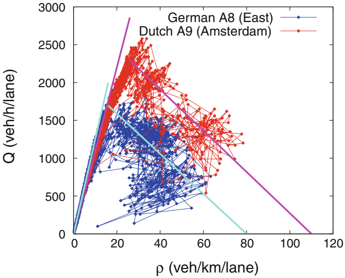
CE 269
Traffic Engineering

Lecture 12

Extensions of the LWR Model

Previously on Traffic Engineering

The flow-density curves often tend to exhibit different behaviour in the un-congested and congested portions.



Phenomena such as *capacity drop* and *dispersion* are commonly observed. This motivates the need for using more parameters or different functions for different regimes of the fundamental diagram.

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Having the fundamental diagram now gives us three sets of equations, which when solved will give the speed, density, and flow in the domain of interest.

1. $q = kv$
2. $\frac{\partial k}{\partial t} + \frac{\partial q}{\partial x} = 0$
3. $q = f(k)$

Plugging the fundamental diagram equation in the conservation law, we get a PDE purely in terms of the density

$$\frac{\partial k}{\partial t} + \frac{\partial f(k)}{\partial x} = 0$$

$$\frac{\partial k}{\partial t} + f'(k)\frac{\partial k}{\partial x} = 0$$

This equation is also called **first-order hyperbolic conservation law**.

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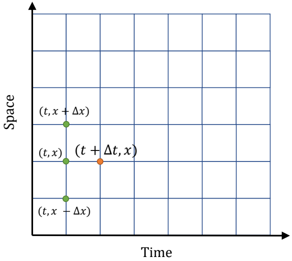
The LWR model PDE $\frac{\partial k}{\partial t} + \frac{\partial f(k)}{\partial x} = 0$ can be approximated using Lax Friedrich-type finite difference method in the following way

$$\frac{k(t+\Delta t, x) - k(t,x)}{\Delta t} + \frac{f(k(t, x+\Delta x)) - f(k(t, x-\Delta x))}{2\Delta x} = 0$$



There are other efficient ways to approximate the PDE, which will be discussed now.

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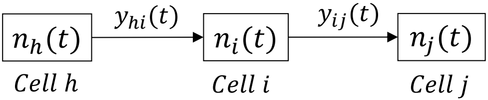
The variables used in CTM are:

- $y_{ij}(t)$: Denotes the flow from cell i to cell j in $[t, t+\Delta t] \equiv [t, t+1]$.
- $n_i(t)$: Number of vehicles in cell i at time t
- N_i: Maximum number of vehicles that can fit in cell i.



Conservation of flow requires

$$n_i(t+1) = n_i(t) + y_{hi}(t) - y_{ij}(t)$$

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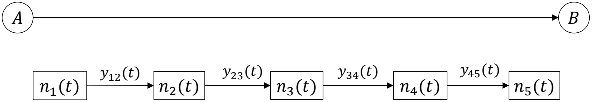
Remember that these iterates give us the flow between cells on a link.



In fact, one can think of cells as miniature links in series and notice that the sending and receiving flow expressions are captured in what we derived.

$$y_{ij}(t) = \min\left\{n_i(t), q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f}\right\}$$

$$y_{ij}(t) = \min\left\{\min\left\{n_i(t), q_{max}\Delta t\right\}, \min\left\{q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f}\right\}\right\}$$

The first minimum is the sending flow of cell i and the second minimum is the receiving flow of cell j.

Lecture Outline

1. Multi-class LWR
2. Multi-lane LWR

Multi-class LWR

So far, we have considered homogeneous traffic using macroscopic models. In a couple of examples related to moving bottlenecks, we had trucks in the traffic which played a limited role.

