CE 205A
Transportation Logistics

Lecture 6

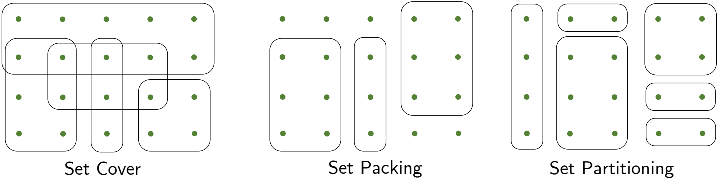
Vehicle Routing Problem

Previously on Transportation Logistics

Consider a set $S = \{1, \ldots, m\}$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_n\}$ be a collection of subsets of S. These subsets could for instance satisfy some property.

- A collection $X \subseteq \mathcal{S}$ is a *cover* of S if $\cup_{S_i \in X} S_i = S$
- X is a *packing* if $S_i \cap S_j = \emptyset \, \forall \, S_i, S_j \in X$
- X is a *partition* if it is both a cover and a packing.

E.g., Let $S = \{1, 2, 3, 4, 5\}$. What are the members of $\mathcal{S}$ if every element of it has a cardinality of at least 2?



Construct an example of a cover, packing, partition for the above example?

Previously on Transportation Logistics

Let y_i be 1 if bin i is used and is 0 otherwise. Let x_{ij} take a value 1 if item j is assigned to bin i and is 0 otherwise.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} y_i & \\
\text{s.t.} \quad & \sum_{j=1}^{n} w_j x_{ij} \leq c y_i & \forall i = 1, \ldots, n \\
& \sum_{i=1}^{n} x_{ij} = 1 & \forall j = 1, \ldots, n \\
& y_i \in \{0, 1\} & \forall i = 1, \ldots, n \\
& x_{ij} \in \{0, 1\} & \forall i = 1, \ldots, n, j = 1, \ldots, n
\end{aligned}
$$

The first constraint is an example of *forcing constraints* of the type $x \leq My$, where one of the variables is allowed to take a non-negative quantity only if the other is active.

Previously on Transportation Logistics

For directed graphs, the shorthand A-δ notation can be extended by additionally defining the following symbols

- $\delta^+(S)$: Set of arcs with tail node in S and head node in S^c.
- $\delta^-(S)$: Set of arcs with tail node in S^c and head node in S.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} & \\
\text{s.t.} \quad & x(\delta^+(i)) = 1 & \forall\, i \in N \\
& x(\delta^-(i)) = 1 & \forall\, i \in N \\
& x(A(S)) \leq |S| - 1 & \forall\, S \subset N, S \neq \emptyset, |S| \geq 2 \\
& x_{ij} \in \{0,1\} & \forall\, (i,j) \in A
\end{aligned}
$$

The SEC constraint can be replaced with $x(\delta^+(S)) + x(\delta^-(S)) \geq 2$. How many variables and constraints are present in this DFW formulation?

Previously on Transportation Logistics

Alternately, Miller, Tucker, and Zemlin's model (MTZ) can be used which keeps track of u_i which is the sequence in which city i is visited. Wlog, let node 1 be the origin of the tour, i.e., $u_1 = 1$.

$$
\begin{aligned}
\min \quad & \sum_{(i,j) \in A} c_{ij} x_{ij} & \\
\text{s.t.} \quad & x(\delta^+(i)) = 1 & \forall\, i \in N \\
& x(\delta^-(i)) = 1 & \forall\, i \in N \\
& u_i - u_j + 1 \leq (n-1)(1 - x_{ij}) & \forall\, i, j \in \{2, \dots, n\}, i \neq j \\
& 2 \leq u_i \leq n & \forall\, i \in N \setminus \{1\} \\
& x_{ij} \in \{0, 1\} & \forall\, (i,j) \in A
\end{aligned}
$$

What happens to the SEC when x_{ij} equals 1 and 0? Note: The SEC constraint appears in slightly different formats in different papers.

Previously on Transportation Logistics

In the TSPTW, each city i in a directed graph $G = (N, A)$ has an associated time window $[a_i, b_i]$ within which it must be visited. Suppose the travel time to go from i to j is t_{ij}.

