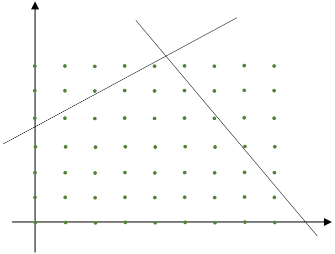
CE 205A
Transportation Logistics

Lecture 9

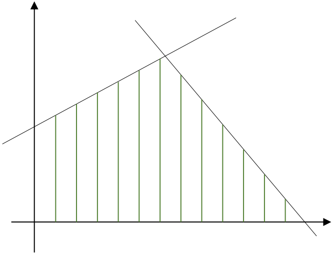
Valid Inequalities

Previously on Transportation Logistics

Integer Program



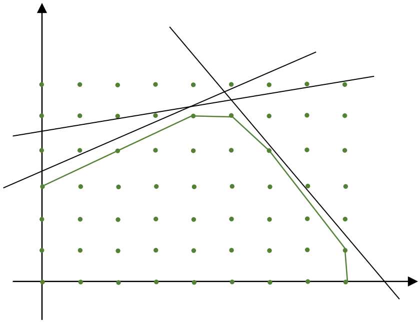
Mixed Integer Program



Are ‘corner points’ solutions optimal? How does the convex hull of the MIP problem look like?

Previously on Transportation Logistics

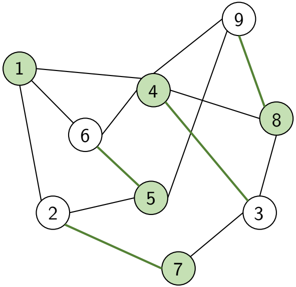
Most methods for solving integer programs rely on relaxations and LP solutions.



An ideal LP relaxation coincides with the convex hull of feasible points. (Why?)

Previously on Transportation Logistics

Consider an undirected graph $G = (V, E)$. A *matching* $M \subseteq E$ is a set of disjoint edges (edges that do not have a node in common). A *node cover* is a set $N \subseteq V$ such that every edge has at least one end point in N.



Formulate the maximum cardinality matching and minimum cardinality cover problems using the set cover/packing/partitioning framework.

Previously on Transportation Logistics

Consider three formulations for the knapsack problem.

$$P_1 = \{x \in [0,1]^4 : 83x_1 + 61x_2 + 49x_3 + 20x_4 \leq 100\}$$
$$P_2 = \{x \in [0,1]^4 : 4x_1 + 3x_2 + 2x_3 + 1x_4 \leq 4\}$$
$$P_3 = \{x \in [0,1]^4 : 4x_1 + 3x_2 + 2x_3 + 1x_4 \leq 4, x_1 + x_2 + x_3 \leq 1, x_1 + x_4 \leq 1\}$$

Do all of these formulations contain the same set of integer solutions? Can you order them on the basis of the strength of the formulations? How are their LP relaxation solutions ordered?

Previously on Transportation Logistics

min problem		max problem
ith constraint $\geq$	$\leftrightarrow$	ith variable ≥ 0
ith constraint $\leq$	$\leftrightarrow$	ith variable ≤ 0
ith constraint $=$	$\leftrightarrow$	ith variable is unrestricted
jth variable ≥ 0	$\leftrightarrow$	jth constraint $\leq$
jth variable ≤ 0	$\leftrightarrow$	jth constraint $\geq$
jth variable is unrestricted	$\leftrightarrow$	jth constraint $=$

Use the above rules and write the dual of the following primal LP:

$$\begin{aligned} \max\ & 8x_1 + 3x_2 - 2x_3 \\ \text{s.t. } & x_1 - 6x_2 + x_3 \geq 2 \\ & 5x_1 + 7x_2 - 2x_3 = -4 \\ & 2x_1 - 3x_2 + 3x_3 \leq 3 \\ & x_1 \leq 0, x_2 \geq 0, x_3 \text{ unrestricted} \end{aligned}$$

Lecture Outline

1. Valid Inequalities
2. Chávtal-Gomory Inequalities
3. Graph-Based Valid Inequalities
4. Cover Inequalities
5. Disjunctive Inequalities and Mixed Integer Rounding

Lecture Outline

Valid Inequalities

Suppose the constraint space X of a MIP contains vectors $(\mathbf{x}, \mathbf{y})$ which satisfy

$$\mathbf{A}\mathbf{x} + \mathbf{G}\mathbf{y} \leq \mathbf{b}$$
$$\mathbf{y} \in \mathbb{R}_+^n, \mathbf{x} \in \mathbb{Z}_+^p$$

If the values in **A**, **G**, and **b** are rational, it is possible to find a convex hull

$$Conv(X) = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^{(n+p)} : \mathbf{A}^{conv}\mathbf{x} + \mathbf{G}^{conv}\mathbf{y} \leq \mathbf{b}^{conv}\}$$

The idea behind studying valid inequalities is to get hyperplanes that are closer to the convex hull.

