every traffic simulation. They recommend how a follower vehicle responds to the actions of a lead vehicle.

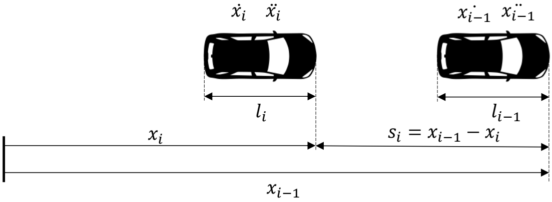
Some of the earliest models were proposed in the 50s and later models attempted to address the drawbacks of those proposed in the previous generations.

Many of the car-following models, in the limit, lead to equilibrium fundamental diagrams that we discussed before.

# Pre-Modern Models

## Introduction

Throughout this lecture, we will refer to the lead vehicle using $i-1$ and the follower using $i$. The symbol $x$ denotes the distance from a reference point and $s_i$ indicates the spacing between the front ends of the vehicles.




The time gap of the following vehicle will be denoted using $g_i$ and the distance between the rear end of the lead vehicle and front end of the following vehicle will be represented as $g_i^x$.

Pipes suggested that vehicles travel such that the following minimum space gap is always maintained.

$$g_i^x(t) \geq \frac{\dot{x}_i(t)}{4.47} l_i$$

This rule was proposed based on a driving rule from a California vehicle code which suggested to maintain a gap of at least the length of a car between your vehicle and the vehicle ahead of you for every ten mile per hour of speed at which you are travelling.

# Pre-Modern Models

## Pipes-Forbes Car Following Theory

Forbes proposed a similar model but the minimum gap was prescribed with reference to the reaction time.

$$g_i(t) \approx h_i(t) - \frac{l_i}{\dot{x}_i(t)} \geq \tau_i$$

where $\tau_i$ is the reaction time of vehicle $i$.

Can you write these expressions in terms of the spacing $s_i(t)$? Assuming vehicles have the same length 6 m and reaction time of 1.5 s,

$$\begin{aligned} s_i(t) &\geq 1.34\dot{x}_i(t) + 6 \qquad &\text{(Pipes)} \\ s_i(t) &\geq 1.50\dot{x}_i(t) + 6 \qquad &\text{(Forbes)} \end{aligned}$$

# Pre-Modern Models

## Evaluating a Car Following Model

What makes a car-following model good?

- It has to be able to different regimes of traffic flow accurately. Particularly, they should
  - Be able to speed up when there are no vehicles in front.
  - Slow down when approaching a stationary vehicle and stop at a safe distance.
  - Follow and trail a leading vehicle in motion.
  - React to sudden decrease in spacing because of lane changes.
- The fundamental diagrams implied by them should resemble that from real-world traffic.

Can you spot any issues with the Pipes-Forbes model?

# Pre-Modern Models

## Evaluating a Car Following Model

Mathematically, let the acceleration of vehicle $i$, $\ddot{x}_i(s_i, v_i, \Delta v_i)$ be written as a function of the spacing $s_i$, velocity of the current vehicle $v_i$, and the speed differential $\Delta v_i = v_i - v_{i-1}$.

Dependence on $t$ is not shown in the above expressions but is implicitly assumed. The required properties can be expressed as

- As vehicles travel faster, they tend to accelerate less

$$\frac{\partial \ddot{x}_i(s_i, v_i, \Delta v_i)}{\partial v_i} < 0$$

- If there is no vehicle in front, drivers prefer to travel at a desired speed

$$\lim_{s_i \to \infty} \ddot{x}_i(s_i, v_i^{max}, \Delta v_i) = 0$$

# Pre-Modern Models

## Evaluating a Car Following Model

- If a lead vehicle is far away, the following vehicle must accelerate

$$\frac{\partial \ddot{x}_i(s_i, v_i, \Delta v_i)}{\partial s_i} \geq 0, \lim_{s_i \to \infty} \frac{\partial \ddot{x}_i(s_i, v_i, \Delta v_i)}{\partial s_i} = 0$$

- Acceleration decreases with increase in speed differential

$$\frac{\partial \ddot{x}_i(s_i, v_i, \Delta v_i)}{\partial \Delta v_i} \leq 0$$

# Pre-Modern Models

## Drawbacks

The Pipes-Forbes model only specifies the spacing to be maintained between the leader and the follower pair. However,

- The velocity of the leader does not feature in the choice of the following vehicle.
- If the spacing is large, the model can suggest unrealistic velocities.
- For a given set of model parameters, the model may not successfully slow down behind a stationary vehicle.