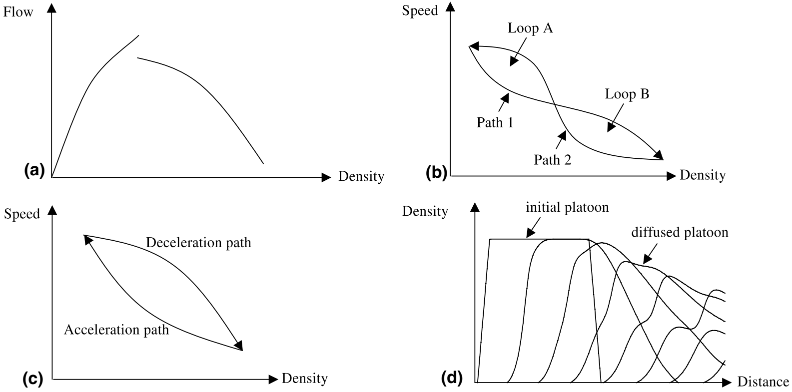
In reality, traffic is more heterogeneous with different types of vehicles and driver behaviours. In this lecture, we tweak the LWR theory to help model such scenarios.

The multi-class LWR model can also explain interesting empirical phenomena that cannot be addressed by the single-class LWR model.

Multi-class LWR

Traffic Phenomena

Some peculiar traffic phenomenon observed in practice include capacity-drop (a), hysteresis (b and c), and platoon dispersion (d).



Multi-class LWR

Traffic Phenomena

Capacity drops or the presence of two-regimes (also called the *reverse-lambda* fundamental diagram) distinguishes the un-congested and congested regions of traffic.

Hysteresis refers to differences in the speed-density profiles (which typically form a loop) during acceleration and deceleration phases that happen during queue buildup and dissipation.

Platoon of traffic (regions with constant density) do not continue for long periods but instead break off due to heterogeneity in vehicle mix/driving styles.

Suppose there are M classes of road users on an uninterrupted highway and let $q_m(t,x)$, $k_m(t,x)$, and $v_m(t,x)$ be the flow, density, and speed of vehicles of type m.

For each class, we can write $q_m(t,x) = k_m(t,x)v_m(t,x)$ using the idea of 'streams' proposed by Wardrop.

Define the total density on the highway as $k(t,x) = \sum_{m=1}^{M} k_m(t,x)$.

Additionally, using the cumulative counts or the other methods used earlier, conservation law for each class of vehicles can be shown to hold.

$$\frac{\partial k_m(t,x)}{\partial t} + \frac{\partial q_m(t,x)}{\partial x} = 0 \; \forall \, m = 1, \ldots, M$$

Multi-class LWR

Modified LWR

In order to find the densities for each class over space and time, we need a third relationship, the fundamental diagram.

The speed for each class can be assumed to depend on a vector densities, i.e., $v_m(t,x) = V_m(k_1, k_2, \dots, k_M) \, \forall \, m = 1, \dots, M$, or simply on the total density of vehicles

$$v_m(t,x) = V_m(k) \, \forall \, m = 1, \dots, M$$

Combining these three equations yields a family of PDEs, one for each class, purely in terms of the densities as shown below.

$$\frac{\partial k_m(t,x)}{\partial t} + \sum_{n=1}^{M} c_{mn}(t,x)\frac{\partial k_n(t,x)}{\partial x} = 0\,\forall\, m = 1,\ldots,M$$

where

$$c_{mn}(t,x) = V_m(t,x)\delta_{mn} + k_m(t,x)\frac{\partial V_m(t,x)}{\partial k_n(t,x)}\,\forall\, m,n = 1,\ldots,M$$

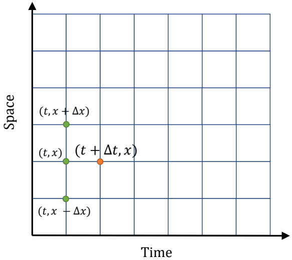
is the equivalent of the derivative of the fundamental diagram or the slope of the characteristics. The value of δ_{mn} is 1 if $m = n$ and is 0 otherwise.

Does this reduce to the single-class LWR when $M = 1$?