Valid Inequalities

Definition

Definition

An inequality $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0$ is a valid inequality for $X \subseteq \mathbb{R}^n$ if $\mathbf{w}^\mathsf{T}\mathbf{x} \leq w_0 \ \forall \ \mathbf{x} \in X$. A valid inequality is also denoted as $(\mathbf{w}, w_0)$.

Valid inequalities are usually grouped into families based on how they are identified. While there are general results that hold across all IP problems, much of the theory is best understood using examples.

We will see in subsequent lectures that not all valid inequalities are useful. Those that are closest to the convex hull will help discover an optimum solution faster.

Valid Inequalities
Examples

Sketch the feasible region for the following examples and construct appropriate valid inequalities. Assume $M, b > 0$.

- $X = \{(x, y) : x \leq My, 0 \leq x \leq b, y \in \{0, 1\}\}$
- $X = \{(x, y) : x \leq My, 0 \leq x \leq b, y \in \mathbb{Z}_+\}$
- $X = \{x : x \leq b, x \in \mathbb{Z}_+\}$

How many valid inequalities can you construct for each of the above sets?

Can you identify valid inequalities for the following set?

$$X = \{\mathbf{x} \in [0,1]^5 : 3x_1 - 4x_2 + 2x_3 - 3x_4 + x_5 \leq -2\}$$

If x_2 and x_4 are zero, then, the LHS cannot be ≤ -2. Hence, we can impose the constraint $x_2 + x_4 \geq 1$.

Can $x_1 = 1$ and $x_2 = 0$? This suggest that we can add another valid inequality $x_1 \leq x_2$.

The above example shows that we have to logically answer what if questions to arrive at these valid inequalities. This is also referred to as *probing* and is sometimes used during the preprocessing phase.

Valid Inequalities

Examples

Construct a valid inequality for a polyhedron described by the following inequalities:

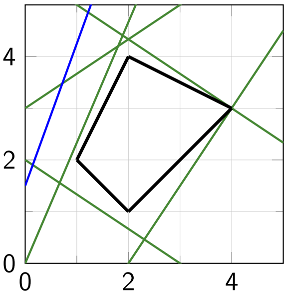
$$
\begin{aligned}
-7x_1 + 3x_2 &\leq 0 \\
-2x_1 - 3x_2 &\leq -6 \\
3x_1 - 2x_2 &\leq 6 \\
-2x_1 + 3x_2 &\leq 9 \\
-2x_1 - 3x_2 &\leq 17 \\
x_1, x_2 &\geq 0
\end{aligned}
$$

What are the inequalities that describe the convex hull? What would you get if you multiplied the inequalities with (2,0,1,0,0) and added them? Is the resulting inequality valid?

Valid Inequalities

Examples

Let $X = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{x}\mathbf{A} \leq \mathbf{b}\}$. Non-negative linear combinations of the constraints, $\boldsymbol{\lambda}^\mathsf{T}\mathbf{A} \leq \boldsymbol{\lambda}^\mathsf{T}\mathbf{b}$ generate valid inequalities of P.

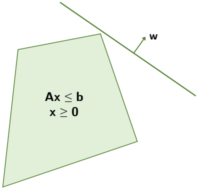


Subtracting a positive quantity from the LHS and adding a positive quantity to the RHS will keep the inequality valid. Hence $\boldsymbol{\lambda}^\mathsf{T}\mathbf{A}\mathbf{x} - \boldsymbol{\mu}^\mathsf{T}\mathbf{x} \leq \boldsymbol{\lambda}^\mathsf{T}\mathbf{b} - d$, for $\boldsymbol{\lambda} \in \mathbb{R}^m_+, \boldsymbol{\mu} \in \mathbb{R}^n_+, d \geq 0$.

Valid Inequalities

Examples

How do you check if a given inequality, e.g., $-11x_1 + 4x_2 \leq 6$ is valid?



