

# An inexact optimization approach for river water-quality management

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## Abstract

A previously developed fuzzy waste load allocation model (FWLAM) for a river system is extended to address uncertainty involved in fixing the membership functions for the fuzzy goals of the pollution control agency (PCA) and the dischargers using the concept of grey systems. The model provides flexibility for the PCA and the dischargers to specify their goals independently, as the parameters for membership functions are considered as interval grey numbers instead of deterministic real numbers. An inexact or a grey fuzzy optimization model is developed in a multiobjective framework, to maximize the width of the interval valued fractional removal levels for providing latitude in decision-making and to minimize the width of the goal fulfillment level for reducing the system uncertainty. The concept of an acceptability index for order relation between two partially or fully overlapping intervals is used to get a deterministic equivalent of the grey fuzzy optimization model developed. The improvement of the optimal solutions over a previously developed grey fuzzy waste load allocation model (GFWLAM) is shown through an application to a hypothetical river system. The fuzzy multiobjective optimization and fuzzy goal programming techniques are used to solve the deterministic equivalent of the GFWLAM.

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## 1. Introduction

A waste load allocation (WLA) model for decision-making in water-quality management of a river system, in general, integrates a water-quality simulation model with an optimization model to provide best compromise solutions acceptable to both the pollution control agency (PCA) and dischargers. A number of WLA models have been developed in the past for optimal allocation of assimilative capacity of a river system (Loucks and Lynn, 1966; Loucks et al., 1981; Loucks, 1983; Fugiwara et al., 1988; Tung and Hathhorn, 1989). Uncertainty due to randomness has been addressed extensively in the models for water-quality management of river systems, starting with the pioneering work of Loucks and Lynn (1966). Another type of uncertainty in water resources problems is the imprecision in management goals and model para-

meters, which has been in general addressed with fuzzy sets (Bogardi et al., 1983; Bardossy and Disse, 1993; Shreshtha et al., 1996; Fontane et al., 1997; Teegavarapu and Simonovic, 1999). The concept of fuzzy decision (Bellman and Zadeh, 1970) has been used in water-quality management problems in recent work (e.g., Hathhorn and Tung, 1989; Chang et al., 1997; Sasikumar and Mujumdar, 1998, 2000; Clark, 2002; Mujumdar and Subbarao, 2004; Ning and Chang, 2004; Subbarao et al., 2004).

Sasikumar and Mujumdar (1998) developed a fuzzy waste load allocation model (FWLAM) for water-quality management of a river system, to incorporate the imprecisely defined conflicting goals of PCA and dischargers in a fuzzy optimization framework. The solution provides a set of optimal fractional removals ( $\hat{X}$ ) of the pollutants and the maximum value of goal fulfillment level ( $\hat{\lambda}$ ), which is a measure of the degree of fulfillment of the fuzzy goals. A major limitation in the models of Sasikumar and Mujumdar (1998, 2000), and Mujumdar and Sasikumar (2002) is that the membership parameters for management goals are assumed fixed and values are assigned based on some judgement (such as, for example, a lower bound of

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5 mg/L and an upper bound of 9 mg/L for DO concentration). Choice of appropriate values of membership parameters is an important issue in any fuzzy optimization model, as these are highly subjective on the choice made by the decision-maker. In FWLAM, the parameters of the membership functions defining the goals of PCA depend on the desirable and maximum permissible level of water-quality. In practical situations different water-quality standards for surface water are used for different uses for a water-quality indicator. For example, standards for public water supply, industrial water supply, agricultural water supply, fish propagation and wild life may all be different for the same water-quality indicator, DO (Hammer, 1986). This results in an uncertainty in the membership parameters and leads to a second level of fuzziness in the model, with the membership functions themselves being imprecisely stated. Karmakar and Mujumdar (2004a, b) developed a grey fuzzy waste load allocation model (GFWLAM), which considers the uncertainty in the bounds and shape of the membership functions in fuzzy optimization models for water-quality management. The model is aimed at relaxing the membership parameters by treating them as interval grey numbers. A terminology “imprecise membership function” is used to represent the membership functions with uncertain parameters. Consideration of imprecise membership functions in the fuzzy optimization model imparts flexibility in the solutions as the solutions are obtained as interval grey numbers. A set of interval optimal fractional removal levels ( $\hat{X}^{\pm}$ ) are obtained corresponding to a range of optimal values of goal fulfillment level ( $\hat{\lambda}^{\pm}$ ), whereas FWLAM gives only a single set of optimal fractional removal levels of pollutants corresponding to the maximum goal fulfillment level. With the GFWLAM solutions, the fractional removal levels can be adjusted within their optimal grey intervals by the decision-maker in the final decision scheme as required in a particular situation. The width of the interval optimal fractional removal level thus plays an important role in the GFWLAM, as more width in the fractional removals implies a wider choice to the decision-makers.

In the present work a modified GFWLAM is developed to maximize the width of the interval valued fractional removal levels of pollutants [i.e.,  $(X^+ - X^-)$ ] using an inexact or grey fuzzy optimization technique (Huang et al., 1993; Huang and Loucks, 2000) in a multiobjective framework. This enhances the flexibility and applicability in decision-making, as the decision-maker gets a wider range of optimal fractional removals than those of solutions obtained from GFWLAM (Karmakar and Mujumdar, 2004a, b). Similar to GFWLAM, the upper and lower bounds of the goal fulfillment level (i.e.,  $\lambda^+$  and  $\lambda^-$ ) are maximized, but in the modified GFWLAM the system uncertainty has also been reduced by minimizing the width of the degree of goal fulfillment level [i.e.,  $(\lambda^+ - \lambda^-)$ ]. In GFWLAM the order of consideration of the goals of PCA and the dischargers, along with the selection of bounds of decision variables create two different

situations of model formulation, which are termed as Cases 1 and 2. Each case is further subdivided into two submodels to obtain two extreme values of  $\lambda^{\pm}$ , which give the solutions for two extreme cases encompassing all intermediate possibilities. Therefore, in GFWLAM, four submodels are solved to obtain the range of optimal fractional removal levels. The present work develops a multiobjective optimization model, which maximizes the width of the interval valued fractional removal levels and reduces the system uncertainty involved in the optimization model by minimizing the width of the interval valued goal fulfillment level in a single optimization model, avoiding intermediate submodels. The next section presents a brief description of GFWLAM, developed in the earlier work.

