Optimal Supply Control for Shared Mobility with Logit Mode-Choice Dynamics

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***Abstract*— In this paper, we discuss the evolution of mode shares over time in a stylised single origin-destination setting with fixed demand and linear non-separable latency functions. One of the modes is assumed to resemble a ride-hailing service whose operator can control its supply over time. The operator is assumed to be interested in reducing costs while trying to achieve a target market share and the problem is formulated as a finite-horizon continuous optimal control problem. We derive the conditions on supply that ensure a unique stochastic user equilibrium and present a necessary condition that achieves the operator's targets. The framework is demonstrated using simulation.**

I. Introduction

In recent times, ride-hailing and sharing services such as Lyft, Ola, and Uber, have dominated the shared mobility space with their disruptive pricing and significantly altered the mode choices of travellers. Such phenomena, though common to many competitive markets, have not been been fully understood. In this paper, we explore how user preferences in a transportation network change over time due to the actions of a ride-hailing operator. We present an abstract single origin-destination (OD) transportation model in which the demand captured by each mode can be viewed as a flow on multiple physical links between the OD pair. Mode 1 is assumed to represent a ride-hailing service whose operator controls the supply for a fixed time horizon and seeks a guaranteed mode-share level in the long run. We formulate the problem as a finite-horizon optimal control framework with logit dynamics to capture mode shifts due to change in the supply of the ride-hailing services.

Literature Review: The stochastic user equilibrium (SUE) [1] is an alternative to deterministic Wardrop equilibrium and assumes that the travellers' perceived utilities for selecting routes are random variables. This framework has been widely used along with different choice dynamics such as the logit, cross nested logit, C-logit, path size logit, and multinomial probit [2]–[4]. The focus in these models has been to account for correlations in alternatives and to find tractable solutions to the SUE problem.

Another line of research in traffic assignment involves the study of the evolution of traveller choices under route switching mechanisms [5]–[8]. These day-to-day traffic models understand the evolution of user choices and analyse the conditions in which the rest points coincide with traffic equilibria. Several randomized dynamics with link-flow versions of the problem have been shown to converge to the SUE solutions using Lyapunov analysis [9], [10]. It has also been shown in [11] that there can potentially exist multiple equilibria with different characteristics. Hence, additional assumptions on the delay functions that guarantee uniqueness are typically employed. Variants of the day-to-day framework have also been effectively used in understanding the evolution of traffic after network disruptions [12]–[14] and congestion control using tolls [15]–[17]. A more general discussion on the logit dynamics can be found in [18]. We approach the problem of interest using a single OD pair and multiple modes stylised model, much like the bottleneck model [19], [20] used for departure-time choice modelling, linear mono-centric models [21] and two-mode dynamic [22] and other evolutionary game models for the mode choice problem [23], [24].

Contributions: In this paper, we model and analyse a dynamic mode choice scenario, when a new operator-controlled mode enters the market. We formulate the problem as an optimal control problem, where the goal of the operator is to minimize the cost of operations over a finite horizon while trying to reach a supply level that remains fixed after the end of the time horizon. Travellers' mode choices are assumed to follow logit dynamics at all time steps. The operator is assumed to control the rate of change of supply to achieve the following desideratum. First, the uncontrolled dynamics, subsequent to an initial control over a finite time duration, must converge to a SUE. Second, the operator should be able to capture a pre-specified market share in the limit/at equilibrium. Assuming linear latency functions, we also consider the existence and uniqueness conditions of the limiting behaviour of the logit dynamics and its connections with SUE. We present a simple necessary condition for the feasibility of the operator's optimal control problem. We also present a simulation setup to consider the effects of various types of interacting modes and demonstrate the effects of the input parameters. While most literature on day-to-day traffic focuses on route choice behaviour, temporal analysis of modes that travellers choose has received limited attention, primarily because mode changes are usually less frequent. However, in the presence of ride-hailing modes, supply-side attributes of alternatives are subject to discernible fluctuations which in turn influence choice probabilities and mode shares. Our goal in this paper is to employ day-to-day traffic models to capture these shifts and study related problems of interest from an operator's point of view.

Notation: Let $\mathbb{Z}$ denote the set of integers. We use the notation $[a, b]_{\mathbb{Z}}$ to denote $[a, b] \cap \mathbb{Z}$. $\mathbf{v}_{[a:b]}$ denotes a slice of

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a vector $\mathbf{v}$ for indices between a and b. $\mathbf{A}_{[a:b,c:d]}$ denotes a sub-matrix of $\mathbf{A}$ with rows between a and b and columns between c and d.