Proposition

An inequality $\mathbf{w}^\top \mathbf{x} \leq w_0$ *is a valid inequality for* $X = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \Leftrightarrow \exists\ \mathbf{y} \geq \mathbf{0}$, *such that* $\mathbf{A}^\top \mathbf{y} \geq \mathbf{w}$ *and* $\mathbf{b}^\top \mathbf{y} \leq w_0$.

Proof.

$\mathbf{w}^\top \mathbf{x} \leq w_0$ is a valid inequality $\Leftrightarrow$

$$w_0 \geq \max\ \mathbf{w}^\top \mathbf{x}$$
$$\text{s.t. } \mathbf{A}\mathbf{x} \leq \mathbf{b}$$
$$\mathbf{x} \geq \mathbf{0}$$

The dual problem of the above LP is $\min \mathbf{b}^\top \mathbf{y}$ s.t., $\mathbf{A}^T \mathbf{y} \geq \mathbf{w}, \mathbf{y} \geq 0$. ■

Chávtal-Gomory Inequalities

Chávtal-Gomory Inequalities

Introduction

Chávtal-Gomory inequalities generalize the observations made in the earlier examples and combine it with rounding methods.

They were originally proposed by Chávtal, but they are closely related to Gomory's cuts which will be discussed in subsequent lectures.

Consider the set $X = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{Ax} \leq \mathbf{b}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}_+$ and $\boldsymbol{\lambda} \geq \mathbf{0}$. Recall that the following inequality is valid for X.

$$\sum_{i=1}^{m} \lambda_i \mathbf{A}_{i.} \mathbf{x} \leq \sum_{i=1}^{m} \lambda_i b_i$$

Chávtal-Gomory Inequalities

Introduction

The inequality can also be written as

$$\sum_{j=1}^{n} \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{A}_{.j} x_j \leq \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{b}$$

Since, $\mathbf{x} \geq 0$, rounding the coefficients still makes it a valid inequality for X

$$\sum_{j=1}^{n} \lfloor \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{A}_{.j} \rfloor x_j \leq \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{b}$$

Finally, the following inequality is valid for $X \cap \mathbb{Z}_{+}^{n}$ since the variables are integral.

$$\sum_{j=1}^{n} \lfloor \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{A}_{.j} \rfloor x_j \leq \lfloor \boldsymbol{\lambda}^{\mathsf{T}} \mathbf{b} \rfloor$$

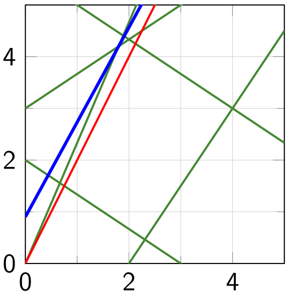
What if we had a mixed integer program? Choose $\boldsymbol{\lambda}$ such that $\boldsymbol{\lambda}^{\mathsf{T}} \hat{\mathbf{A}} \geq \mathbf{0}$ and apply the above steps for the integer variables.

Chávtal-Gomory Inequalities

Introduction

Apply the CG procedure for the earlier example using $\boldsymbol{\lambda} = (7/30, 0, 0, 1/10, 0)$.

$$-\frac{49}{30}x_1 + \frac{7}{10}x_2 \leq 0$$
$$-\frac{2}{10}x_1 + \frac{3}{10}x_2 \leq \frac{9}{10}$$



Adding the above inequalities,

$$-\frac{55}{30}x_1 + x_2 \leq \frac{9}{10}$$

Rounding the LHS and RHS, we get $-2x_1 + x_2 \leq 0$, which is one of the inequalities describing the convex hull.

Chávtal-Gomory Inequalities

Introduction

One can create new valid inequalities using the valid inequalities generated from previous rounds.

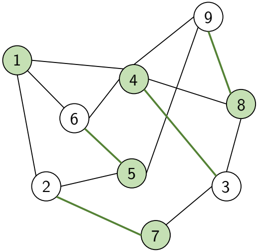
Proposition

Every valid inequality of X for pure integer programs can be derived from repeated application of Chávtal-Gomory for a finite number of times.

Chávtal-Gomory Inequalities

Examples

Consider the matching problem using the set packing formulation. The constraints are of the form $x(\delta(i)) \leq 1 \; \forall \; i \in V$.



What is the maximum number of edges within a set S containing 3 nodes, i.e., what is $x(E(S))$?