## 2. Grey fuzzy waste load allocation model (GFWLAM)

A general river system is considered for developing the GFWLAM. The system consists of a set of dischargers, who are allowed to release pollutants into the river after removing some fraction of the pollutants. Fractional removal is necessary to maintain acceptable water-quality conditions in the river as prescribed by the PCA. Acceptable water-quality conditions are ensured by checking the water-quality in terms of water-quality indicators (e.g., DO-deficit, hardness, nitrate–nitrogen concentration) at a finite number of locations, which are referred to as checkpoints. The goals of PCA are to ensure that pollution is within an acceptable limit by imposing water-quality and effluent standards. On the other hand, the dischargers prefer to use the assimilative capacity of the river system to minimize the waste treatment cost, by assigning an aspiration level (minimum desirable treatment) and maximum fractional removal level for different pollutants. These goals are imprecise and subjective, and are addressed in the model through a fuzzy mathematical framework by assigning membership functions. The concentration of a water-quality indicator is expressed as a function of the fractional removal levels of the pollutants using an appropriate pollutant transport model (such as, for example Streeter-Phelps model, QUAL2E, WASP4, etc.). It is assumed, in the model development that: (1) steady-state flow conditions prevail; an instantaneous mixing of the pollutants takes place at the discharge point, (2) river characteristics are homogeneous and the river parameters for a river reach remain unchanged, (3) water-quality indicators are such that the desirable level is less than the permissible level (eg. DO-deficit, toxic pollutant concentration, etc.), (4) a pollutant affects one or more than one water-quality indicator; but the pollutants are chemically non-reactive with each other, and (5) industrial and municipal effluents are pretreated at the site prior to discharge into the river. Similar to FWLAM, conflict between the fuzzy goals of PCA and dischargers is modeled using the concept of fuzzy decision, but because of treating the membership parameters as interval grey numbers, the notion of “fuzzy decision” leads to the notion of “grey

fuzzy decision”. A terminology grey fuzzy decision is used to represent the fuzzy decision resulting from the imprecise membership functions. This terminology was used earlier by Luo et al. (1999) to define a ‘grey fuzzy motion decision’ combining grey prediction and fuzzy logic control theories. The notion of “grey fuzzy decision” presented in this paper is different from that used by Luo et al. (1999).

An overview of the basic concepts of grey systems, fuzzy decision and grey fuzzy decision is given in the following subsections, as a prerequisite to the development of GFWLAM.

### 2.1. Grey systems theory

Grey systems theory was first proposed by Deng (1982). A grey system is a system other than white (system with completely known information) and black (system with completely unknown information) systems, and thus has partially known and partially unknown characteristics (Liu and Lin, 1998). In reality, many processes of interest in environmental management are in the grey stage due to inadequate and fuzzy information. A “grey number” is such a number whose exact value is unknown but a range within which the value lies is known (Liu and Lin, 1998; Yang and John, 2003). Let ‘ $a$ ’ denote a closed and bounded set of real numbers. A grey number ( $a^\pm$ ) is defined as an interval with known lower ( $a^-$ ) and upper ( $a^+$ ) bounds but unknown distribution information for  $a$  (Huang et al., 1995).

$$a^\pm = [a^-, a^+] = [t \in a | a^- \leq t \leq a^+]. \quad (1)$$

$a^\pm$  becomes a “deterministic number” or “white number” when  $a^\pm = a^- = a^+$ . When  $a^\pm = [a^-, a^+] = (-\infty, +\infty)$ ,  $a^\pm$  is called a “black number”. An “interval number” (Moore, 1979) or “interval grey number” ( $a^\pm = [a^-, a^+]$ ) is one among several classes of grey numbers (Liu and Lin, 1998). The “grey degree” is a measure, useful for quantitatively evaluating the quality of uncertain input or output information for mathematical models (Huang et al., 1995). The “grey degree” of an interval grey number is defined as its width [ $a_w = (a^+ - a^-)$ ] divided by its whitened mid value (WMV) [ $a_m = \frac{1}{2}(a^- + a^+)$ ] (Huang et al., 1995), and is expressed in percentage (%) as follows:

$$\text{Gd}(a^\pm) = (a_w/a_m) \times 100\%, \quad (2)$$

where  $\text{Gd}(a^\pm)$  is the grey degree of  $a^\pm$ . Model outputs with considerably high grey degree have high width ( $a_w$ ) of output variables, which are considered as less useful and of poor quality by the decision-makers. As the grey degree of the objective function of an optimization model decreases, implying decreasing system uncertainties, the effectiveness of the grey model increases. Therefore, a lower value of the grey degree of the optimal objective function (e.g., goal fulfillment level,  $\lambda^\pm$ , of a fuzzy optimization model) results in a more acceptable grey solution. It should be noted for clarification that the notion of grey systems modelling is different from the notion of sensitivity analysis. The

sensitivity analysis is considered as a post-optimality analysis, where systematic variation of input parameters, considering variation of a single parameter (i.e., univariate sensitivity analysis) or a group of parameters (i.e., multivariate sensitivity analysis) at a time in a model is performed to assess the effect of uncertainties or variation in these parameters on the model output. But the grey systems modelling directly addresses the uncertainties of all uncertain model parameters in a single mathematical framework and gives the solutions as stable intervals, which can be directly used for generating decision alternatives.