Organisation: We first describe the problem setup in the Section II. We derive sufficient conditions for uniqueness of equilibrium points in Section III. We follow up with a discussion on existence of a feasible solution to the operator's optimal control problem in Section IV. Section V contains a demonstration of the proposed model and a simulation setup and in Section VI, we discuss the limitations and scope for future work.

II. Problem Setup

Consider a single OD pair that is being serviced by m modes, as shown in Figure 1. Let d be the total demand between O and D. Let $\mathbf{x} = [x_1, x_2, \ldots, x_m]^{\mathrm{T}}$ represent the vector of mode shares with x_i representing the share of demand choosing mode i. The set of feasible shares, Ω, can be defined as

$$\Omega := \left\{ \mathbf{x} \mid \sum_{i=1}^{m} x_i = d,\ \mathbf{x} \geq \mathbf{0} \right\}. \tag{1}$$

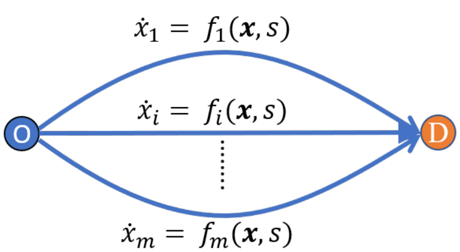


Fig. 1: The supply of the ride-hailing mode is s. Mode shares are assumed to evolve according to logit dynamics.

The decision maker in this paper is the service operator of one of the modes, say mode 1. We assume that the operator seeks to control x_1, the mode-share on mode 1. To this end, the operator dynamically chooses u, the *rate of change of supply* (of vehicles) of mode 1. We denote the *supply* on mode 1 at time t by $s(t)$. Under this setting, we analyse the day-to-day evolution of traffic between the OD pair given the actions of the operator. Then, we also determine the optimal supply rate trajectory so that the operator may minimize costs and achieve a targeted demand for mode 1.

We assume that the mode shares evolve according to the logit dynamics, with a linear generalised cost function. Thus with uncertainty factor, θ, and reluctance factor, α, the dynamics are defined as

$$\dot{x}_i = f_i(\mathbf{x}, s) = \alpha \left(\hat{x}_i(\mathbf{x}, s) - x_i \right), \ \forall i \in [1, m]_{\mathbb{Z}} \tag{2}$$

$$\hat{x}_i(\mathbf{x}, s) := d \frac{\exp(-\theta c_i(\mathbf{x}, s))}{\sum_{j=1}^{m} \exp(-\theta c_j(\mathbf{x}, s))} \tag{3}$$

$$\dot{s}(t) = u(t), \tag{4}$$

where c_i is the generalised cost incurred by users of mode i and u is the rate of change of supply s on mode 1.

We next model the constraints faced by the operator. In practice it is not possible to recruit drivers or reduce/increase the supply in a unit time without restrictions. Hence, we denote the maximum and minimum of u as $\bar{u}$ and $\underline{u}$, respectively. Note that for $s = 0$, there cannot be any service for mode 1. Therefore, we assume that it is at least equal to a small value, ϵ, at all times.

$$u(t) \in [\underline{u}, \bar{u}], \quad \forall t \geq 0 \tag{5}$$

$$s(t) \geq \epsilon, \quad \forall t \geq 0. \tag{6}$$

When entering a new market, operators tend to aggressively advertise and recruit only for a fixed time window, which we denote by T. That is, we assume that the operator alters supply until time T and the final supply is chosen keeping the long-run equilibrium shares in mind since they ultimately influence their long-term profits. Operators may choose to alter the supply at a later time after observing changes to the market, in which case the current model can be extended with minor modifications. With these parameters defined, we next define the generalised cost structure and the complete formulation.

A. Model of Linear Generalised Cost, $\mathbf{c}(\mathbf{x}, s)$

For mode 1, the actual market share and the supply are distinguished to account for empty miles travelled due to searching, rebalancing, and dead-heading trips. The congestion costs due to mode 1 thus depend on the supply s in addition to actual customers x_1. We assume that the generalized costs are linear and non-separable since shares/supply of one mode can influence the travel time of the other. To capture this, we use a *congestion matrix* of constants, $\mathbf{K}$, and define a component of the delay functions as

$$\hat{\mathbf{c}}(\mathbf{x}, s) := \mathbf{K}\mathbf{z}(\mathbf{x}, s) \quad \text{where, } \mathbf{z}(\mathbf{x}, s) = [s, x_2, \ldots, x_m]^{\mathrm{T}}. \tag{7}$$

To these delays, we add a vector $\mathbf{b}$, which can be viewed as the out-of-pocket costs of using each mode. More details on the generalized costs for the ride-hailing mode and other modes are given below.