Some of the drawbacks can be fixed by defining additional rules that limit the velocity and acceleration.

Suppose there are $1, \ldots, I$ cars in a traffic stream. Let $\bar{a}_i$ and $\underline{a}_i$ be the maximum acceleration and deceleration of vehicle $i$ and suppose that vehicle $i$ has a desired speed $v_i$.

**Algorithm 1** DISCRETE TIME SIMULATION OF PIPES MODEL

**for** $i = 1, \ldots, I$ **do**
- $s_i(t) \leftarrow x_{i-1}(t-1) - x_i(t-1)$
- $s_i^{min}(t) \leftarrow l_i[\dot{x}_i(t-1)/4.47 + 1]$
- **if** $s_i(t) < s_i^{min}(t)$ **then**
  - $v_i(t) \leftarrow \max\{0, v_i(t-1) - \underline{a}_i \Delta t\}$
- **else**
  - $v_i(t) \leftarrow \min\{v_i, v_i(t-1) + \bar{a}_i \Delta t\}$
- **end if**
- $x_i(t) \leftarrow x_i(t-1) + v_i(t)\Delta t$

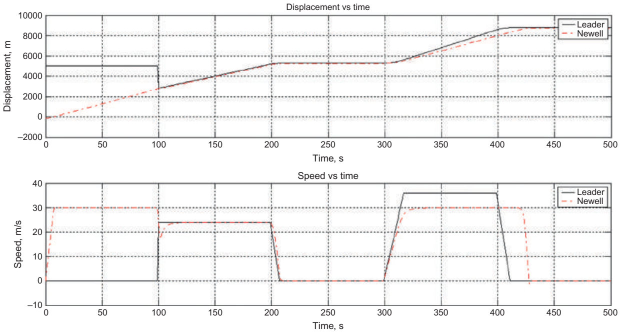
**end for**

# Pre-Modern Models

## Example

The spreadsheet shared with you has positions of the lead vehicle over time. Use the pseudocode to simulate the positions of the follower vehicle.

Assume that the follower vehicle starts at $x = -102$, has a maximum acceleration and deceleration of 4 m/s$^2$ and $-6$ m/s$^2$. Suppose that the desired speed is 30 m/s.



# Pre-Modern Models

## Equilibrium Analysis

The microscopic models can be simulated to generate scatter plots of $q$, $k$, and $v$ and these can be compared with the ones from the field.

Alternately, we can assume steady state conditions and homogeneous traffic and write

$$s = \frac{v}{4.47} l + l \Rightarrow \frac{k_j}{k} = \frac{v}{4.47} + 1$$

Alternately,

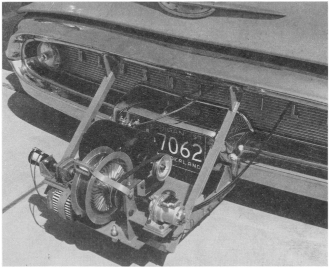
$$v = 4.47 \left( \frac{k_j}{k} - 1 \right)$$

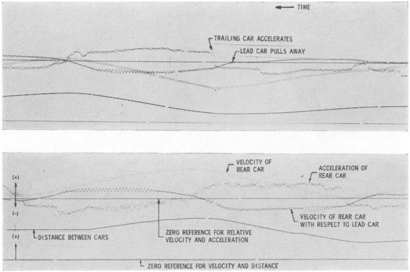
Does this fundamental diagram have any drawbacks?

# Pre-Modern Models

## GM Models

In the 50s, a team of researchers General Motors conducted a series of experiments to study the effect of response of the following vehicle to a lead vehicle.






Also called the GM models or Gazis-Herman-Rothery (GHR) model, they also analyzed the stability of car following models and served as a benchmark for later studies.