### 2.2. Fuzzy decision

The fuzzy decision ( $Z$ ) for a water-quality management problem may be defined using the general concept of fuzzy decision proposed by Bellman and Zadeh (1970). They proposed, a broad definition of the fuzzy decision as a confluence of fuzzy goals and fuzzy constraints. The term “confluence” is context dependent and for a particular situation a justifiable aggregation should be performed to determine a meaningful fuzzy decision. In the water-quality management problem it is appropriate that the aggregating function should correspond to the aggregation operation “logical and”, which corresponds to the “set theoretic intersection”. Noting that the decision space is defined by the intersection of different fuzzy goals, the fuzzy decision ( $Z$ ) is written as follows:

$$Z = F_1 \cap F_2, \quad (3)$$

where fuzzy sets  $F_1$  and  $F_2$  represent the two fuzzy goals. The membership function of the fuzzy decision ( $Z$ ) is given by

$$\mu_Z(x) = \lambda = \min[\mu_{F_1}(x), \mu_{F_2}(x)], \quad (4)$$

where  $\lambda$  is the measuring variable corresponding to the membership function of fuzzy decision ( $Z$ ), which reflects the degree of fulfillment of the system goals. The term “goal fulfillment level” is used throughout the paper to represent the variable  $\lambda$ .  $x$  is the argument of the fuzzy goals  $F_1$  and  $F_2$ . The solution  $\hat{x}$  corresponding to the maximum value of the membership function of the resulting  $Z$  is the optimum solution. That is

$$\mu_Z(\hat{x}) = \hat{\lambda} = \max_{x \in Z} [\mu_Z(x)], \quad (5)$$

where  $\hat{\lambda}$  is the optimal goal fulfillment level. Fig. 1 shows the concept of a fuzzy decision, where  $F_1$  and  $F_2$  are non-increasing and non-decreasing membership functions, respectively.

### 2.3. Grey fuzzy decision

In the concept of a fuzzy decision, the arguments of  $F_1$  and  $F_2$  are deterministic real numbers ( $x$ ). When the goals  $F_1^\pm$  and  $F_2^\pm$  are imprecise fuzzy goals or grey fuzzy goals and the corresponding arguments are interval grey

numbers ( $x^\pm$ ), the fuzzy decision leads to a “grey fuzzy decision”. Fig. 2 illustrates the concept of grey fuzzy decision considering the confluence of two imprecise membership functions for  $F_1^\pm$  and  $F_2^\pm$ . Considering “logical and”, corresponding to the “set theoretic intersection” as an aggregation operator, the grey fuzzy decision is determined. In Fig. 2 the decision  $Z^\pm$  is not a fixed space (as shown in Fig. 1 for a fuzzy decision). It is a flexible space, whose lower and upper boundaries are shown in Fig. 2 as A’FNGH’ and A’ECMCHH’, respectively. The solutions  $\hat{x}^\pm$  corresponding to the maximum value of the membership function of the resulting grey fuzzy decision ( $Z^\pm$ ) is an interval in the space CMC’N (Fig. 2). Mathematically the grey fuzzy decision ( $Z^\pm$ ) for  $F_1^\pm$  and  $F_2^\pm$  can be defined with the imprecise membership function:

$$\mu_{Z^\pm}^-(x^\pm) = \lambda^- = \min \left[ \min \left\{ \mu_{F_1^\pm}^-(x^-), \mu_{F_2^\pm}^-(x^-) \right\}; \min \left\{ \mu_{F_1^\pm}^-(x^+), \mu_{F_2^\pm}^-(x^+) \right\} \right], \quad (6)$$

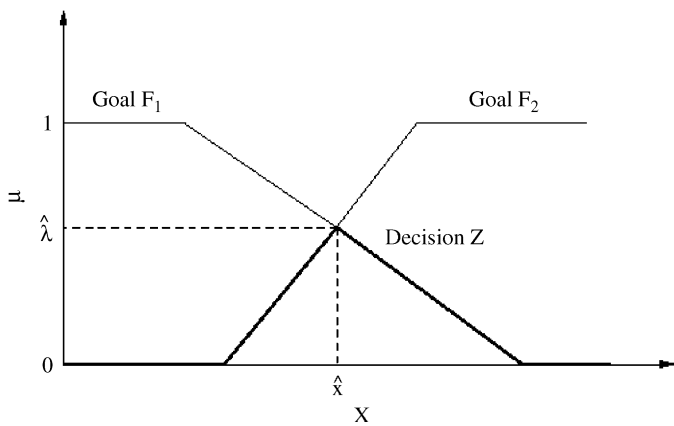


Fig. 1. Concept of fuzzy decision.

$$\mu_{Z^\pm}^+(x^\pm) = \lambda^+ = \max \left[ \min \left\{ \mu_{F_1^\pm}^+(x^-), \mu_{F_2^\pm}^+(x^-) \right\}; \min \left\{ \mu_{F_1^\pm}^+(x^+), \mu_{F_2^\pm}^+(x^+) \right\} \right] \quad (7)$$

where  $\mu_{Z^\pm}^-(x^\pm)$  and  $\mu_{Z^\pm}^+(x^\pm)$  are lower and upper bounds of the imprecise membership functions for an interval  $[x^-, x^+]$ , respectively, encompassing all intermediate possibilities of membership values [i.e.,  $\mu_{Z^\pm}^\pm(x^\pm)$ ]. Eqs. (6) and (7) are valid for all combinations of imprecise membership functions (i.e., non-increasing, non-decreasing, or a combination of the two). In GFWLAM two sets of imprecisely defined and conflicting goals (i.e., goals of the PCA and dischargers) are addressed through an optimization model by using the concept of grey fuzzy decision.

2.4. Goals of the PCA

The PCA sets the desirable concentration level ( $c_{jl}^D$ ) and maximum permissible concentration level ( $c_{jl}^H$ ) of the water-quality indicator  $j$  (e.g., DO-deficit, hardness, nitrate–nitrogen concentration) at the water-quality checkpoint  $l$  ( $c_{jl}^D \leq c_{jl}^H$ ). The goal  $E_{jl}$ , of the PCA, is to make the concentration level ( $c_{jl}$ ) of water-quality indicator  $j$  at the checkpoint  $l$  as close as possible to the desirable level,  $c_{jl}^D$ , so that the water-quality at the checkpoint  $l$  is enhanced with respect to the water-quality indicator  $j$ , for all  $j$  and  $l$ . These goals are represented by a membership function. For example, if DO-deficit is the water-quality indicator, a non-increasing membership function suitably reflects the goals of the PCA with respect to DO-deficit at a checkpoint. The uncertainty associated with membership parameters ( $c_{jl}^D$  and  $c_{jl}^H$ ) is addressed using interval grey numbers, and the membership parameters are expressed as  $c_{jl}^{D\pm}$  and  $c_{jl}^{H\pm}$ . Using non-increasing imprecise membership functions, the grey fuzzy goals of PCA (i.e.,  $E_{jl}^\pm$ ) are