The traveler can reach early, in which case they wait till the start of the time window. Suppose 0 is the starting city and y_i is the time at which customer i is visited, then the TSPTW can be formulated as

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij} x_{ij} & \\
\text{s.t.} \quad & x(\delta^+(i)) = 1 & \forall\, i \in N \\
& x(\delta^-(i)) = 1 & \forall\, i \in N \\
& y_j \geq y_i + t_{ij} - M(1 - x_{ij}) & \forall\, i, j \in N, j \neq 0 \\
& a_i \leq y_i \leq b_i & \forall\, i \in N \\
& x_{ij} \in \{0, 1\} & \forall\, (i, j) \in A
\end{aligned}
$$

Lecture Outline

Lecture Outline

Vehicle Routing Problem

The VRP involves serving a set of customer demands at different locations using a vehicle fleet located at one or more depots.

Assume all vehicles are homogeneous and have the same capacity. Consider a graph $G = (N, A)$. Let $N = \{0, 1, \dots, n\}$, where 0 is the depot and vertices 1 to n are customer locations.

The cost of traversing an edge (i, j) is c_{ij}. Customer i's demand is denoted by d_i and a set of vehicles K each with capacity $C \geq d_i, \forall i \in V \backslash \{0\}$ are assumed to be available.

The goal is to find a partition of customers $S_1, \dots, S_{|K|}$ and the routes taken by a vehicle that serves each partition to minimize the overall cost.

Vehicle Routing Problem

Greatest Hits

Dantzig and Ramser (1959)

Clarke and Wright (1964)

Balinski and Quandt (1964)

Gillett and Miller (1974)

Christofides et al. (1981)

Laporte, Mercure, and Nobert (1986)

Desrochers, Desrosiers, and Solomon (1992)

Cornuejols and Harche (1994)

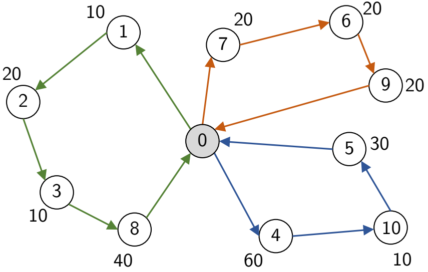
Feillet et al. (2004)

Most common formulations for the capacitated vehicle routing problem (CVRP) are

- Two-index formulation
- Three-index formulation
- Commodity flow formulation
- Set-Partitioning models

The first two formulations are also referred to as vehicle-flow formulations.

Let $r(S)$ as the minimum number of vehicles required to serve the demand of a subset of customers $S \subseteq N \backslash \{0\}$.



Suppose the capacity of the truck in the above example is 100. What is $r(\{3, 4, 6\})$?

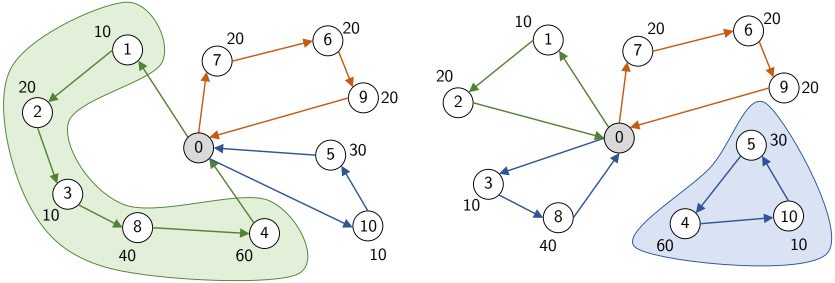
Solve a bin-packing problem with S as the set of items and $|S|$ bins, each with capacity C. A lower bound for $r(s)$ is $\lceil \sum_{i \in S} d_i / C \rceil$ (Why?).

The two-index formulation keeps track of binary variables x_{ij} which is 1 if arc (i, j) is used and is 0 otherwise.

$$\begin{aligned}
\min \quad & \sum_{(i,j)\in A} c_{ij}x_{ij} \\
\text{s.t.} \quad & x(\delta^+(0)) = x(\delta^-(0)) = |K| \\
& x(\delta^+(i)) = 1 && \forall\, i \in N\backslash\{0\} \\
& x(\delta^-(i)) = 1 && \forall\, i \in N\backslash\{0\} \\
& x(\delta^+(S)) \geq r(S) && \forall\, S \subseteq N\backslash\{0\}, S \neq \emptyset \\
& x_{ij} \in \{0, 1\} && \forall\, (i, j) \in A
\end{aligned}$$

The fourth constraint, also called as the *capacity-cut constraints* (CCC) addresses both capacity limits as well as prevents sub-tours.