# Pre-Modern Models

## GM Car Following Theory

The GM models track the accelerations (*response*) of the follower vehicle as a function of *sensitivity* and *stimuli* such as reaction time, velocity differential, and spacing.

$$\ddot{x}_i(t+\tau_i) = \alpha_{l,m} \frac{\left(\dot{x}_i(t+\tau_i)\right)^m}{\left(x_{i-1}(t) - x_i(t)\right)^l} \left(\dot{x}_{i-1}(t) - \dot{x}_i(t)\right)$$

where $\alpha_{l,m}$ is referred to as the sensitivity coefficient and $l$ and $m$ are the speed and spacing exponents.

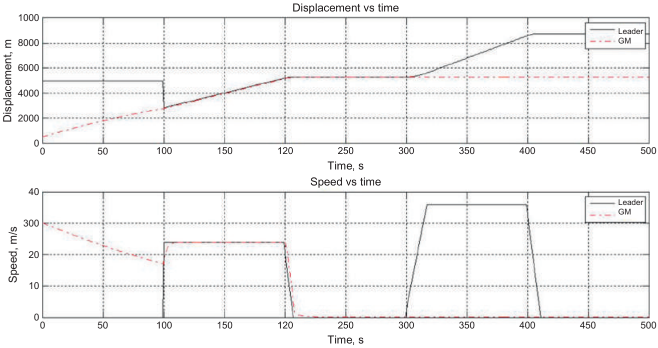
While the GM models take the velocity differential into account, they still have a few drawbacks.

Use the spreadsheet to simulate the GM model with $\tau = 1$ s, $\alpha = 0.8$, $m = 0$, $l = 1$, starting $x = 467$ m, and starting velocity is 30 m/s$^2$.

**Algorithm 2** DISCRETE TIME SIMULATION OF GM MODEL

**for** $i = 1, \ldots, l$ **do**
  $\dot{x}_i(t) \leftarrow \max\{0, \dot{x}_i(t-1) + \ddot{x}_i(t)\Delta t\}$
  $\Delta v \leftarrow \dot{x}_{i-1}(t) - \dot{x}_i(t)$
  $x_i(t) \leftarrow x_i(t-1) + \dot{x}_i(t)\Delta t$
  $s_i(t) \leftarrow x_{i-1}(t) - x_i(t)$
  $\ddot{x}_i(t+\tau) \leftarrow \alpha \dot{x}_i(t) \frac{\Delta v}{s_i(t)}$
**end for**

Note that if the parameters of the model are changed, the predicted trajectory would also be different.




In this GM model, what happens when the spacing between two vehicles travelling at the same speed becomes very small?

# Pre-Modern Models

## Equilibrium Analysis

Suppose $m = 0$ and $l = 1$.

$$\ddot{x}_i(t + \tau_i) = \alpha_{l,m} \frac{\dot{x}_{i-1}(t) - \dot{x}_i(t)}{x_{i-1}(t) - x_i(t)}$$

In steady state,

$$\ddot{x}_i = \alpha_{l,m} \frac{\dot{x}_{i-1} - \dot{x}_i}{x_{i-1} - x_i}$$

Integrating this on both sides with respect to $t$,

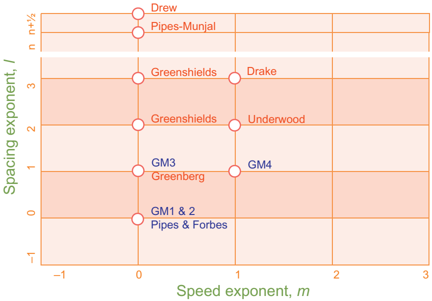
$$\begin{aligned} \int \ddot{x}_i dt &= \int \alpha_{l,m} \frac{\dot{x}_{i-1} - \dot{x}_i}{x_{i-1} - x_i} dt \\ \dot{x}_i dt &= \int \alpha_{l,m} \frac{1}{x_{i-1} - x_i} d(x_{i-1} - x_i) \\ \dot{x}_i dt &= \alpha_{l,m} \log(x_{i-1} - x_i) + C \end{aligned}$$

The LHS can be viewed as $q$ and the RHS contains $\log s = \log\left(\frac{k_j}{k}\right)$. This is equivalent to the Greenberg fundamental diagram.