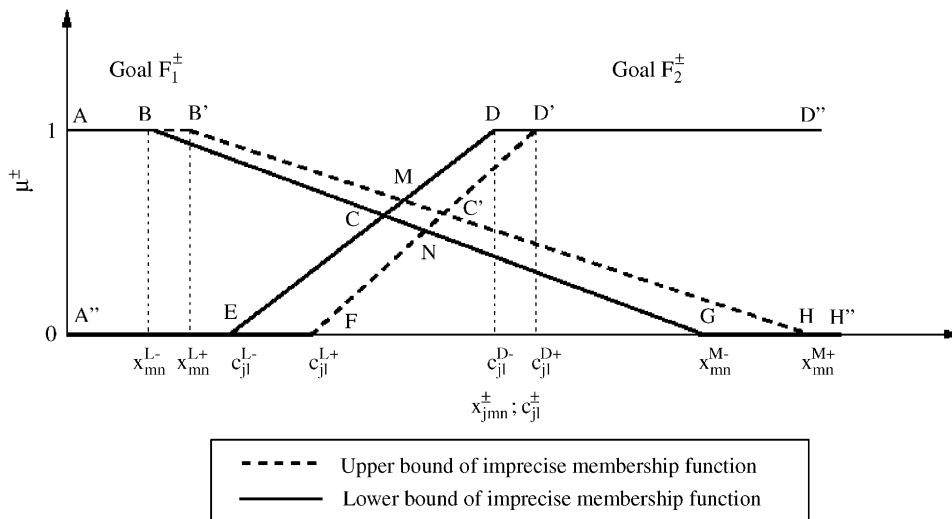


Fig. 2. Concept of grey fuzzy decision.

represented as

$$\mu_{E_{jl}^\pm}(C_{jl}^\pm) = \begin{cases} 1 & c_{jl}^+ < c_{jl}^{D-} \\ \left[ \frac{c_{jl}^{H\pm} - c_{jl}^\pm}{c_{jl}^{H\pm} - c_{jl}^{D\pm}} \right]^{\alpha_{jl}} & c_{jl}^{D\pm} \leq c_{jl}^\pm \leq c_{jl}^{H\pm} \\ 0 & c_{jl}^- > c_{jl}^{H+} \end{cases} \quad (8)$$

where  $c_{jl}^\pm$  is the uncertain concentration level of water-quality indicator  $j$  at the checkpoint  $l$ , represented as an interval grey number. The exponent  $\alpha_{jl}$  is a nonzero positive real number. Assignment of a numerical value to this exponent is subject to the desired shape of the membership functions. A value of  $\alpha_{jl} = 1$  leads to a linear imprecise membership function, as shown in Fig. 3.

### 2.5. Goals of the dischargers

The goal of discharger  $m$ ,  $F_{jmn}^\pm$ , is to make the fractional removal level ( $x_{jmn}^\pm$ ) as close as possible to the aspiration level ( $x_{jmn}^{L\pm}$ ), to minimize the waste treatment cost of pollutant  $n$  influencing water-quality indicator  $j$ . The grey fuzzy goals of the dischargers (i.e.,  $F_{jmn}^\pm$ ) are similarly represented as:

$$\mu_{F_{jmn}^\pm}(x_{jmn}^\pm) = \begin{cases} 1 & x_{jmn}^+ < x_{jmn}^{L-} \\ \left[ \frac{x_{jmn}^{M\pm} - x_{jmn}^\pm}{x_{jmn}^{M\pm} - x_{jmn}^{L\pm}} \right]^{\beta_{jmn}} & x_{jmn}^{L\pm} \leq x_{jmn}^\pm \leq x_{jmn}^{M\pm} \\ 0 & x_{jmn}^- > x_{jmn}^{M+} \end{cases} \quad (9)$$

where the aspiration level and maximum acceptable level of the fractional removal of the pollutant  $n$  at discharger  $m$  are represented as  $x_{jmn}^{L\pm}$  and  $x_{jmn}^{M\pm}$ , respectively ( $x_{jmn}^{L\pm} \leq x_{jmn}^{M\pm}$ ). Similar to the exponent  $\alpha_{jl}$  in Eq. (8),  $\beta_{jmn}$  is a non zero

positive real number. A value of  $\beta_{jmn} = 1$  leads to a linear imprecise membership function, as shown in Fig. 4.

### 2.6. GFWLAM formulation

In GFWLAM the grey fuzzy decision ( $Z^\pm$ ) is expressed by using the concept of fuzzy decision [Eq. (3)]:

$$Z^\pm = \left( \bigcap_{j,l} E_{jl}^\pm \right) \cap \left( \bigcap_{j,m,n} F_{jmn}^\pm \right), \quad (10)$$

where  $E_{jl}^\pm$  and  $F_{jmn}^\pm$  represent the grey fuzzy goals of PCA and dischargers, respectively (as shown in Figs. 3 and 4). The arguments of  $E_{jl}^\pm$  and  $F_{jmn}^\pm$  are  $c_{jl}^\pm$  and  $x_{jmn}^\pm$ , respectively. The intersection of these non-increasing imprecise membership functions is shown in Fig. 5, where ABCDD'GH and ABC'D'GH are lower and upper boundaries of the grey fuzzy decision, and the optimal solution  $\hat{x}^\pm$  is an interval in the space MC'NC. The concept of grey fuzzy decision is used to model the conflicting nature of the goals of PCA and dischargers in the optimization model. The formulation of GFWLAM is written as

$$\text{Max } \lambda^\pm \quad (11)$$

$$\text{Subject to } \mu_{E_{jl}^\pm}(c_{jl}^\pm) = \left[ \frac{c_{jl}^{H\pm} - c_{jl}^\pm}{c_{jl}^{H\pm} - c_{jl}^{D\pm}} \right]^{\alpha_{jl}} \geq \lambda^\pm \quad \forall j, l, \quad (12)$$

$$\mu_{F_{jmn}^\pm}(x_{jmn}^\pm) = \left[ \frac{x_{jmn}^{M\pm} - x_{jmn}^\pm}{x_{jmn}^{M\pm} - x_{jmn}^{L\pm}} \right]^{\beta_{jmn}} \geq \lambda^\pm \quad \forall j, m, n, \quad (13)$$

$$c_{jl}^{D\pm} \leq c_{jl}^\pm \leq c_{jl}^{H\pm} \quad \forall j, l, \quad (14)$$

$$x_{jmn}^{L\pm} \leq x_{jmn}^\pm \leq x_{jmn}^{M\pm} \quad \forall j, m, n, \quad (15)$$