Verify that the following solutions violate the cut-capacity constraints.



It can be also reformulated in the following form familiar SEC constraint form which imply that at least $r(S)$ arcs/vehicles must go out of S.

$$x(A(S)) \leq |S| - r(S) \qquad S \subseteq N \backslash \{0\}, S \neq \emptyset$$

Can you show that this is equivalent to CCC? Use the fact that the number of arcs in a solution is $n + |K|$.

Vehicle Routing Problem

Two-index Formulation

What if some of the vehicles are unused? If $|K| > r(N \setminus \{0\})$, then we can replace the first inequality with $\leq$. Does having more vehicles than needed imply that the objective is higher?

One can also add fixed costs to the problem by introducing costs for the edges out of the depot.

Excess vehicles can also be routed outside the network. That is, create two node for the depot 0 and $n+1$ and connect them by a zero cost edge.

Vehicle Routing Problem

Two-index Formulation

Just as in TSP, the SEC constraints can also be modeled using MTZ constraints

$$
\begin{aligned}
& d_i \leq u_i \leq C && \forall\ i \in N \setminus \{0\} \\
& u_i - u_j + d_j \leq C(1 - x_{ij}) && \forall\ (i,j) \in A(N\setminus\{0\}) : d_i + d_j \leq C
\end{aligned}
$$

where u_i is a continuous variable which represents the load on a vehicle after visiting customer i.

How are sub-tour constraints met? Note that if $x_{ij} = 1$, $u_j > u_i$. What happens if you had a sub-tour $i_1, i_2, \ldots, i_1$?

As before, the LP relaxations of the MTZ version are often weak.

Vehicle Routing Problem

Two-index Formulation

Two-index formulation uses fewer variables. Its solutions are also unique unlike the subsequent models in which multiple solutions can be generated by re-numbering vehicles.

But it does not tell us which vehicle travels on which link. This is a problem if vehicle fleet is heterogeneous or if the cost depends on vehicle type (i.e., capacities are different, or some are EVs).

Vehicle Routing Problem

Three-index Formulation

In this formulation, x_{ijk} is 1 if vehicle k uses link (i,j) and y_{ik} is 1 if customer i is served by vehicle k.

$$
\begin{array}{lll}
\min & \sum_{(i,j)\in A} \sum_{k\in K} c_{ijk} x_{ijk} & \\
\text{s.t.} & \sum_{k\in K} y_{0k} = |K| & \\
& \sum_{k\in K} y_{ik} = 1 & \forall i \in N \setminus \{0\} \\
& x_k(\delta^+(i)) = x_k(\delta^-(i)) = y_{ik} & \forall\, i \in N, k \in K \\
& \sum_{i\in N} d_i y_{ik} \leq C_k & \forall\, k \in K \\
& x_k(\delta^+(S)) \geq y_{hk} & \forall\, S \subseteq N \setminus \{0\}, h \in S, k \in K \\
& y_{ik} \in \{0,1\} & \forall\, i \in N, k \in K \\
& x_{ijk} \in \{0,1\} & \forall\, (i,j) \in A, k \in K
\end{array}
$$

Verify that the capacity and SEC constraints are violated for the earlier examples.

Vehicle Routing Problem

Three-index Formulation

The SECs can also be written as

$$x_k(A(S)) \leq |S| - 1 \qquad \forall\ S \subseteq N \setminus \{0\}, |S| \geq 2, k \in K$$

which implies that at least one arc leaves the set S.

These exponential number of constraints can also be replaced by the following MTZ constraints.

$$d_i \leq u_{ik} \leq C_k \qquad \forall\ i \in N \setminus \{0\}, k \in K$$
$$u_{ik} - u_{jk} + d_j \leq C_k(1 - x_{ijk}) \qquad \forall\ (i,j) \in A(N \setminus \{0\}) : d_i + d_j \leq C_k, k \in K$$

All the above formulations can be written for the symmetric version.

Vehicle Routing Problem

Set-Partitioning Formulation

Suppose J is a set of tours which are feasible (satisfy capacity constraints). Let a_{ij} be 1 if tour j visits customer i and is 0 otherwise.