# Pre-Modern Models

## Equilibrium Analysis

Many other fundamental diagrams can be obtained using different parameters in the GM model.

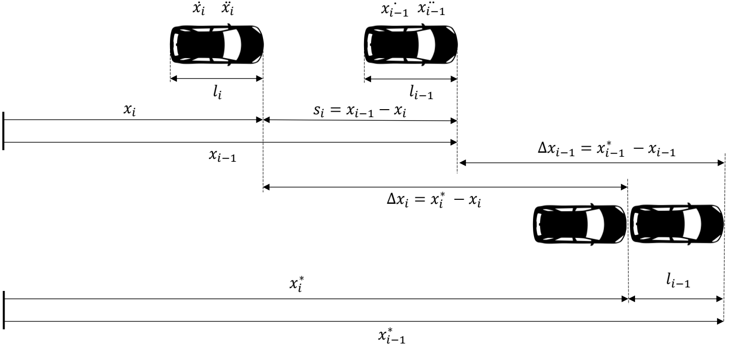


# Modern Models

# Modern Models

## Gipps Model

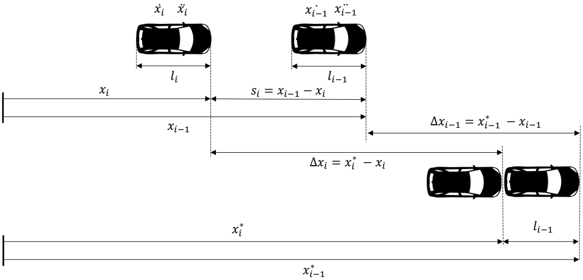
Gipps model is also derived using a safety distance idea but by assuming that the follower chooses actions such that they can brake even if the lead vehicle hard brakes and comes to a complete stop.



# Modern Models

## Gipps Model

Assume that the lead vehicle brakes by decelerating at $\underline{a}_{i-1}^{max}$. It travels a distance of $\Delta x_{i-1}^{*} = \frac{-v_{i-1}^{2}(t)}{2\underline{a}_{i-1}^{max}}$ before coming to a complete stop.




The follower is assume to decelerate at a rate $\underline{a}_i$ after $\tau_i$, the reaction time. Hence, the distance covered by the follower can be written as

$$\Delta x_i = \frac{v_i(t) + v_i(t+\tau_i)}{2}\tau_i - \frac{-v_i^2(t+\tau_i)}{2\underline{a}_i}$$

Gipps suggested that $x_i^* \leq x_{i-1}^* - l_{i-1}$. Plugging the values of $x*$ in this expression, we get

$$s_i(t) \geq \frac{v_i(t) + v_i(t+\tau_i)}{2}\tau_i - \frac{-v_i^2(t+\tau_i)}{2\underline{a}_i} + \frac{v_{i-1}^2(t)}{2\underline{a}_{i-1}^{max}} + l_{i-1}$$

In this inequality, the driver gets to choose $v_i(t+\tau_i)$. In addition to $\tau_i$, Gipps added a safety buffer to the reaction time to allow for some gap if the two vehicles were come to a stop.

Solving the quadratic expression with the safety buffer,

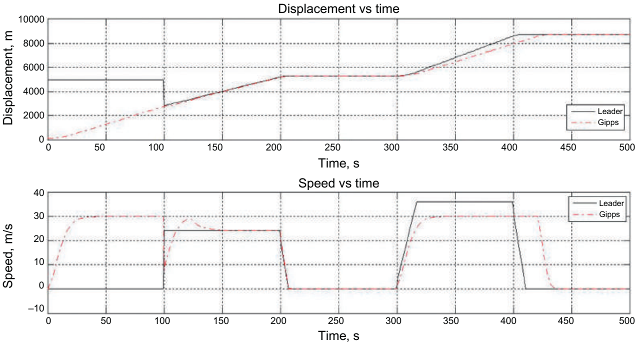
$$v_i(t+\tau_i) \leq -\underline{a}_i\tau_i + \sqrt{\underline{a}_i^2\tau_i^2 - \underline{a}_i\left(-v_i(t)\tau_i\frac{-v_{i-1}^2(t)}{\underline{a}_{i-1}^{max}} - 2l_{i-1} + 2s_i(t)\right)}$$