Multi-class LWR

Modified LWR

These PDEs can be solved using CTM-like updates or the Lax Friedrich-type finite difference method



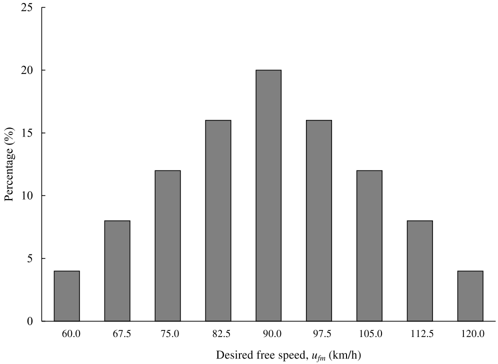
$$k_m(t + \Delta t, x) = \frac{1}{2}\Big(k_m(t, x + \Delta x) + k_m(t, x - \Delta x)\Big) - \frac{\Delta t}{2\Delta x}\Big(q_m(t, x + \Delta x) - q_m(t, x - \Delta x)\Big)$$

How do you get the q values in the above equation?

Multi-class LWR

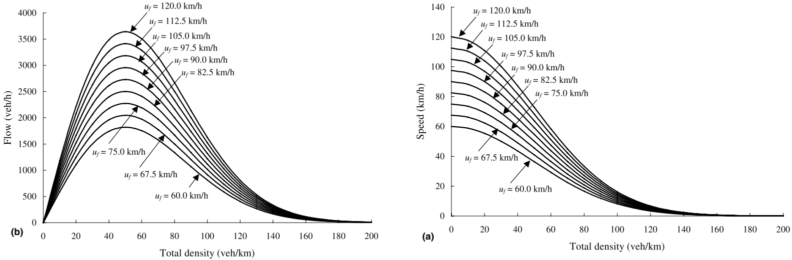
Simulation Inputs

Here are some results from simulations on a highway with 9 classes with the following distribution.



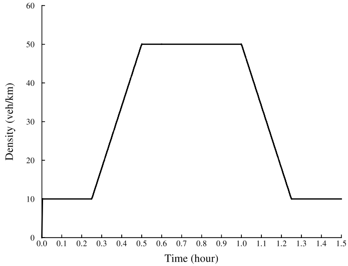
Each class has a different desired speed and the speed-density relationships follow a modified Drake's model as shown below.

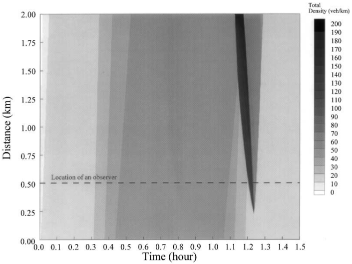
The associated class-specific fundamental diagrams as a function of the total density shows that as the density increases, the variance in the speeds and flows decreases.



Under congested conditions, the distinction between multiple classes is not pronounced.

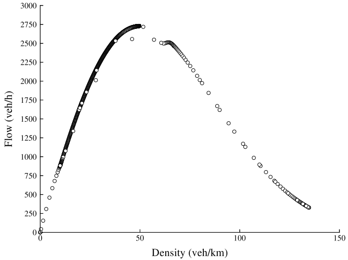
In the first experiment, the highway is assumed to be empty to begin with and a trapezoidal profile for initial density is assumed at the upstream end of the link.

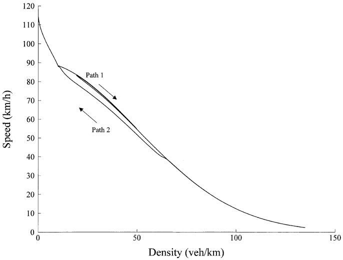




The figure on the right shows the simulated density when the outflow is blocked for a period of 3 min.

The fundamental diagrams for different time-aggregated bins for a point 1.5 km upstream of the bottleneck is shown below.





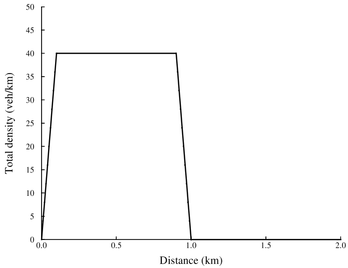
The model replicates the reverse-lambda pattern and the hysteresis loop. As density increases, faster moving vehicles contribute to higher capacity.