- $c_1(\mathbf{x}, s)$: The generalised cost of mode 1 is assumed to depend on the congestion and the prices charged by the operator. The prices are assumed to depend on the supply s and demand for mode 1, i.e., x_1. Rides on mode 1 will get expensive as more travelers shift to it. Likewise, they get cheaper as the supply starts to increase. Therefore, we assume that prices are inversely-proportional to the supply, s and directly proportional to the mode share, x_1. Thus, $c_1(\mathbf{x}, s)$ is defined as

  $$c_1(\mathbf{x}, s) = \frac{\bar{K}_{11} x_1}{s} + \hat{c}_1(\mathbf{x}, s) + b_1, \tag{8}$$

  where $\bar{K}_{11}$ is a model parameter which we call the *surge-price factor* that reflects additional costs due to supply-demand imbalances.
- $c_j(\mathbf{x}, s), j \neq 1$: We assume that the generalised cost is the combination of the congestion cost and the price

  $$c_j(\mathbf{x}, s) = \hat{c}_j(\mathbf{x}, s) + b_j. \tag{9}$$

Thus, the linear generalised-cost vector, $\mathbf{c}(\mathbf{x}, s)$, can be summarised as follows:

$$\mathbf{c}(\mathbf{x}, s) = \bar{\mathbf{K}}(s)\mathbf{x} + \bar{\mathbf{b}}(s) \tag{10a}$$

$$\text{where, } \bar{\mathbf{K}}(s) := [\bar{\mathbf{K}}_1(s)\ \mathbf{K}_2\ \ \ldots \mathbf{K}_m] \tag{10b}$$

$$\bar{\mathbf{K}}_1(s) = [\bar{K}_{11}/s, 0 \ldots 0]^{\mathrm{T}} \tag{10c}$$

$$\bar{\mathbf{b}}(s) := s\mathbf{K}_1 + \mathbf{b}. \tag{10d}$$

Here, $\mathbf{K}_i$ represents the i^{th} column of *congestion matrix*, $\mathbf{K}$. Note that the linear structure used in this sub-section simplifies the problem while capturing many aspects of a multi-modal transportation system. Other cost structures, though richer in features, may not be useful in deriving analytical guarantees similar to those proposed in this paper.

B. The Objective Function

With the controls and dynamics defined above, the operator desires to achieve a "*long-term occupancy*" for mode 1 of at least d_1. Mathematically, assuming that the equilibrium point of the dynamical system is $\mathbf{x}^*$, we want

$$x_1^* = \lim_{t\to\infty} x_1(t) \geq d_1 \tag{11}$$

To this end, the operator changes the supply in a time window $[0, T]$ and minimizes the cost of operations in this period.

Specifically, we assume that to change the supply at a rate u, the operator spends γu^2 amount of cost per unit time. This cost is assumed to be incurred for both adding and removing supply and it penalizes drastic changes in the supply of mode 1. Further, we assume an operating cost of R amount per unit time that is required to operate and maintain a unit supply. Thus, the operator spends $Rs(t)$ units of money per unit time to maintain a supply of $s(t)$. In summary, the operation costs for the transient period can be written as

$$J(u, s) = \gamma \int_0^T u(t)^2 dt + R \int_0^T s(t) dt. \tag{12}$$

The *operator cost minimization* problem is thus defined as

$$\min_{u,s} \quad J(u, s)$$
$$\text{Subject to: (2), (4), (5), (6), (10), (11).} \tag{13}$$

In practice, an operator may model the problem using other definitions of $J(u, s)$ without affecting the analytical results in Sections III and IV as long as the *linear generalised cost* structure in (8) and (9) is assumed. Analytical solutions of the cost minimisation problem are out of reach due to the use of logit equations for modal split. After time period T, the system is assumed to continue to follow logit dynamics with a constant supply of $s(T)$. With this setup, we analyse the conditions on the control variables that ensure the existence and uniqueness of the equilibrium point. Moreover, we present necessary conditions for the optimal control problem (13) to be feasible.

III. Conditions on Supply For Uniqueness Of SUE

Recall that according to (11), the operator seeks to achieve a certain long-term (asymptotic) occupancy of mode 1. In general, there could be multiple equilibrium points or even limit cycles for the logit dynamics. Hence, we choose a supply that results in a unique SUE, which we hope would be asymptotically stable with the entire simplex (1) in its region of attraction. Thus, we first establish the conditions on the supply, s, that are necessary and sufficient for uniqueness of the SUE of the logit dynamics.