In addition, to account for the scenarios where there is no vehicle in front, Gipps used empirical data to suggest another function for the velocity.

$$v_i(t+\tau_i) = \min \begin{cases} v_i(t) + 2.5\bar{a}_i\tau_i \left(1 - \frac{v_i(t)}{v_{max}}\right) \left[0.025 + \frac{v_i(t)}{v_{max}}\right]^{1/2} \\ -\underline{a}_i\tau_i + \left[\underline{a}_i^2\tau_i^2 - \underline{a}_i \left(-v_i(t)\tau_i \frac{-v_{i-1}^2(t)}{\underline{a}_{i-1}^{max}} - 2l_{i-1} + 2s_i(t)\right)\right]^{1/2} \end{cases}$$

# Modern Models

## Gipps Model



# Modern Models

## Optimal Velocity Model

In the mid 90s a group of Japanese researchers Bando et al. proposed the optimal velocity model in which the driver is assumed to reach a speed that depends on the spacing. Specifically,

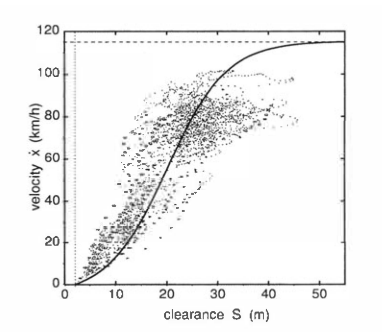
$$\ddot{x}_i(t+\tau_i) = \frac{V^{opt}(s_i(t)) - v_i(t)}{\tau}$$

where $\tau$ is a parameter called the adaptation time.

# Modern Models

## Optimal Velocity Model

The exact shape of the optimal velocity function can be obtained from data.



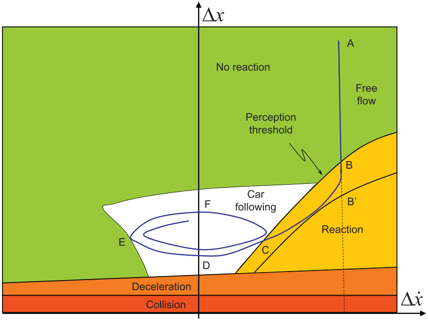

The model has a few drawbacks since the acceleration does not depend on the speed difference. That is, it does not matter if the leader is faster or slower than the following vehicle.

# Post-Modern Models

# Post-Modern Models

## Widermann Car-Following Models

Widermann in 1974 and 1999 proposed two models that has multiple regimes and several parameters to reflect different behaviors. This model is used in VISSIM.




The x-axis represents the speed differential and the y-axis is the spacing.

The intelligent driver model (IDM) by Treiber, Hennecke, and Helbing (2000) predicts the acceleration of a following vehicle using the velocity of the current vehicle and the desired spacing $s^*$ that depends on the speed differential.

$$\ddot{x}_i(t+\tau_i) = \bar{a}_i \left[ 1 - \left( \frac{v_i}{v_i^{max}} \right)^{\delta} - \left( \frac{s_i^*(t)}{s_i(t)} \right)^2 \right]$$

$$s_i^*(t) = s_0 + \max \left( 0, v_i(t) T_i + v_i(t) \frac{v_i(t) - v_{i-1}(t)}{2\sqrt{b_i \bar{a}_i}} \right)$$

Where
$v_i^{max}$ is the desired speed
$T$ is the time gap
$s_0$ is the minimum gap
$\delta$ is the acceleration exponent
$b_i$ is a comfortable value of deceleration.

Advantages of the IDM model:

- Breaking is smoother because of $b_i$ and at the same time, it allows for hard breaking in the case of an emergency.
- Rate of change of acceleration *jerk* is finite.
- The model is parsimonious since each parameter describes only one aspect of driving behaviour which makes it easy to calibrate it.

# Your Moment of Zen