What if S contained 5 nodes? Odd number of nodes?

The following is a valid inequality, also called the *odd cut inequalities*, if S has odd cardinality.

$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}$$

Chávtal-Gomory Inequalities

Examples

We can arrive at the same result using CG inequalities. Consider the $x(\delta(i)) \leq 1 \; \forall \; i \in V$ constraints of the matching problem.

Set the weights to $1/2$ for the constraints associated with nodes in S and 0 otherwise.

$$x(E(S)) + \frac{1}{2} x(\delta(S, S^c)) \leq \frac{|S|}{2}$$

Since $x(\delta(S, S^c)) \geq 0$, we can conclude that $x(E(S)) \leq \frac{|S|}{2}$. As the LHS is fractional, we can round the RHS to get $x(E(S)) \leq \lfloor \frac{|S|}{2} \rfloor$. If $|S|$ is odd, we can thus write

$$\sum_{e \in E(S)} x_e \leq \frac{|S| - 1}{2}$$

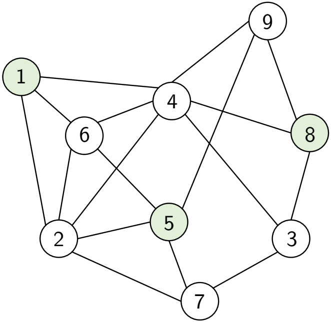
Lecture Outline

Graph-Based Valid Inequalities

Many graph-based problems have specialized valid inequalities that are specific to the problem structure.

Consider a node packing problem in which the goal is to select a subset of vertices but no two chosen vertices must be connected by an edge.

The feasible region is $X = \{x \in [0,1]^n : x_i + x_j \leq 1 \ \forall \ \{i,j\} \in E\}$. Can you construct valid inequalities for this set?



Graph-Based Valid Inequalities

Set Packing Polytope

For every clique C in the graph, only one of the nodes can be active. For example, the valid inequality from $C = \{1, 4, 6\}$ is

$$x_1 + x_4 + x_6 \leq 1$$

Can you spot other clique inequalities?

Note that $C = \{1, 2, 4, 6\}$ is a maximal clique. That is, it cannot be extended to another clique by adding another vertex.

Clique inequalities using maximal cliques are stronger than those from the sub-cliques. (Why?)

Graph-Based Valid Inequalities

Set Packing Polytope

Like the matching problem, we can use subsets of nodes with odd cardinality to create other valid inequalities.

Of special interest is the *odd hole inequalities* which are defined by $H \subset V$ and $|H| \geq 5$ for which H is a "chordless" cycle.

For example, $H = \{5, 9, 8, 3, 7\}$ is an odd hole. $H = \{1, 2, 7, 5, 6\}$ is a cycle of length 5 but has chords $(2, 6)$ and $(2, 5)$.

If H is an odd hole then the following inequality is valid

$$\sum_{i \in H} x_i \leq \frac{|H| - 1}{2}$$

Graph-Based Valid Inequalities

Conflict Graphs

The ideas seen in the probing example and the clique inequalities can be applied to other problems involving binary variables.

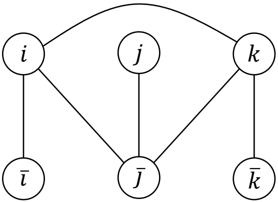
Consider two binary variables x_i and x_j. There are four logical relationships between them.

$$
\begin{aligned}
x_i = 1 \Rightarrow x_j = 1 \Leftrightarrow x_i + x_j \leq 1 \\
x_i = 0 \Rightarrow x_j = 0 \Leftrightarrow (1 - x_i) + x_j \leq 1 \\
x_i = 1 \Rightarrow x_j = 1 \Leftrightarrow x_i + (1 - x_j) \leq 1 \\
x_i = 0 \Rightarrow x_j = 1 \Leftrightarrow (1 - x_i) + (1 - x_j) \leq 1
\end{aligned}
$$

One can create valid inequalities using a *conflict graph* in which each variable is represented by two nodes i and $\bar{i}$, indicating x_i and $(1 - x_i)$, respectively.

We add edges between nodes if both of them cannot be one at the same time. Construct a conflict graph for the following inequalities.

$$x_i + (1 - x_j) \leq 1$$
$$x_i + x_k \leq 1$$
$$(1 - x_j) + x_k \leq 1$$



Graph-Based Valid Inequalities

Conflict Graphs

Using the notion of conflict graphs, we can derive valid inequalities for the set-packing polytope $\mathbf{Ax} = \mathbf{1}$ that we saw in VRPs and crew scheduling.

