CE 269
Traffic Engineering

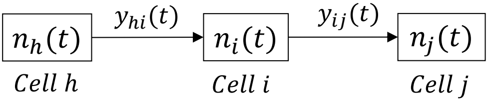
Lecture 11

Node Models

Previously on Traffic Engineering

The variables used in CTM are:

- $y_{ij}(t)$: Denotes the flow from cell i to cell j in $[t, t + \Delta t] \equiv [t, t + 1]$.
- $n_i(t)$: Number of vehicles in cell i at time t
- N_i: Maximum number of vehicles that can fit in cell i.



Conservation of flow requires

$$n_i(t + 1) = n_i(t) + y_{hi}(t) - y_{ij}(t)$$

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$$q = \min \left\{ v_f k, q_{max}, (k_j - k)w \right\}$$

Multiply both sides of the equation with Δt

$$q\Delta t = \min \left\{ v_f k\Delta t, q_{max}\Delta t, (k_j\Delta t - k\Delta t)w \right\}$$

The equivalent formula in terms of y and n variables is given by

$$y_{ij}(t) = \min \left\{ n_i(t), q_{max}\Delta t, \left(N_j - n_j(t) \right) \frac{w}{v_f} \right\}$$

Can you explain this expression in words? The number of vehicles which move from cell i to cell j is limited by the

- Existing number of vehicles $n_i(t)$
- Flow at capacity $q_{max}\Delta t$
- Available space in the downstream cell j. Remember this is congested, so the velocity is lower and hence the factor w/v_f.

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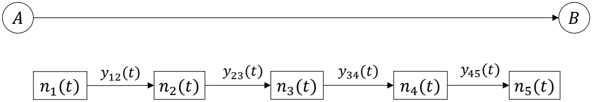
Remember that these iterates give us the flow between cells on a link.



In fact, one can think of cells as miniature links in series and notice that the sending and receiving flow expressions are captured in what we derived.

$$y_{ij}(t) = \min\left\{n_i(t), q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f}\right\}$$

$$y_{ij}(t) = \min\left\{\min\left\{n_i(t), q_{max}\Delta t\right\}, \min\left\{q_{max}\Delta t, \left(N_j - n_j(t)\right)\frac{w}{v_f}\right\}\right\}$$

The first minimum is the sending flow of cell i and the second minimum is the receiving flow of cell j.

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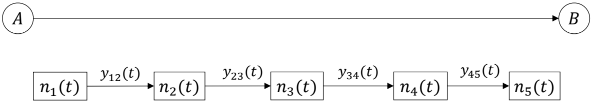
The sending and receiving flows are calculated for the last and the first cells of the link



For instance, the sending flow of link (A, B) is calculated from cell 5.

$$S_{AB}(t) = \min\left\{n_5(t), q_{max}\Delta t\right\}$$

$$R_{AB}(t) = \min\left\{q_{max}\Delta t, \left(N_1 - n_1(t)\right)\frac{w}{v_f}\right\}$$

Using the S and R expressions and the network topology, we update the flows across cells in the next link.

All cell occupancies n and flows y at time step t are updated with values from time step $t-1$ and so on.

Lecture Outline

1. Merges
2. Diverges
3. Example
4. Generalized Models

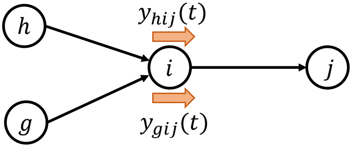
Lecture Outline

Merges

Merges

Introduction

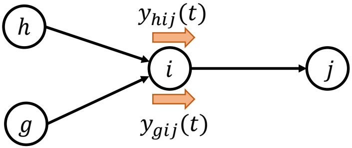
A merge node is when two or more links join into a single link.



We denote using S_{hi} and S_{gi}, the sending flows on links (h, i) and (g, i). Let R_{ij} represent the receiving flow on link (i, j).

The actual flows y_{hij} and y_{gij} are calculated differently for different cases.

Case I: Suppose $S_{hi} + S_{gi} \leq R_{ij}$



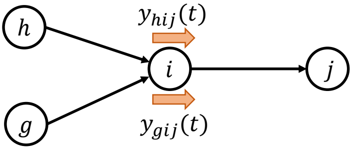
There is no obstruction since the downstream arc can take all the sending flow.

$$y_{hij} = S_{hi}$$

$$y_{gij} = S_{gi}$$

Suppose $q_{gi}^{max} = 1200$ and $q_{hi}^{max} = 2400$. What are the y values when $S_{hi} = 500$, $S_{gi} = 1000$, and $R_{ij} = 2000$?