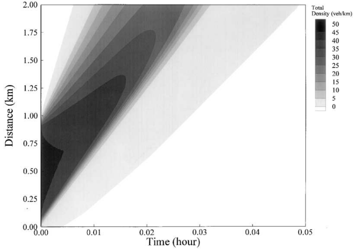
Past the critical density, when the queue builds up, a significant number of slow moving vehicles are in the traffic stream. And when the queue dissipates, the flows and speeds are lower.

Multi-class LWR

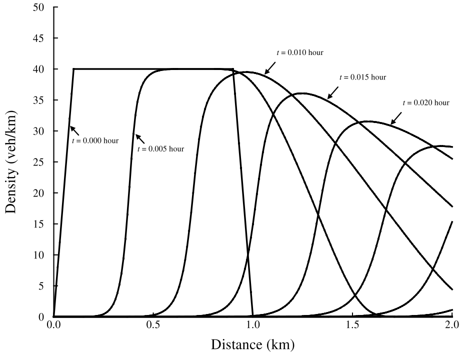
Experiment 2

To replicate, platoon dispersion, consider another experiment where the density profile at $t = 0$ is trapezoidal and there is no incoming traffic at the upstream end over time.





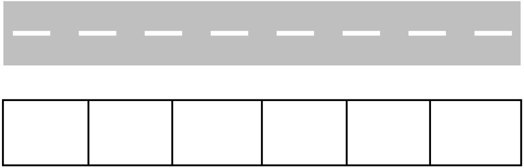
The following figure shows a cross-section of the $k(t, x)$ plots for different time slices.



As can be seen from the figure, the portion of constant density quickly starts to disappear.

Multi-lane LWR

The CTM that we saw so far does not distinguish between lanes and does not capture lane shifts.



The maximum number of vehicles that can fit in each cell would be appropriately adjusted if there are multiple lanes.

Multi-lane LWR

LWR Extension

Lane changing vehicles can act as temporary moving bottlenecks. To model their effect, Munjal and Pipes and Michalopoulos et al. extended the LWR equation for multiple lanes as follows

$$\frac{\partial k_\ell}{\partial t} + \frac{\partial q_\ell}{x} = \Phi_\ell \, \forall \, \ell = 1, \ldots, n$$

where $k_\ell(t, x)$ and $q_\ell(t, x)$ are the density and flow on lane ℓ, and Φ_ℓ is the net lane-changing rate onto ℓ.

If $\Phi_{\ell\ell'}$ denotes the lane-changing rate from ℓ to ℓ', then

$$\Phi_\ell = \sum_{\ell \neq \ell'} \Phi_{\ell'\ell} - \Phi_{\ell\ell'}$$

Multi-lane LWR

In addition to these lane-changing rates, let q_ℓ be the actual flow rate of vehicles continuing on lane ℓ. These two variables may be viewed as the flow variables y in CTM. To estimate them, we can re-use ideas of sending and receiving flows.

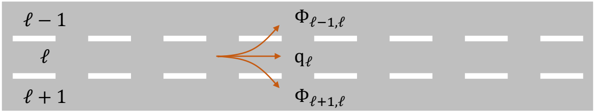
First, we define the following quantities. Let $k(t, x)$ be the vector of densities on all the lanes.

- Desired lane-changing rate $L_{\ell\ell'}(k, t, x)$ from ℓ to ℓ' (i.e., demand)
- Desired flow rate of through movements $T_\ell(k, t, x)$
- Capacity on lane ℓ, $\mu_\ell(k_\ell, t, x)$

Both L and T are similar to sending flows and μ is equivalent to the receiving flows.