A. Cost Monotonicity Condition for Uniqueness of SUE

In general, the uniqueness of an equilibrium point is guaranteed by the strict monotonicity property of the cost function. Zhou et al. in [3] show that for asymmetric cost functions like $\mathbf{c}(\mathbf{x}, s)$ in (10a), it is necessary and sufficient that for a given equilibrium point, $\mathbf{x}^*(s)$, to be a SUE the variational inequality (VI) defined below is satisfied.

$$\begin{aligned} \mathrm{VI}(\eta, \Omega) : \eta(\mathbf{x}^*(s), s)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}^*(s)) \geq 0, \quad \forall \mathbf{x} \in \Omega \\ \eta(\mathbf{x}, s) := \left(\mathbf{c}(\mathbf{x}, s) + \theta^{-1}(1 + \ln(\mathbf{x}))\right). \end{aligned} \tag{14}$$

We assume that the costs satisfy the following *fixed-supply monotonicity* property.

$$(\mathbf{c}(\mathbf{x}_1, s) - \mathbf{c}(\mathbf{x}_2, s))^{\mathrm{T}}(\mathbf{x}_1 - \mathbf{x}_2) \geq 0, \quad \mathbf{x}_1, \mathbf{x}_2 \in \Omega. \tag{15}$$

Since, we assume a linear cost function, one can show that *fixed-supply monotonicity* is sufficient for a unique equilibrium under a fixed-supply. Further, this monotonicity property is guaranteed by the positive semi-definiteness of $\bar{\mathbf{K}}(s)$.

Theorem III.1. *(Sufficient condition on $\bar{\mathbf{K}}(s)$ for a unique SUE). Consider the logit dynamics (2) with the generalized cost $\mathbf{c}$ as in (10). If $\bar{\mathbf{K}}(s) \succeq 0$ then the logit dynamics has a unique equilibrium, which is also the stochastic user equilibrium (SUE).*

Proof. Notice that if $\bar{\mathbf{K}}(s) \succeq 0$ then for $\mathbf{c}$ as in (10), the fixed-supply monotonicity property (15) is satisfied. The equilibrium points and SUE of the logit dynamics (2) are both solutions of $\mathbf{x}^*(s) = \hat{\mathbf{x}}(\mathbf{x}^*(s))$. Further, since $\mathbf{x}^*(s)$ is an SUE if and only if $\mathbf{x}^*(s)$ solves the VI (14), it suffices to show that if $\bar{\mathbf{K}}(s) \succeq 0$, then the VI has a unique solution.

Since the generalised cost $\mathbf{c}$ in (10) is continuous in $\mathbf{x}$, Proposition 2 in Section 2.2 of reference [3] guarantees that the VI has at least one solution. We prove, by contradiction, that the VI has a unique solution. Let us assume there are two distinct fixed points of VI (14) namely $\mathbf{x}_1^*$ and $\mathbf{x}_2^*$. Then,

$$\eta(\mathbf{x}_2^*)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_2^*) \geq 0, \quad \eta(\mathbf{x}_1^*)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}_1^*) \geq 0, \ \forall \mathbf{x} \in \Omega.$$

In particular, we have

$$\eta(\mathbf{x}_2^*)^{\mathrm{T}}(\mathbf{x}_1^* - \mathbf{x}_2^*) \geq 0, \quad \eta(\mathbf{x}_1^*)^{\mathrm{T}}(\mathbf{x}_2^* - \mathbf{x}_1^*) \geq 0,$$

which imply

$$(\eta(\mathbf{x}_2^*) - \eta(\mathbf{x}_1^*))^{\mathrm{T}}(\mathbf{x}_2^* - \mathbf{x}_1^*) \leq 0.$$

Using (14), we can rewrite this as

$$\left(\bar{\mathbf{K}}(s)(\mathbf{x}_2^* - \mathbf{x}_1^*) + \theta^{-1} \ln\left(\frac{\mathbf{x}_2^*}{\mathbf{x}_1^*}\right)\right)^{\mathrm{T}} (\mathbf{x}_2^* - \mathbf{x}_1^*) \leq 0,$$

which implies

$$(\mathbf{x}_2^* - \mathbf{x}_1^*)^{\mathrm{T}}\bar{\mathbf{K}}(s)^{\mathrm{T}}(\mathbf{x}_2^* - \mathbf{x}_1^*) +$$

$$\theta^{-1}\sum_{i\in[1,m]_{\mathbb{Z}}}\ln\left(\frac{x^*_{2i}}{x^*_{1i}}\right)(x^*_{2i}-x^*_{1i})\leq 0.$$