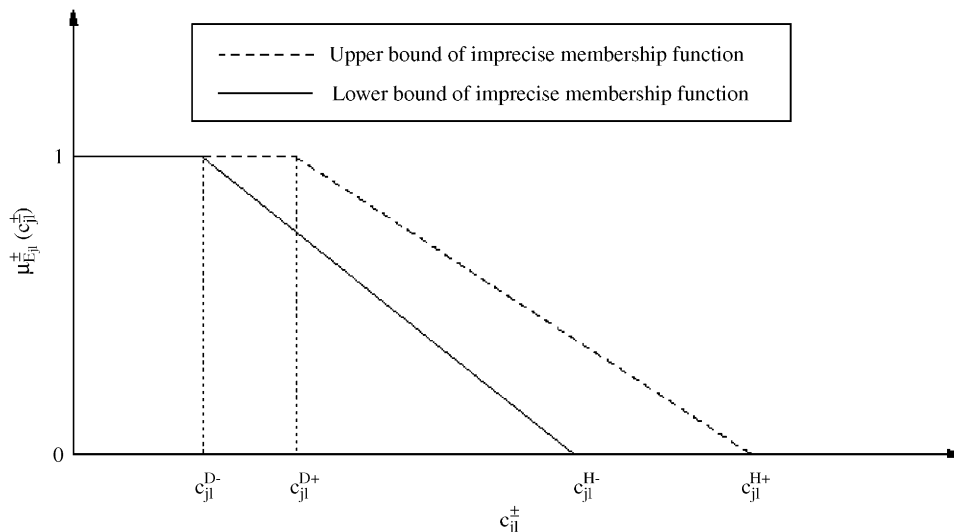


Fig. 3. Linear imprecise membership function for the goals of PCA.

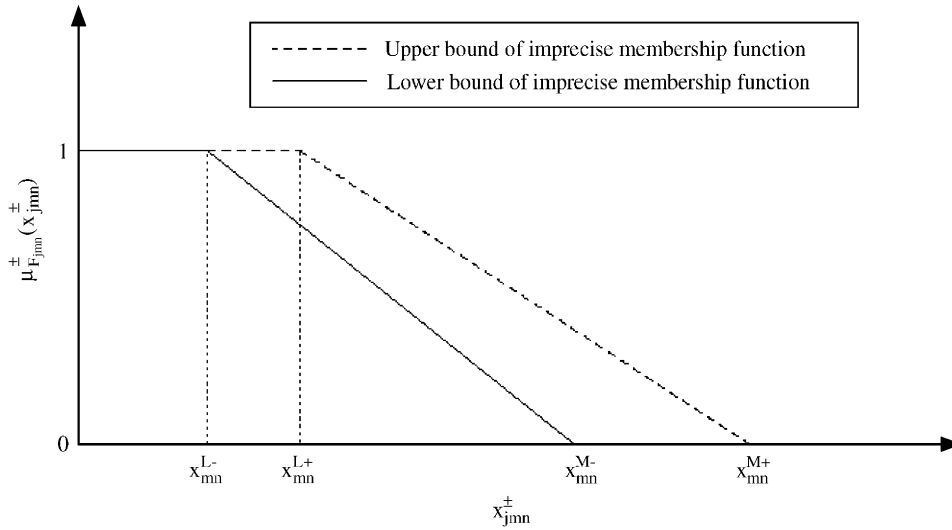


Fig. 4. Linear imprecise membership function for the goals of dischargers.

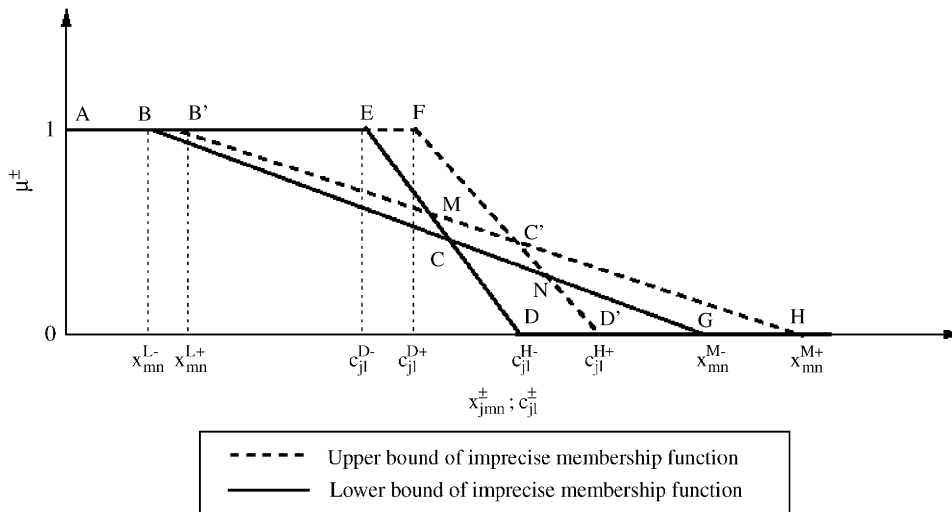


Fig. 5. Grey fuzzy decision resulting from confluence of the goals of PCA and dischargers.