Case II: Suppose $S_{hi} + S_{gi} > R_{ij}$ and both (h, i) and (g, i) are congested.



The total flow moving from (h, i) and (g, i) must equal R_{ij}.

$$y_{gij} + y_{hij} = R_{ij}$$

The ratio in which incoming flow is divided is assumed to be a function of the associated link capacities, i.e.,

$$y_{hij}/y_{gij} = q_{hi}^{max}/q_{gi}^{max}$$

Suppose $q_{gi}^{max} = 1200$ and $q_{hi}^{max} = 2400$. What are the y values when $S_{hi} = 500$, $S_{gi} = 1000$, and $R_{ij} = 300$?

Merges

Case II

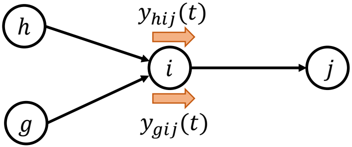
Suppose $q_{gi}^{max} = 1200$ and $q_{hi}^{max} = 2400$. What are the y values when $S_{hi} = 100$, $S_{gi} = 1000$, and $R_{ij} = 300$?

In this case, we end up sending more flow than what we are allowed to. This is treated in a third case, where one of the two approaches can send all its flow and the other is restricted by the merge.

Merges

Case III

Case III: Suppose $S_{hi} + S_{gi} > R_{ij}$ and (h, i) is uncongested and and (g, i) is congested.



Since (h, i) is uncongested, we send all the flow and set $y_{hij} = S_{hi}$ and $y_{gij} = R_{ij} - S_{hi}$.

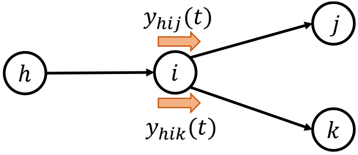
Lecture Outline

Diverges

Diverges

Introduction

A merge node is when one link splits into two or more links.



As before denote using S_{hi} as the sending flow on link (h, i) and R_{ij} and R_{ik} as the receiving flow on link (i, j).

The actual flows y_{hij} and y_{hik} are calculated differently for different cases. A key difference between merges and diverges is that this is where route choice is captured.

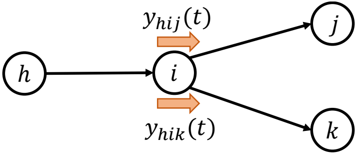
Given the path choices of travelers, we can compute the proportions of those wanting to go to j and to node k. Let us call this p_{ij} and p_{ik} respectively.

Diverges

Case I

Notice that $p_{ij} S_{hi}$ and $p_{ik} S_{hi}$ are the effective sending flows.

Case I: Suppose $p_{ij} S_{hi} \leq R_{ij}$ and $p_{ik} S_{hi} \leq R_{ik}$.



There is no obstruction since both downstream arcs can take all the sending flow.

$$y_{hij} = p_{ij} S_{hi}$$

$$y_{hik} = p_{ik} S_{hi}$$

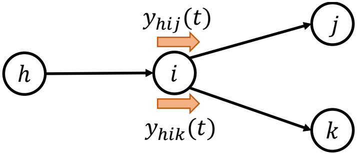
Suppose $p_{ij} = 2/3$. What are the y values when $S_{hi} = 600$, $R_{ij} = 600$, and $R_{ik} = 300$?

Diverges

Case II

Recall that $p_{ij}S_{hi}$ and $p_{ik}S_{hi}$ are the effective sending flows.

Case II: Suppose one of $p_{ij}S_{hi} \leq R_{ij}$ and $p_{ik}S_{hi} \leq R_{ik}$ is violated.



Define ϕ as the proportion of sending flow that can leave (h, i). Then,

$$y_{hij} = \phi p_{ij} S_{hi}$$

$$y_{hik} = \phi p_{ik} S_{hi}$$

What is the value of ϕ? It has to depend on the receiving flows. We try to find the maximum ϕ which preserves $\phi p_{ij}S_{hi} \leq R_{ij}$ and $\phi p_{ik}S_{hi} \leq R_{ik}$.