The left hand side of this inequality is strictly greater than 0 as $\bar{\mathbf{K}}(s)\succeq 0$ and $\ln(\frac{x^*_{2i}}{x^*_{1i}})\,(x^*_{2i}-x^*_{1i})>0$ when $x^*_{2i}\neq x^*_{1i}$. This is a contradiction and the only possible way the above relation holds is if $\mathbf{x}^*_1=\mathbf{x}^*_2$. Thus, the $\mathrm{VI}(\eta,\Omega)$ in (14) has a unique solution. □

In most cost models a strict monotonicity is required, but from Theorem III.1 we observe that under linear cost functions as defined in (8) and (9), semi-definiteness of $\bar{\mathbf{K}}(s)$ is sufficient, which implies that a "non-strict" version of monotonicity is sufficient for uniqueness of the SUE.

B. Conditions on Supply, s, for Uniqueness of Equilibrium

With Theorem III.1, we can now find conditions on supply s that ensure $\bar{\mathbf{K}}(s)\succeq 0$ and thereby the uniqueness of SUE. Consequently, we need to inspect the symmetric component of the matrix $\bar{\mathbf{K}}(s)$. Note that the matrix $\bar{\mathbf{K}}(s)$ in (10b) has the structure

$$\bar{\mathbf{K}}(s):=\begin{bmatrix}\bar{K}_{11}/s & \mathbf{r}_1\\ \mathbf{0} & \mathbf{M}\end{bmatrix},$$

where $\bar{K}_{11}/s$ is a scalar, $\mathbf{r}_1\in\mathbb{R}^{m-1}$ is a row vector, $\mathbf{0}\in\mathbb{R}^{m-1}$ is a column vector and $\mathbf{M}\in\mathbb{R}^{(m-1)\times(m-1)}$ is a square matrix. Then, the symmetric part of $\bar{\mathbf{K}}(s)$ is

$$\widetilde{\mathbf{K}}(s):=\frac{\bar{\mathbf{K}}(s)+\bar{\mathbf{K}}(s)^{\mathrm{T}}}{2}=\begin{bmatrix}\bar{K}_{11}/s & \mathbf{r}_1/2\\ \mathbf{r}_1^{\mathrm{T}}/2 & (\mathbf{M}+\mathbf{M}^{\mathrm{T}})/2\end{bmatrix}. \tag{16}$$

We assume that $\mathbf{M}$ is positive semi-definite, i.e., $\mathbf{y}^{\mathrm{T}}\mathbf{M}\mathbf{y}\geq 0,\forall\mathbf{y}\in\mathbb{R}^{m-1}$. Implicitly, this assumes that the cost functions satisfy the monotonicity property even before the ride-hailing mode was introduced. We can now find constraints on the supply as follows.

Lemma III.2. *Suppose* $\bar{K}_{11}>0$, $\mathbf{r}_1\neq 0$, *and* $\bar{\mathbf{M}}:=\frac{(\mathbf{M}+\mathbf{M}^T)}{2}\succ 0$. $\bar{\mathbf{K}}(s)\succ 0\Leftrightarrow s\in\big(0,(4\bar{K}_{11})/(\mathbf{r_1}\bar{\mathbf{M}}^{-1}\mathbf{r_1}^T)\big)$.

Proof. First note that $\bar{\mathbf{K}}(s)\succ 0$ iff $\widetilde{\mathbf{K}}(s)\succ 0$. From the Sylvester's Criterion[1], the symmetric matrix $\widetilde{\mathbf{K}}(s)\succ 0$ iff all its leading principal minors are positive. As $\bar{\mathbf{M}}\succ 0$, this implies that $\det(\widetilde{\mathbf{K}}(s))>0$ is necessary and sufficient for $\widetilde{\mathbf{K}}(s)\succ 0$. Using the Schur complement of $\bar{\mathbf{M}}$ in $\widetilde{\mathbf{K}}(s)$, we have

$$\det(\widetilde{\mathbf{K}}(s))=\det(\bar{\mathbf{M}})\det\left(\frac{\bar{K}_{11}}{s}-\frac{1}{4}\mathbf{r_1}\bar{\mathbf{M}}^{-1}\mathbf{r_1}^{\mathrm{T}}\right)>0.$$

As $\bar{\mathbf{M}}\succ 0$ and as the Schur complement of $\bar{\mathbf{M}}$ in $\widetilde{\mathbf{K}}(s)$ is a 1×1 matrix, and the result follows. □

Note that such a condition for uniqueness need not be applied for all values of $t\in[0,T]$. This is because $\mathbf{x}^*$ depends only on $s(T)$ and a value of $s(T)$ that respects these conditions is sufficient to ensure uniqueness of $\mathbf{x}^*$.