$$0 \leq \lambda^{\pm} \leq 1. \tag{16}$$

Constraints (12) and (13) are formulated from imprecise membership functions for the goals of PCA and dischargers, respectively, and define the minimum goal fulfilment level ( $\lambda^{\pm}$ ). The crisp constraints (14) and (15) are based on the water-quality requirements set by the PCA, and acceptable fractional removal levels by the dischargers, respectively. Constraint (16) presents the bounds on the parameter  $\lambda^{\pm}$ . In the expression for the goals of PCA [constraint (12)], the concentration level  $c_{jl}^{\pm}$ , of water-quality indicator  $j$  at checkpoint  $l$ , may be expressed as

$$c_{jl}^{\pm} = f(x_{jmn}^{\pm}), \tag{17}$$

where the transfer function ( $f$ ) indicates the aggregate effect of all pollutants and dischargers, located upstream of checkpoint  $l$  on the water-quality indicator  $j$  at that checkpoint. The transfer function can be evaluated using appropriate mathematical models that determine the

spatial distribution of the water-quality indicator due to pollutant discharge into the river system from point sources (Fugiwara et al., 1987, 1988; Sasikumar and Mujumdar, 1998). For most water-quality indicators, a high level of fractional removal of pollutants (e.g., BOD loading, toxic pollutant concentration, etc.) results in a low level of water-quality indicator (e.g., DO-deficit, nitrate-nitrogen concentration, etc.). The lower bound of water-quality indicator ( $c_{jl}^{-}$ ) is therefore expressed in terms of the upper bound of the fractional removal level ( $x_{jmn}^{+}$ ) using Eq. (17).

$$c_{jl}^{-} = f(x_{jmn}^{+}). \tag{18}$$

Similarly,

$$c_{jl}^{+} = f(x_{jmn}^{-}). \tag{19}$$

Eqs. (18) and (19) are the deterministic equivalent of the Eq. (17), which contains interval grey numbers (i.e.,  $c_{jl}^{\pm}$  and  $x_{jmn}^{\pm}$ ) in both the sides. The order of consideration of

constraints (12) and (13), along with the selection of bounds of decision variables ( $x_{jmn}^-$ ,  $x_{jmn}^+$ ) creates two different cases of model formulation, which are termed as Case 1 and Case 2, with each problem divided into two submodels. Submodel 1 maximizes the upper bound,  $\lambda^+$ , and Submodel 2 maximizes the lower bound,  $\lambda^-$ . Lower and upper bounds of the decision ( $x_{jmn}^-$  and  $x_{jmn}^+$ ) are obtained from these two submodels. These four submodels (two each for Cases 1 and 2) are the deterministic equivalent of the grey fuzzy optimization model given in (11)–(16). The four submodels are solved to obtain the set of optimal values of fractional removal levels of the pollutants (denoted by  $\hat{X}^\pm = \{\hat{x}_{jmn}^\pm\}$ ) as a set of flexible policies in the form of interval grey numbers.

The present work is also aimed at addressing the uncertainty in the assignment of membership functions for management goals of the PCA and dischargers as addressed in GFWLAM, but the multiobjective approach adopted to obtain the deterministic equivalent of the grey fuzzy optimization model is simpler and the solutions are more useful to the decision-makers as a wider choice in the interval valued optimal fractional removals is obtained. The optimal value of goal fulfilment level ( $\hat{\lambda}^\pm$ ) may be viewed as a measure of compromise between the decision-makers in the system (Kindler, 1992), and may also be considered as a measure of conflict existing in the system. A value of  $\hat{\lambda}^\pm = 0$  (i.e.,  $\hat{\lambda}^- = 0$  and  $\hat{\lambda}^+ = 0$ ) indicates a strong conflict scenario, whereas a value of  $\hat{\lambda}^\pm = 1$  (i.e.,  $\hat{\lambda}^- = 1$  and  $\hat{\lambda}^+ = 1$ ) corresponds to a no-conflict scenario. As  $\lambda^\pm$  is a measure of the degree of fulfillment of the goals of PCA and dischargers,  $\lambda^+$  and  $\lambda^-$  are maximized to ensure the highest possibility of fulfillment of the management goals. Secondly, the width of the interval of fractional removal level [i.e.,  $(x_{jmn}^+ - x_{jmn}^-)$ ] is maximized, to ensure more flexibility for the decision-makers in post-optimality decision-making. Finally, the output uncertainty is reduced by minimizing the width of the degree of the goal fulfillment level [i.e.,  $(\lambda^+ - \lambda^-)$ ]. All these objectives [viz., maximization of  $\lambda^+$ ,  $\lambda^-$ ,  $(x_{jmn}^+ - x_{jmn}^-)$ , and minimization of  $(\lambda^+ - \lambda^-)$ ] are addressed in a single optimization model in a multiobjective framework along with the uncertain information addressed through interval grey numbers. The next section gives a description of the modified GFWLAM.

### 3. Modified GFWLAM

The grey fuzzy optimization model given in (11)–(16) forms the basis of the modified GFWLAM. The fuzzy inequality constraints (12) and (13) address the goals of the PCA and dischargers in the optimization model. These are the order relations (e.g., the relations “greater than or equal to” or “less than or equal to”) containing interval grey numbers on both the sides. Determination of meaningful ranking between two partially or fully overlapping intervals in the order relations is a potential research area (e.g., Moore, 1979; Ishibuchi and Tanaka, 1990; Sengupta et al., 2001). Recently, Sengupta et al. (2001) proposed a

satisfactory deterministic equivalent form of inequality constraints containing interval grey numbers by using the acceptability index ( $\mathcal{A}$ ). In the following subsection the formulation of the modified GFWLAM is presented using the concept of acceptability index for comparing the interval grey numbers.