Two variables, x_i and x_j, cannot both be 1 if there is a customer common to both routes. In other words, $\mathbf{A}_{.i}$ and $\mathbf{A}_{.j}$ have at least one 1 in the same position, i.e., $\mathbf{A}_{.i}^{\mathsf{T}}\mathbf{A}_{.j} > 0$.

A conflict graph can be constructed with nodes as routes and edges connect two routes if they cannot both be 1. Clique inequalities in this graph are valid for the set packing polytope.

Lecture Outline

Cover Inequalities

Cover Inequalities

0-1 Knapsack Set

Consider the Knapsack constraint $X = \{\mathbf{x} \in \{0,1\}^n : \sum_{j=1}^{n} a_j x_j \leq b\}$. Let $N = \{1, \dots, n\}$. Assume that $b > 0$ and $a_j > 0$ for all j. Is this restrictive?

Definition (Cover)

A set $C \subseteq N$ is a cover/dependent set if $\sum_{j \in C} a_j > b$. A cover is minimal if $C \setminus \{j\}$ is not a cover or any $j \in C$.

Determine all covers of $2x_1 + 5x_2 + 3x_3 + x_4 \leq 6$.

- Which of these are minimal?
- What kind of valid inequalities are implied by covers?

Cover Inequalities

0-1 Knapsack

Proposition

If $C \subseteq N$ is a cover for X, then $\sum_{j \in C} x_j \leq |C| - 1$ is valid for X.

Proof.

We prove using the contraposition. Suppose a $\mathbf{x}^*$ does not satisfy the cover inequality. Then, $\sum_{j \in C} x_j^* > |C| - 1$, which implies that $x_j^* = 1 \ \forall \ j \in C$.

$$\sum_{j \in N} a_j x_j^* = \sum_{j \in C} a_j + \sum_{j \in N \setminus C} a_j x_j^* > b$$

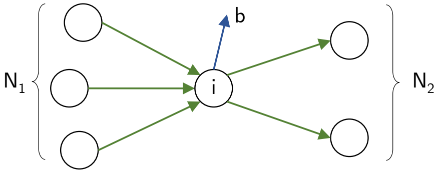
Hence, $\mathbf{x}^* \notin X$. ■

These valid inequalities are also referred to as knapsack 0-1 inequalities.

Flow cover inequalities are valid for mixed 0-1 sets of the following form

$$X = \Big\{ (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^n_+ \times \{0,1\}^n : \sum_{j \in N_1} x_j - \sum_{j \in N_2} x_j \leq b,$$
$$x_j \leq a_j y_j \ \forall\, j \in N_1 \cup N_2 \Big\}$$

The y variables are binary and indicate if the link is allowed to carry flow or not. as can be viewed as capacities.



The knapsack problem is a special case of this with $N_2 = \emptyset$ and $x_j = a_j y_j$.

Cover Inequalities

Mixed 0-1 Sets

Definition (Generalized Cover)

A set $C = C_1 \cup C_2$ where $C_1 \subseteq N_1$ and $C_2 \subseteq N_2$, is a generalized cover for X if $\sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j > b$.

The difference $\lambda = \sum_{j \in C_1} a_j - \sum_{j \in C_2} a_j - b > 0$ is called the *cover excess*.

Identify a generalized cover in the following example.

$$X = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^6_+ \times \{0,1\}^6 : x_1 + x_2 + x_3 - x_4 - x_5 - x_6 \leq 4, \\ x_1 \leq 3y_1, x_2 \leq 3y_2 \\ x_3 \leq 6y_3, x_4 \leq 3y_4 \\ x_5 \leq 5y_5, x_6 \leq y_6\}$$

Cover Inequalities

Mixed 0-1 Sets

Proposition

Let $L_2 \subseteq N_2 \setminus C_2$. Then, the following inequality is valid for X.

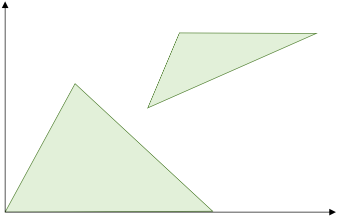
$$\sum_{j \in C_1} x_j + \sum_{j \in C_1} (a_j - \lambda)^+ (1 - y_j) - \sum_{j \in C_2} a_j - \lambda \sum_{j \in L_2} y_j - \sum_{j \in N_2 \setminus (C_2 \cup L_2)} x_j \leq b$$

where $a^+ = \max\{a, 0\}$.