[1]See Theorem 7.2.5 in [25]

IV. Necessary Conditions on Feasibility

In this section, we derive a necessary condition for the feasibility of the optimal control problem (13). Note that all the constraints except (11) are easily satisfied. Thus, we focus on the feasibility of (11). The mode shares $\mathbf{x}^*$ at steady state (if it exists) depends only on the final supply $s(T)$. For a given $s(T)$, an equilibrium is the solution of

$$x^*_i(s(T))=d\frac{\exp(-\theta c_i(\mathbf{x}^*,s(T)))}{D(\mathbf{x}^*,s(T))},\ \forall i\in[1,m]_{\mathbb{Z}}, \tag{17}$$

$$\text{where, } D(\mathbf{x}^*,s(T)):=\sum_{j=1}^{m}\exp(-\theta c_j(\mathbf{x}^*,s(T))). \tag{18}$$

Now, for the feasibility of (11) and hence of the problem (13), it is necessary that $x^*_1(s(T))\geq d_1$, i.e.,

$$\frac{d_1}{d}\leq\frac{x^*_1}{d}=\frac{\exp(-\theta c_1(\mathbf{x}^*,s(T)))}{D(\mathbf{x}^*,s(T))},$$

or equivalently,

$$\ln\left(\frac{d_1}{d}\right)\leq-\theta c_1(\mathbf{x}^*,s(T))-\ln(D(\mathbf{x}^*,s(T))). \tag{19}$$

Since the arithmetic mean is always greater than or equal to the geometric mean, we have

$$\ln\left(\frac{D(\mathbf{x}^*,s(T))}{m}\right)\geq\frac{\sum_{j=1}^m\ln(\exp(-\theta c_j(\mathbf{x}^*,s(T))))}{m},$$
$$-\ln\left(D(\mathbf{x}^*,s(T))\right)\leq\frac{\sum_{j=1}^m\theta c_j(\mathbf{x}^*,s(T))}{m}-\ln(m). \tag{20}$$

Using (19) and (20),

$$\ln\left(\frac{md_1}{d}\right)\leq-\theta c_1(\mathbf{x}^*,s(T))+\frac{\sum_{j=1}^m\theta c_j(\mathbf{x}^*,s(T))}{m},$$
$$c_1(\mathbf{x}^*,s(T))\leq-\theta^{-1}\ln\left(\frac{md_1}{d}\right)+\frac{\sum_{j=1}^m c_j(\mathbf{x}^*,s(T))}{m}$$
$$\leq\frac{\theta^{-1}m}{m-1}\ln\left(\frac{d}{md_1}\right)+\frac{\sum_{j=2}^m c_j(\mathbf{x}^*,s(T))}{m-1}.$$

Note that the first term in this upper bound decreases with increase in d_1. Thus, with more target demand d_1, one expects the cost on mode 1 to be lesser. Since only the elements in the first row and the first column of $\bar{\mathbf{K}}$ depend on $s(T)$, we can separate the terms containing $s(T)$ as follows:

$$\sum_{j=2}^m c_j(\mathbf{x}^*,s(T))=\mathbf{1}^{\mathrm{T}}\left(\bar{\mathbf{K}}(s(T))_{[2:m,:]}\mathbf{x}^*\right)+\mathbf{1}^{\mathrm{T}}\bar{\mathbf{b}}(s)_{[2:m]}$$
$$=\mathbf{1}^{\mathrm{T}}\left(\mathbf{M}\mathbf{x}^*_{[2:m]}+\mathbf{b}_{[2:m]}\right)+s(T)\mathbf{1}^{\mathrm{T}}\mathbf{K}_{1[2:m]}$$

Let

$$a:=\frac{\mathbf{1}^{\mathrm{T}}\mathbf{b}_{[2:m]}}{m-1}+\frac{\theta^{-1}m\ln(\frac{d}{md_1})}{m-1}-b_1$$

Substituting $c_1(\mathbf{x},s(T))$ from (8) and a

$$\frac{\bar{K}_{11}x^*_1}{s(T)}+K_{11}s(T)+\sum_{j=2}^m K_{1j}x^*_j\leq a+\frac{\mathbf{1}^{\mathrm{T}}\mathbf{M}\mathbf{x}^*_{[2:m]}+s(T)\mathbf{1}^{\mathrm{T}}\mathbf{K}_{1[2:m]}}{m-1} \tag{21}$$