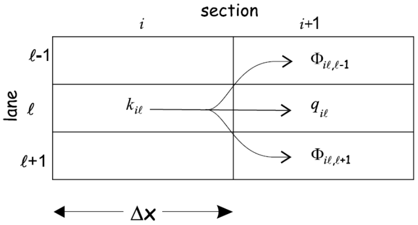
Given the set of demands for lane changes and through movements, assume that we have a mapping which gives the actual lane-change and through movement rates as shown below.

$$(\Phi_{\ell-1,\ell}, q_\ell, \Phi_{\ell+1,\ell}) = \mathcal{F}(L_{\ell-1,\ell}, T_\ell, L_{\ell+1,\ell}, \mu_\ell)$$



The mapping $\mathcal{F}$ can be chosen to capture overtaking rules and nature of lane-changes (discretionary vs. mandatory).

Just as done in CTM, we create cells of size $\Delta x = u\Delta t$, where u is the free flow speed on the link. Suppose we index cells as shown below.



The discretized version of the PDE is

$$\frac{k_\ell(t+1,i) - k_\ell(t,i)}{\Delta t} + \frac{q_\ell(t,i) - q_\ell(t,i-1)}{\Delta x} = \sum_{\ell \neq \ell'} \Phi_{\ell'\ell}(t,i-1) - \Phi_{\ell\ell'}(t,i)$$

Each lane is assumed to have its own (triangular) fundamental diagram.

Multi-lane LWR

Sending and Receiving Flows

The sending flow is the minimum of how many vehicles are available and the capacity (adjusted by the time step).

$$S_{i\ell}(t) = \Delta t \min\{uk_{i\ell}(t), Q\}$$

To split this across downstream cells, we need an routing matrix since we have a diverge-like scenario.

$$L_{i\ell\ell'}(t)\Delta t\Delta x = \pi_{i\ell\ell'}(t)\Delta t S_{i\ell}(t)$$

where $\pi_{i\ell\ell'}(t)$ indicates the proportion of traffic that intends to shift from lane ℓ to ℓ'.

It can be defined based on the velocity differential in the two lanes along with some parameter τ that can be calibrated.

$$\pi_{i\ell\ell'}(t) = \frac{\Delta v_{i\ell\ell'}(t)}{u\tau}$$

Multi-lane LWR

Sending and Receiving Flows

The demand for the through traffic can also be derived from the π values.

$$T_{i\ell}(t)\Delta t = \left(1 - \sum_{\ell' \neq \ell} \pi_{i\ell\ell'}(t)\Delta t\right) S_{i\ell}(t)$$

For the receiving flows, the available capacity μ is defined using the triangular fundamental diagram and backward wave speed

$$\mu_{i\ell}(t) = \min\{w(\kappa - k_{i\ell}(t), Q\}$$

where κ is the jam density.

Multi-lane LWR

Sending and Receiving Flows

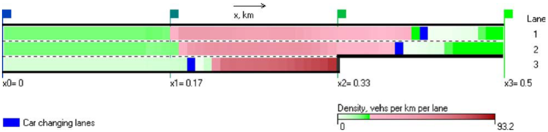
Finally, the flows that move from one cell to the next are computed using the ϕ like notation (which we will denote using γ here) that we used in diverge scenarios of CTM.

$$\gamma_\ell = \min\left\{1, \frac{\mu_\ell}{T_\ell + \sum_{\ell \neq \ell'} \Delta x L_{\ell'\ell}}\right\}$$

Thus, the actual lane-changing and through flow is given by

$$\Phi_{\ell'\ell} = \gamma_\ell L_{\ell'\ell}$$

$$q_\ell = \gamma_\ell T_\ell$$



Additional Reading

Wong, G. C. K., & Wong, S. C. (2002). A multi-class traffic flow model-an extension of LWR model with heterogeneous drivers. Transportation Research Part A: Policy and Practice, 36(9), 827-841.

Levin, M. W., & Boyles, S. D. (2016). A multiclass cell transmission model for shared human and autonomous vehicle roads. Transportation Research Part C: Emerging Technologies, 62, 103-116.

Laval, J. A., & Daganzo, C. F. (2006). Lane-changing in traffic streams. Transportation Research Part B: Methodological, 40(3), 251-264.

Your Moment of Zen