#### 3.1. Model formulation

Acceptability index ( $\mathcal{A}$ ) is defined as the grade of acceptability of the premise the “first interval grey number ( $a^\pm$ ) is inferior to the second ( $b^\pm$ )”, denoted as  $a^\pm (<) b^\pm$ . Here, the term “inferior to” (“superior to”) is analogous to “less than” (“greater than”). The acceptability index ( $\mathcal{A}$ ) is expressed as (Sengupta et al., 2001)

$$\mathcal{A}[a^\pm (<) b^\pm] = [m(b^\pm) - m(a^\pm)] / [w(b^\pm) + w(a^\pm)], \quad (20)$$

where  $[w(b^\pm) + w(a^\pm)] \neq 0$ ;  $w(a^\pm)$  is the half-width of  $a^\pm = 1/2 (a^+ - a^-)$ ;  $m(a^\pm)$  is the mean of  $a^\pm = 1/2 (a^- + a^+)$ . Notations are similarly defined for the interval grey number  $b^\pm$ . The grade of acceptability of  $a^\pm (<) b^\pm$  may be classified and interpreted further on the basis of the comparative position of the mean and half-width of interval  $b^\pm$  with respect to those of interval  $a^\pm$ . Let us consider an interval inequality relation  $a^\pm x \geq b^\pm$ , where  $x$  is a deterministic variable. A satisfactory deterministic equivalent form of the interval inequality relation  $a^\pm x \geq b^\pm$ , is proposed as

$$a^\pm x \geq b^\pm \Rightarrow \{a^- x \geq b^- \text{ and } \mathcal{A}[a^\pm x (<) b^\pm] \leq \alpha \in [0, 1]\}, \quad (21)$$

where  $\alpha$  is interpreted as an optimistic threshold fixed by the decision-maker. A brief description of the concept of acceptability index is given in the appendix. The deterministic equivalent of the grey fuzzy optimization model given in (11)–(16) is formulated using the expression given in Eq. (21). By using the attributes mean, width and acceptability index of the interval grey numbers, the grey fuzzy optimization model is reduced to a deterministic multiobjective optimization model, as follows

$$\text{Max } \lambda^+, \quad (22)$$

$$\text{Max } \lambda^-, \quad (23)$$

$$\text{Min } [(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)] \quad (24)$$

$$\text{Subject to } \mu_{E_j^\pm}^-(c_{jl}^\pm) = [(c_{jl}^{H-} - c_{jl}^+) / (c_{jl}^{H+} - c_{jl}^{D-})] \geq \lambda^- \quad \forall j, l, \quad (25)$$

$$\mu_{F_{jmn}^\pm}^-(x_{jmn}^\pm) = [(x_{jmn}^{M-} - x_{jmn}^+) / (x_{jmn}^{M+} - x_{jmn}^{L-})] \geq \lambda^- \quad \forall j, m, n, \quad (26)$$

$$\mathcal{A}[(c_{jl}^{H\pm} - c_{jl}^\pm) / (c_{jl}^{H\pm} - c_{jl}^{D\pm}) (<) \lambda^\pm] \leq \alpha_1 \in [0, 1] \quad \forall j, l, \quad (27)$$

$$A[(x_{jmn}^{M\pm} - x_{jmn}^{\pm}) / (x_{jmn}^{M\pm} - x_{jmn}^{L\pm})] (\lambda^{\pm}) \leq \alpha_2 \in [0, 1] \quad \forall j, m, n, \tag{28}$$

$$c_{jl}^{D-} \leq c_{jl}^+ \leq c_{jl}^{H+}; c_{jl}^{D-} \leq c_{jl}^- \leq c_{jl}^{H+} \quad \forall j, l, \tag{29}$$

$$x_{mn}^{L-} \leq x_{jmn}^+ \leq x_{mn}^{M+}; x_{mn}^{L-} \leq x_{jmn}^- \leq x_{mn}^{M+} \quad \forall j, m, n, \tag{30}$$

$$c_{jl}^- \leq c_{jl}^+ \quad \forall j, l, \tag{31}$$

$$x_{jmn}^- \leq x_{jmn}^+ \quad \forall j, m, n, \tag{32}$$

$$0 \leq \lambda^+ \leq 1; 0 \leq \lambda^- \leq 1, \tag{33}$$

$$\lambda^- \leq \lambda^+. \tag{34}$$

Constraints (25)–(28) are the deterministic equivalent of constraints (12)–(13). The acceptability index in constraints (27) and (28) compares the interval grey numbers in the inequality constraints (12) and (13). In constraints (27) and (28),  $\alpha_1$  and  $\alpha_2$  are the optimistic thresholds fixed by the decision-maker. The goals of the PCA and dischargers are represented by linear imprecise membership functions [i.e., substituting  $\alpha_{jl}$  and  $\beta_{jmn} = 1$  in Eqs. (8) and (9), respectively], as the Eq. (21) with acceptability index for ranking the interval grey numbers in the inequality constraints is applicable only for linear programming problems (Sengupta et al., 2001). In water-quality management problems, any continuous monotonically non-increasing membership function will address reasonably well the imprecision in the management goals of the PCA and dischargers, but the placement of membership parameters on the universe of discourse is important. The mathematical expression of acceptability index given in Eq. (20) is substituted in the constraints (27) and (28), and algebraic operations on the interval grey numbers (Moore, 1979; Liu and Lin, 1998) are performed to get a simplified form. The final expressions of constraints (27) and (28) are given as

$$\begin{aligned} & \{[(\lambda^+ + \lambda^-) - (c_{jl}^{H-} - c_{jl}^+) / (c_{jl}^{H+} - c_{jl}^{D-}) \\ & - (c_{jl}^{H+} - c_{jl}^-) / (c_{jl}^{H-} - c_{jl}^{D+})] / \{(\lambda^+ - \lambda^-) \\ & + (c_{jl}^{H+} - c_{jl}^-) / (c_{jl}^{H-} - c_{jl}^{D+}) - (c_{jl}^{H-} - c_{jl}^+) / (c_{jl}^{H+} - c_{jl}^{D-})\} \} \\ & \leq \alpha_1 \in [0, 1] \quad \forall j, l \end{aligned} \tag{35}$$

$$\begin{aligned} & \{[(\lambda^+ + \lambda^-) - (x_{mn}^{M-} - x_{jmn}^+) / (x_{mn}^{M+} - x_{jmn}^{L-}) \\ & - (x_{mn}^{M+} - x_{jmn}^-) / (x_{mn}^{M-} - x_{jmn}^{L+})] / \{(\lambda^+ - \lambda^-) \\ & + (x_{mn}^{M+} - x_{jmn}^-) / (x_{mn}^{M-} - x_{jmn}^{L+}) - \\ & (x_{mn}^{M-} - x_{jmn}^+) / (x_{mn}^{M+} - x_{jmn}^{L-})\} \} \\ & \leq \alpha_2 \in [0, 1] \quad \forall j, m, n. \end{aligned} \tag{36}$$