Apply the above result to develop a valid inequality for the previous example using $C_1 = \{1, 3\}$, $C_2 = \{4\}$, and $L_2 = \{5\}$.

Disjunctive Inequalities and MIR

Consider a disjunction $X = X_1 \cup X_2$, where $X_1, X_2 \subset \mathbb{R}^n_+$. We encountered such feasible regions in problems in either-or-or type constraints.



$$X_1 = \{\mathbf{x} \in \mathbb{R}^2_+ : -x_1 + x_2 \leq 1, \quad x_1 + x_2 \leq 5\}$$

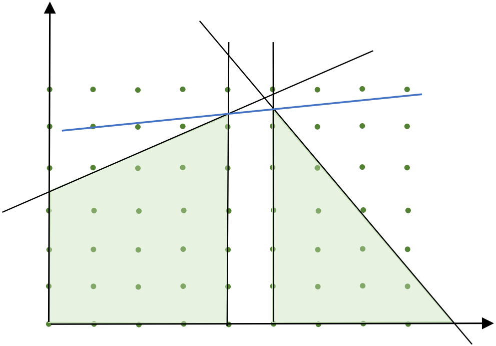
$$X_2 = \{\mathbf{x} \in \mathbb{R}^2_+ : x_2 \leq 4, \quad -2x_1 + x_2 \leq -6 \quad -3x_2 \leq -2\}$$

Can you identify a valid inequality in the above example? Note that a valid inequality of X is valid for X_1 and X_2 but not vice versa.

Disjunctive Inequalities and MIR

Introduction

Disjunction sets also appear in the Branch and Bound-type decomposition. Suppose $X = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}\mathbf{x} \le \mathbf{b}\}$.



$$X_1 = X \cap \{\mathbf{x} \in \mathbb{R}^n_+ : x_1 \le \lfloor d \rfloor\}$$

$$X_2 = X \cap \{\mathbf{x} \in \mathbb{R}^n_+ : x_1 \ge \lfloor d \rfloor\}$$

Proposition

Let $X_i = \{\mathbf{x} \in \mathbb{R}^n_+ : \mathbf{A}^i\mathbf{x} \leq \mathbf{b}^i\}$ *for* $i = 1, 2$. *If* $(\mathbf{w}^1, w_0^1)$ *is valid for* X_1 *and* $(\mathbf{w}^2, w_0^2)$ *is valid for* X_2*, then the following inequality is valid for* $X = X_1 \cup X_2$

$$\sum_{j=1}^{n} \min\{w_j^1, w_j^2\} x_j \leq \max\{w_0^1, w_0^2\}$$

Proof.

(WTS) $\mathbf{x} \in X$ satisfies the given valid inequality. Since $\mathbf{x} \in X_1$ or $\mathbf{x} \in X_2$,

$$\sum_{j=1}^{n} w_j^1 x_j \leq w_0^1 \text{ or } \sum_{j=1}^{n} w_j^2 x_j \leq w_0^2$$

Disjunctive Inequalities and MIR

Contd.

$\min\{w_j^1, w_j^2\} \leq w_j^1$ and $\min\{w_j^1, w_j^2\} \leq w_j^2$ for all $j = 1, \ldots, n$. Since all xs are ≥ 0,

$$\sum_{j=1}^{n} \min\{w_j^1, w_j^2\} x_j \leq \sum_{j=1}^{n} w_j^1 x_j \leq w_0^1$$

$$\text{or } \sum_{j=1}^{n} \min\{w_j^1, w_j^2\} x_j \leq \sum_{j=1}^{n} w_j^2 x_j \leq w_0^2$$

Thus, $\sum_{j=1}^{n} \min\{w_j^1, w_j^2\} x_j \leq \max\{w_0^1, w_0^2\}$. ■

How do you check if a given inequality $(\mathbf{w}, w_0)$ is valid for a disjunction $X = X_1 \cup X_2$? Solve the following LPs for $i = 1, 2$.

$$
\begin{aligned}
z_i \geq \max\ & \mathbf{w}^\mathsf{T}\mathbf{x} \\
\text{s.t. } & \mathbf{A}^i\mathbf{x} \leq \mathbf{b}^i \\
& \mathbf{x} \geq \mathbf{0}
\end{aligned}
$$

$(\mathbf{w}, w_0)$ is valid for $X \Leftrightarrow w_0 \geq \max\{z_1, z_2\}$. This condition is equivalent to the existence of $\mathbf{y}^1, \mathbf{y}^2 \geq \mathbf{0}$, such that $\mathbf{A}^{i\mathsf{T}}\mathbf{y}^i \geq \mathbf{w}$ and $\mathbf{b}^{i\mathsf{T}}\mathbf{y}^i \leq w_0$ for $i = 1, 2$.