Let

$$g(s(T), x_1^*) := \frac{\bar{K}_{11} x_1^*}{s(T)} + s(T)\left(K_{11} - \frac{\sum_{j=2}^{m} K_{j1}}{m-1}\right) \tag{22a}$$

$$f(\mathbf{x}^*) := \frac{\mathbf{1}^{\mathrm{T}}\left(\mathbf{M}\mathbf{x}^*_{[2:m]}\right)}{m-1} - \sum_{j=2}^{m} K_{1j} x_j^* + a \tag{22b}$$

Then, (21) can be rewritten as

$$g(s(T), x_1^*) \leq f(\mathbf{x}^*). \tag{23}$$

This is a necessary condition for $\mathbf{x}^*$ to be an SUE that satisfies (11) for a given $s(T)$. With this analysis, we give a conservative but simple to calculate necessary condition for the feasibility of (11) and hence of the problem (13) in the following result.

Theorem IV.1. *Let $\bar{\Omega} := \{\mathbf{x} | x_1 \geq d_1, \ \sum x_i = d, x_i \geq 0\}$ be the set of desired mode shares. Denote the set of feasible final supplies with $\mathcal{S} := [\max\{s(0) + \underline{u}T, \epsilon\}, s(0) + \bar{u}T]$. Suppose that $\bar{K}_{11} > 0$. The problem (13) is feasible only if $\exists\, s \in \mathcal{S}$ such that $g(s, d_1) \leq \max_{\mathbf{y} \in \bar{\Omega}} f(\mathbf{y})$*

Proof. The constraints (4)-(6) imply that we must have $s(T) \in \mathcal{S}$ for feasibility. Suppose $\exists\, s(T) \in \mathcal{S}$ such that $\mathbf{x}^* \in \bar{\Omega}$. Then, by the derivation in Section IV, we see that (23) must be satisfied for $\mathbf{x}^*$. Now, for this $s = s(T)$ and the resultant $\mathbf{x}^* \in \bar{\Omega}$, we have

$$g(s, d_1) \leq g(s, x_1^*) \leq f(\mathbf{x}^*) \leq \max_{\mathbf{y} \in \bar{\Omega}} f(\mathbf{y}),$$

where the first inequality is true because $g(s, .)$ is an increasing function in the second argument and the second inequality is true because of (23). □

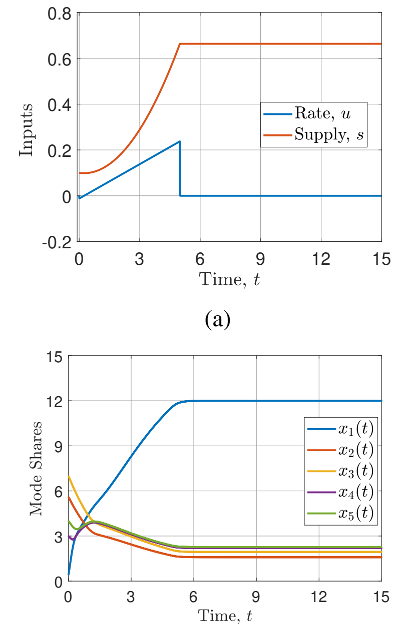
V. Simulations

In this section, we demonstrate the proposed framework on an example that is solved using GEKKO [26].

We consider a simulation with 5 modes of transportation, arranged into 2 blocks. Modes 2 and 3 could represent personal mobility modes that share road space with ride-hailing services. On the other hand, modes 4 and 5 could include BRT and metro which operates on a network that is not shared by ride-hailing mode. We assume an uncertainty factor $\theta = 1$, reluctance factor $\alpha = 0.5$, and the objective weights $\gamma = 1$, $\epsilon = 0.001$ and $R = 0.1$. We let $\bar{u} = -\underline{u} = 1$. We let the values of $\mathbf{K}$ and $\mathbf{b}$ be as follows

$$\mathbf{K} = \begin{bmatrix} 1 & 0.15 & 0.2 & 0 & 0 \\ 1.5 & 2 & 2 & 0 & 0 \\ 2 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0.5 \\ 0 & 0 & 0 & 0.5 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0.295 \\ 0.975 \\ 0.090 \\ 0.975 \\ 0.771 \end{bmatrix}.$$

The value of $\bar{K}_{11}$ is varied with every simulation to test the effects of various prices on the mode-choice equilibrium. Figure 2a shows the evolution of the rate of change of supply and the supply as functions of time. Figure 2b shows the evolution of the mode shares for the optimal control determined in Figure 2a. In this example, $\bar{K}_{11} = 0.3$, $s(0) = 0.1$, and the target mode share for mode 1 is $d_1 = 12$.