Define c_j as the cost of the tour and x_j as a binary variable which is 1 if tour j is chosen.

$$
\begin{aligned}
\min \quad & \sum_{j \in J} c_j x_j \\
\text{s.t.} \quad & \sum_{j \in J} x_j = |K| \\
& \sum_{j \in J} a_{ij} x_j = 1 && \forall\, i \in V \setminus \{0\} \\
& x_j \in \{0, 1\} && \forall\, j \in J
\end{aligned}
$$

The advantage of this formulations is that the cost of a tour need not be equal to the sum of constituent link costs. It can also depend on the order of customers visited.

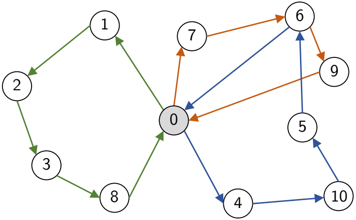
The set-partitioning model can be naturally extended to problems with time windows since they are not explicitly modeled.

The SPP formulation tends to have strong LP relaxations. However, the number of variables in this model is exponential and requires the use of column generation to solve the LP relaxations.

Vehicle Routing Problem

Set-Partitioning Formulation

If the costs satisfy triangle inequality, the problem can be formulated as a covering problem.



$$\sum_{j \in J} a_{ij} x_j \geq 1 \; \forall \; i \in V \backslash \{0\}$$

The solution can be repaired by adding a short-cut (7,9) or (5,0).

The advantage of using the covering version is that the feasible tours can be replaced with *maximal-feasible tours*. For example, we need not have tours 0-4-0, 0-4-10-0, and 0-4-10-5-0, in the above example.

Further, the dual space is more constrained for the covering problem.

Lecture Outline

Variants

In the VRP with time windows, each customer has a time window $[a_i, b_i]$ within which they must be served. If i is reached early, the vehicle must wait till the start time of the time window.

In addition to the demand, a service time of customer i is s_i units. The travel time between customer pair (i, j) is denoted as t_{ij}.

Assume that we have a copy of the depot 0, $n+1$. That is, $N = \{0, 1, \ldots, n, n+1\}$. We set $d_0 = s_0 = d_{n+1} = s_{n+1} = c_{0,n+1} = t_{0,n+1} = 0$.

Note that an arc can be deleted if $a_i + s_i + t_{ij} > b_j$ or if $d_i + d_j > \max_{k \in K} C_k$.

Variants

VRP with Time Windows (VRPTW)

Let decision variable w_{ik} indicate the start time of service at customer i by vehicle k.

$$\min \sum_{k \in K} \sum_{(i,j) \in A} c_{ijk} x_{ijk}$$

$$\begin{aligned}
\text{s.t.} \quad & \textstyle\sum_{k \in K} x_k(\delta^+(i)) = 1 && i \in N \backslash \{0, n+1\} \\
& x_k(\delta^+(0)) = 1 && \forall\, k \in K \\
& x_k(\delta^+(i)) = x_k(\delta^-(i)) && \forall\, i \in N \backslash \{0, n+1\}, k \in K \\
& x_k(\delta^-(n+1)) = 1 && \forall\, k \in K \\
& w_{jk} \geq w_{ik} + s_i + t_{ij} - M_{ij}(1 - x_{ijk}) && \forall\, k \in K, (i,j) \in A \\
& a_i \leq w_{ik} \leq b_i && \forall\, k \in K, i \in N \\
& \textstyle\sum_{i \in N} d_i x_k(\delta^+(i)) \leq C_k && \forall\, k \in K \\
& x_{ijk} \in \{0, 1\} && \forall\, (i,j) \in A, k \in K
\end{aligned}$$

What is a good choice of M_{ij}? $b_i + s_i + t_{ij} - a_j$.

Variants

VRP with Pickup, Dropoff, and Time Windows (VRPPDTW)

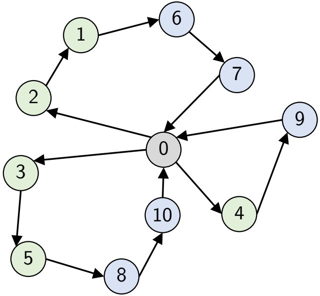
Vehicle fleet must serve requests, each of which has a pick-up and a drop-off locations. The demand could be goods/people (think Swiggy and Ola).