Diverges

Case II

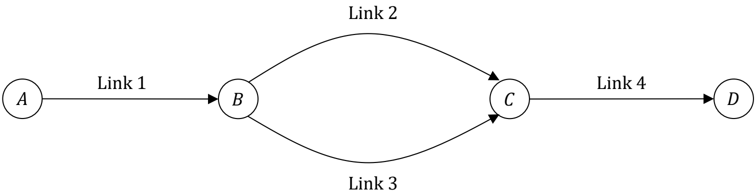
Suppose $p_{ij} = 2/3$. What are the y values when $S_{hi} = 1200$, $R_{ij} = 800$, and $R_{ik} = 300$?

Example

Example

Four-Link Network

Using a spreadsheet, find the cell occupancies for the diverge-merge network shown below.



If the sending and receiving flows are non-zero, different cases in the diverge and merge formula can also be written as

$$y_{gij} = \text{med}\left\{ S_{gi}, R_{ij} - S_{hi}, \frac{q_{gi}^{max}}{q_{gi}^{max} + q_{hi}^{max}} R_{ij} \right\}$$

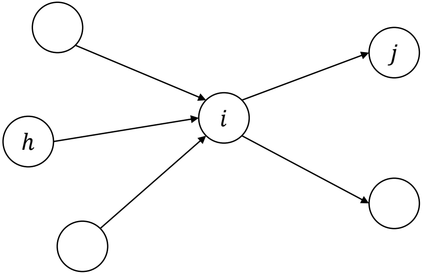
$$\phi = \min\left\{ 1, \frac{R_{ij}}{p_{ij} S_{hi}}, \frac{R_{ik}}{p_{ik} S_{hi}} \right\}$$

Generalized Models

Generalized Models

Introduction

In a network, there could be more than two links coming in and going out of a node.



The concepts we saw earlier can be extended to this setting as well.

The routing decisions are captured using proportions just as done in the diverge case.

At each node, we keep track of an *allocation* or *routing* matrix $P_i(t)$ of size $|A_i| \times |A_i^{-1}|$ which specifies what fraction of travelers on an outgoing link come from different incoming links.

Let A_i and A_i^{-1} be the adjacency and the inverse adjacency list. The elements of the matrix are $p_{ij,hi}(t)$ and satisfy $\sum_j p_{ij,hi}(t) = 1 \; \forall \; h \in A_i^{-1}$

Generalized Models

Allocation Matrices

Suppose the sending flows on the incoming links be denoted by $S_{hi}(t)$ and receiving flows by $R_{ij}(t)$.

Let the actual flows on the incoming and outgoing links be represented by y_{hi} and y_{ij}, respectively.

Then, the following constraints must hold.

$$y_{hi}(t) \leq S_{hi}(t) \; \forall \; h \in A_i^{-1}$$

$$y_{ij}(t) \leq R_{ij}(t) \; \forall \; j \in A_i$$

How are the y variables related to the allocation matrix?

$$y_{ij}(t) = \sum_{h \in A_i^{-1}} p_{ij,hi}(t) y_{hi}(t)$$

These constraints form a polyhedron but do not uniquely identify the flows.

Hence, we add an objective that maximizes the outflow at each node $\max \sum_{j \in A_i} y_{ij}(t)$.

For the earlier example, suppose the sending flows were 100, 200, and 300 and the allocation matrix splits flow equally among the two downstream links. If the receiving flows are 100 and 100, find the y values.

Using this optimization problem, derive the results that we saw earlier for a pair of links in series and a simple merge and a diverge.

$$\begin{aligned} \max \sum_{j \in A_i} & y_{ij}(t) && \\ \text{s.t.} \quad & y_{ij}(t) = \sum_{h \in A_i^{-1}} p_{ij,hi}(t) y_{hi}(t) && \forall (i,j) \in A \\ & y_{hi}(t) \leq S_{hi}(t) && \forall h \in A_i^{-1} \\ & y_{ij}(t) \leq R_{ij}(t) && \forall j \in A_i \\ & y_{ij}(t) \geq 0 && \forall (i,j) \in A \end{aligned}$$

Does this model handle scenarios where the sending and receiving flows are zero in a merge and diverge?

CTM has also been extended to signalized intersection with priority rules.

Your Moment of Zen