(a)

(b)

Fig. 2: Evolution of control-inputs and mode-shares with time, t. Here, $\bar{K}_{11} = 0.3$, $s(0) = 0.1$, $d_1 = 12$.

In Figure 3, we show the variation of the optimal objective with different values of $\bar{K}_{11}$. These were simulated for different values of initial supply and a fixed initial mode 1 mode-share. We observe that with increasing surge-price factor an operator needs to spend more to ensure that the target mode share, d_1, is reached. This is because with increase in prices, there is a tendency to shift away from mode 1. Thus, the operator needs to vary $s(t)$ so that the supply first decreases causing costs c_1 to reduce and thereby increasing later so that $x_1^* \geq d_1$ is satisfied asymptotically.

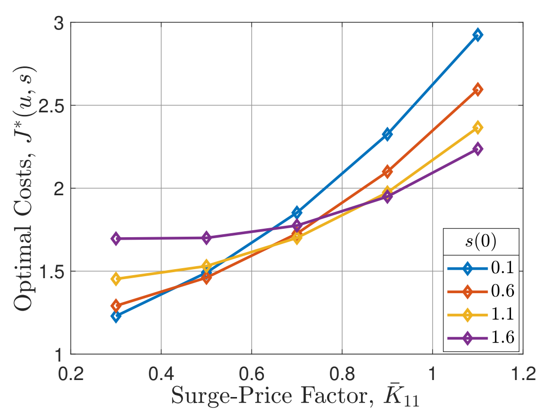


Fig. 3: Variation of the optimal costs with $\bar{K}_{11}$ for different initial supplies, for a target mode-share, $d_1 = 12$.

Next, in Figure 4, we show the optimal costs for different target mode-share, d_1. For lower values of d_1, we see that the costs are very similar. The costs incurred are due to the phenomenon that is described earlier, i.e. the supply initially drops and finally increases to $s(T)$. As expected, as d_1 increases, the operator needs to increase the supply to meet the target. Also, as one can expect, for higher values of d_1 and $\bar{K}_{11}$, the problem (13) becomes infeasible. This is because of the implications from Theorem IV.1.

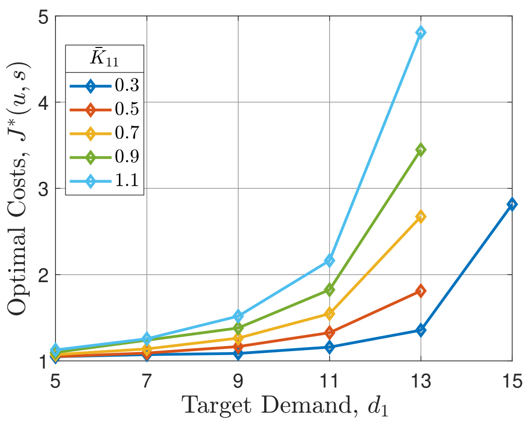


Fig. 4: Variation of optimal costs for $s(0) = 0.1$ and different surge-price factors $\bar{K}_{11}$. The simulations were done for $d_1 \in [5, 15]_{\mathbb{Z}}$ in steps of 2. For higher surge-price factors the problem does not converge for higher d_1.

Solver Limitations: While simulating the problem (13) using GEKKO, we found that IPOPT with APMonitor sometimes fails to converge to the optimal solution. Many of these instances are due to the calculation of the exponents in the logit dynamics. The solver also fails to converge when the user equilibrium tends to be close to the boundary of Ω. The results presented here converged without such issues.

VI. Conclusions

In this paper, we proposed a novel optimal control based mode-choice model for a fixed-demand travelling between a single OD pair. The analysis allows an operator to make an informed decision on the rate of supply, given the parameters of the modes, the initial supply, and the maximum rates at which supply can be changed. Using simulations, we also showed how the supply and prices can be used to attract more customers. This framework can be utilised to model the entry of a disruptive ride-sharing mode, a phenomenon that is common in emerging markets. An immediate extension of this work is to find sufficient conditions for feasibility of the optimal control problem and proving the convergence of the dynamics to SUE in the asymmetric costs scenario chosen in this paper. Another direction of future research is to extend this work to multi-OD case with an underlying network. Finally, we could explore other solution techniques that circumvent numerical issues posed by solvers.

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