The problem involving people is also called Dial-a-Ride-Problem (DARP). The constraints include visiting each pickup and dropoff node just once and the pickup must precede the dropoff.

Variants

VRP with Pickup, Dropoff, and Time Windows (VRPPDTW)

Each request is represented by a pick-up node i and a drop-off node $n+i$. The set of pick-up and drop-off nodes are indicated by $P = \{1, 2, \ldots, n\}$ and $D = \{n+1, n+2, \ldots, 2n\}$, respectively.



We assume that there are $|K|$ vehicles, where the kth vehicle has an origin o_k and destination d_k. $N = P \cup D \cup_{k \in K} \{o_k, d_k\}$. Vehicles are assumed to have a capacity C_k.

Variants

VRP with Pickup, Dropoff, and Time Windows (VRPPDTW)

The decision variables for this model include the x_{ijk} variables which is 1 if vehicle k uses link (i, j) and is 0 otherwise.

The time at which vehicle k starts service at i is recorded by w_{ik}. The load on vehicle k after visiting node i is indicated by q_{ik}.

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in A} \sum_{k\in K} c_{ijk} x_{ijk} & \\
\text{s.t.} \quad & \sum_{k\in K} x_k(\delta^+(i)) = 1 & i \in P \\
& x_k(\delta^+(i)) = x_k(\delta^-(n+i)) & \forall i \in P, k \in K \\
& x_{ijk} \in \{0, 1\} & \forall (i, j) \in A, k \in K
\end{aligned}
$$

Variants

VRP with Pickup, Dropoff, and Time Windows (VRPPDTW)

Flow must be conserved for each vehicle.

$$\sum_{j \in P \cup \{d_k\}} x_{o_k j k} = 1 \qquad \forall\, k \in K$$

$$\sum_{j \in P \cup D \cup \{d_k\}} x_{ijk} - \sum_{j \in P \cup D \cup \{o_k\}} x_{jik} = 0 \qquad \forall\, k \in K, i \in P \cup D$$

$$\sum_{i \in D \cup \{o_k\}} x_{i d_k k} = 1 \qquad \forall\, k \in K$$

The time windows and precedence constraints must be satisfied.

$$w_{jk} \geq w_{ik} + s_i + t_{ij} - M_{ij}(1 - x_{ijk}) \qquad \forall\, k \in K, (i,j) \in A$$

$$a_i \leq w_{ik} \leq b_i \qquad \forall\, k \in K, i \in P \cup D$$

$$w_{ik} + t_{i,n+1,k} \leq w_{n+1,k} \qquad \forall\, k \in K, i \in P$$

Variants

VRP with Pickup, Dropoff, and Time Windows (VRPPDTW)

Finally, capacity constraints must be met.

$$
\begin{aligned}
& q_{jk} \geq q_{ik} + d_i - O_{ij}(1 - x_{ijk}) && \forall\, k \in K, (i, j) \in A \\
& d_j \leq q_{ik} \leq C_k && \forall\, k \in K, i \in P \\
& 0 \leq q_{n+i,k} \leq C_k - d_i && \forall\, k \in K, n + i \in D
\end{aligned}
$$

In the case of DARP, one may also impose ride-time constraints to avoid long detours.

Variants

VRP with Backhauls (VRPB)

VRPs with backhauls involve both deliveries and pickups. Customers where demand is dropped off are called *linehaul* customers and those where supply is picked up are called *backhaul* customers.

Vehicles begin and end at the depot. Backhaul customers are served after visiting all the linehaul customers to make within truck operations easier.

Capacity constraints can thus be applied to both set of customers independently. The problem can be easily solved using the regular VRP model by just modifying the network.

Variants

VRP with Backhauls (VRPB)

The set of linehaul customers are denoted by $L = \{1, \dots, n\}$, and backhauls by $B = \{n+1, \dots, n+m\}$. Let $N = L \cup B \cup \{0\}$. As before, d_i is the demand to be delivered or collected, where $i \in L \cup B$.

The arc set can exclude the arcs from B to L and from 0 to B because of the precedence constraints.

The minimum number of vehicles required can be found by solving two separate bin packing problems involving L and B.