Disjunctive Inequalities and MIR

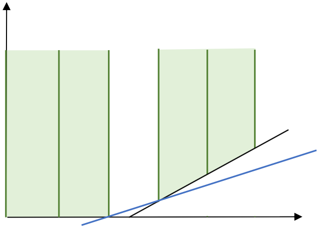
Rounding

As before, non-negative linear combinations of MIP constraints yield valid inequalities. These are not very useful to get closer to the convex hull. As done in the case of the pure integer version, we can round certain terms.

To motivate this, consider the mixed-integer set $X = \{(x, y) \in \mathbb{Z}_+ \times \mathbb{R}_+ : x - y \leq b\}$. Sketch the feasible region and show that

$$x - \frac{1}{1 - f_0} y \leq \lfloor b \rfloor$$

is valid for X where $f_0 = b - \lfloor b \rfloor$.



Can you spot the disjunction? $x_1 \leq \lfloor b \rfloor$ or $x_1 \geq \lfloor b \rfloor + 1$.

Derive the valid inequality using a non-negative combination of these constraints. In the second case use $\frac{f_0}{1-f_0}$ and $\frac{1}{1-f_0}$ on the new and the linear constraint. Why not round all the continuous terms?

Disjunctive Inequalities and MIR

The idea we just saw can be generalized to any mixed integer inequality.

Proposition

Let $X = \{\mathbf{x} \in \mathbb{Z}_+^n, \mathbf{y} \in \mathbb{R}_+^p : \sum_{j \in N} a_j x_j + \sum_{j \in J} g_j y_j \leq b\}$*, where* $N = \{1, \dots, n\}$*,* $J = \{1, \dots, p\}$*, and* $a_j, g_j, b \in \mathbb{R}$ *for all* j*. Then,*

$$\sum_{j \in N} \lfloor a_j \rfloor x_j + \frac{1}{1 - f_0} \sum_{j \in J_{<0}} g_j y_j \leq \lfloor b \rfloor$$

is valid for X*, where* $J_{<0} = \{j \in J : g_j < 0\}, f_0 = b - \lfloor b \rfloor$.

Proof.

Case I: Let $\sum_{j \in J} g_j y_j > f_0 - 1$

$$\sum_{j \in N} \lfloor a_j \rfloor x_j \leq \sum_{j \in N} a_j x_j \leq b - \sum_{j \in J} g_j y_j < b - (f_0 - 1) = \lfloor b \rfloor + 1$$

Proof.

But $\sum_{j \in N} \lfloor a_j \rfloor x_j$ is integral, hence

$$\sum_{j \in N} \lfloor a_j \rfloor x_j < \lfloor b \rfloor + 1 \Rightarrow \sum_{j \in N} \lfloor a_j \rfloor \leq \lfloor b \rfloor$$

Note that $(1 - f_0) \in (0, 1]$. How do we get the valid inequality from here?

Case II: Let $\sum_{j \in J} g_j y_j \leq f_0 - 1 \Rightarrow \sum_{j \in J_{<0}} g_j y_j \leq f_0 - 1$

$$\begin{aligned} \sum_{j \in N} \lfloor a_j \rfloor x_j + \frac{1}{1 - f_0} \sum_{j \in J_{<0}} g_j y_j &\leq \sum_{j \in N} a_j x_j + \frac{1}{1 - f_0} \sum_{j \in J_{<0}} g_j y_j \\ &= b - \sum_{j \in J} g_j y_j + \frac{1}{1 - f_0} \sum_{j \in J_{<0}} g_j y_j \\ &\leq b + \frac{f_0}{1 - f_0} \sum_{j \in J_{<0}} g_j y_j \leq \lfloor b \rfloor \end{aligned}$$

■

Your Moment of Zen

"Algebra class will be important to you later in life because there's going to be a test six weeks from now."

Source:glasbergen.com