The CCCs are applied for subsets $S \in L \cup B$. We define $r(S)$ as the minimum number of vehicles required to serve all customers in S even if they are a mixture of linehaul and backhaul customers.

Variants

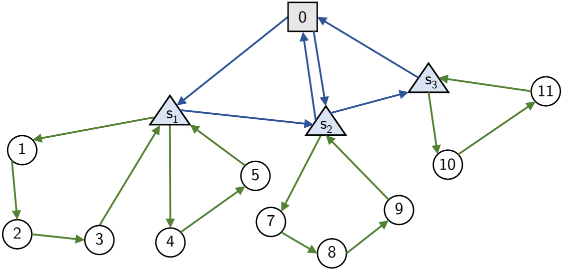
Multi-Echelon VRP

In the 2-echelon variant (2E-CVRP), demand is first routed to satellites which are smaller depots and then routed to customers.

A set of primary vehicles transport demand from the depot to the satellites and secondary vehicles move them from satellites to customers. The satellites can be served by one or more of the primary vehicles and has a capacity.

Each customer, however, is assumed to be served by only one secondary vehicle. The set of satellites is S and the set of customers is N. Depot is indicated by 0.

The arcs at the first level A_1 comprise of connections from the depot to the satellites and between the satellites.



Arcs from the satellites to the customers and between customers are indicated by A_2.

Assume that K_1 and K_{2s} denote the set of primary and secondary vehicles from satellite s with capacities C_1 and C_2, respectively.

Define the decision variables as:

- x_{ij} which is 1 if the first level arc (i, j) is used and is 0 otherwise
- y_{ijs} which is 1 if the arc (i, j) is used from a vehicle starting at satellites s
- z_{si} which is 1 if customer at i is served by satellite s.

Formulate the 2E-CVRP as a MILP.

In LRP depot locations for vehicles are determined along with their routes can improve efficiency and reduce the operating and fixed/leasing costs of locating a facility.

Assumed that there are set of potential depot sites D. Opening a depot at location $i \in D$ costs f_i.

Define variables that assign customer locations to depots using z_{ij}, which is set to 1 if customer at j is served by vehicles starting from depot i. Suppose the capacity of the depot is W_i.

The objective is to minimize the total cost of operations (transportation and fixed costs)

$$\min \sum_{i \in N \cup D} \sum_{j \in N \cup D} \sum_{k \in K} c_{ij} x_{ijk} + \sum_{i \in D} f_i y_i$$

Variants

Location Routing Problem (LRP)

Each customer should be visited from another customer or from the depot by exactly one vehicle. Vehicle entering a node will leave it. Vehicles can begin from different depots due to the third constraint.

$$\sum_{k \in K} \sum_{i \in N \cup D} x_{ijk} = 1 \qquad \forall\, j \in N$$

$$\sum_{j \in N \cup D} x_{ijk} - \sum_{j \in N \cup D} x_{jik} = 0 \qquad \forall\, k \in K, i \in N \cup D$$

$$\sum_{i \in D} \sum_{j \in N} x_{ijk} \leq 1 \qquad \forall\, k \in K$$

Capacity and SEC constraints are enforced using the following inequalities.

$$x_k(A(S)) \leq |S| - 1 \qquad \forall\, S \subseteq N, k \in K$$

$$\sum_{j \in N} \sum_{i \in N \cup D} d_j x_{ijk} \leq C_k \qquad \forall\, k \in K$$

$$\sum_{j \in N} d_j z_{ij} \leq W_i y_i \qquad \forall\, i \in D$$

Variants

Location Routing Problem (LRP)

The z and x variables are linked as shown below. If z_{ij} is zero, then both x_{ih} and x_{gj} terms cannot be 1, else some vehicle from depot i will serve customer j.

$$\sum_{h \in N} x_{ihk} + \sum_{g \in N \cup D \setminus \{j\}} x_{gjk} \leq 1 + z_{ij} \qquad \forall\, i \in D, j \in N, k \in K$$

Finally, the decision variables are required to be binary using

$$\begin{aligned} x_{ijk} &\in \{0, 1\} && \forall\, i \in N \cup D, j \in N \cup D, k \in K \\ y_i &\in \{0, 1\} && \forall\, i \in D \\ z_{ij} &\in \{0, 1\} && \forall\, i \in D, j \in N \cup D \end{aligned}$$

Your Moment of Zen