Finally, (22)–(34), replacing constraints (27) and (28) by (35) and (36), respectively, represent the deterministic equivalent of the grey fuzzy optimization model (11)–(16) in a multiobjective framework. All  $c_{jl}^-$  and  $c_{jl}^+$  terms in (22)–(36) are expressed in terms of  $x_{jmn}^+$  and  $x_{jmn}^-$ , respectively, using the expressions obtained from the

water-quality transport model [similar to Eq. (18) and (19)], which results in  $x_{jmn}^+$  and  $x_{jmn}^-$  as the only decision variables in the optimization model. Objectives (22) and (23) maximize the upper and lower bound of the goal fulfillment level ( $\lambda^{\pm}$ ), respectively, which ensure the maximum possibility of fulfillment of the goals of PCA and dischargers. Objective (24) minimizes the width of the goal fulfillment level [ $\lambda_w = (\lambda^+ - \lambda^-)$ ]. This objective is included as the reduction of width of the goal fulfillment level implies reduction in system uncertainties and increase in effectiveness of the grey model (Huang et al., 1995). Similarly, a higher flexibility (i.e., higher width of the interval) of the decision variables ( $x_{jmn}^{\pm}$ ) is always desirable, as it allows a wider choice to the decision-makers. But objectives (22)–(24) do not address the maximization of the width of decision variables [i.e.,  $(x_{jmn}^+ - x_{jmn}^-)$ ]. The width of the decision variables is maximized along with the objectives (22)–(24) while solving the deterministic equivalent of the grey fuzzy optimization model [i.e., (22)–(34), replacing constraints (27) and (28) by (35) and (36)] using a fuzzy multiobjective optimization technique (Sakawa et al., 1984). The procedure adopted is discussed in the section, ‘Model Application’. The problem is also solved by using a fuzzy goal programming technique (Sakawa et al., 1987; Pal and Moitra, 2003) to examine the consistency of the solutions. A brief description of these optimization techniques is given in the appendix.

#### 4. Model application

The multiobjective optimization model [(22)–(34)] for water-quality management is applied to a hypothetical river system shown in Fig. 6, and solved by using fuzzy multiobjective optimization and fuzzy goal programming techniques. The river network is discretized into four reaches. Four dischargers are considered as the point sources of pollutants. The only pollutant considered in the system is BOD waste-load due to point sources. The water-quality indicator of interest is the DO-deficit at 18 checkpoints in the river system due to the point sources of BOD. The saturation DO concentration is taken as 10 mg/L for all the reaches. A deterministic streamflow of 7 Mcum/day is considered. Details of the effluent flow and imprecise membership functions are given in Tables 1 and 2, respectively. A set of lower and upper bounds of membership parameters ( $c_l^{D\pm}, c_l^{H\pm}, x_m^{L\pm}$  and  $x_m^{M\pm}$ ) are fixed arbitrarily (Table 2) to obtain a set of optimal solutions for demonstration. The notations of different variables are simplified by retaining only the suffixes  $l$  (checkpoints) and  $m$  (dischargers). Optimization model [(22)–(34), with (35) and (36)] for water-quality management of the hypothetical river system is now presented as follows:

$$\text{Max } \lambda^+, \tag{37}$$

$$\text{Max } \lambda^-, \tag{38}$$

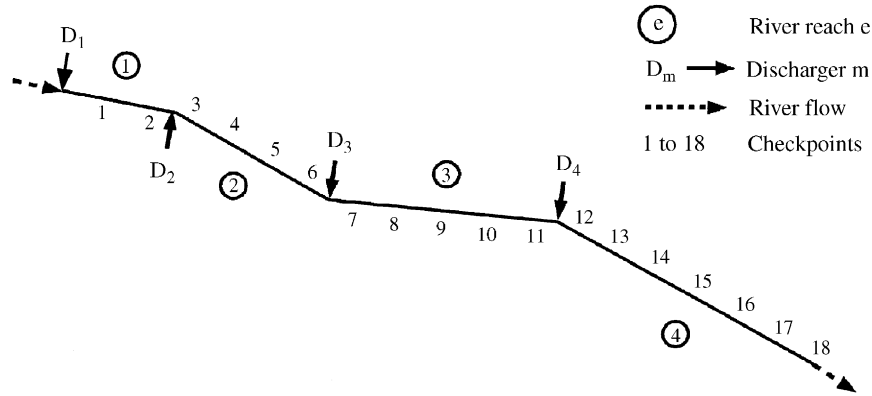


Fig. 6. Hypothetical river system.

Table 1  
Effluent flow data\*

Discharger (1)	Effluent flow rate (10 <sup>4</sup> m <sup>3</sup> /day) (2)	BOD concentration (mg/L) (3)	DO concentration (mg/L) (4)
D <sub>1</sub>	2.134	1250	1.230
D <sub>2</sub>	6.321	1415	2.400
D <sub>3</sub>	7.554	1040	1.700
D <sub>4</sub>	5.180	935	2.160

\*Source of data (Sasikumar, 1998; Mujumdar and Sasikumar, 2002).

$$\text{Min}[(\lambda^+ - \lambda^-)/(\lambda^+ + \lambda^-)], \tag{39}$$

$$\text{Subject to } \mu_{E_i^\pm}^-(c_i^\pm) = [(c_i^{H-} - c_i^+)/ (c_i^{H+} - c_i^{D-})] \geq \lambda^- \quad \forall i, \tag{40}$$

$$\mu_{F_m^\pm}^-(x_m^\pm) = [(x_m^{M-} - x_m^+)/ (x_m^{M+} - x_m^{L-})] \geq \lambda^- \quad \forall m, \tag{41}$$

$$\begin{aligned} & \{ \{ (\lambda^+ + \lambda^-) - (c_i^{H-} - c_i^+) / (c_i^{H+} - c_i^{D-}) \\ & - (c_i^{H+} - c_i^-) / (c_i^{H-} - c_i^{D+}) \} / \{ (\lambda^+ - \lambda^-) \\ & + (c_i^{H+} - c_i^-) / (c_i^{H-} - c_i^{D+}) - (c_i^{H-} - c_i^+) / (c_i^{H+} - c_i^{D-}) \} \} \\ & \leq \alpha_1 \in [0, 1] \quad \forall i, \end{aligned} \tag{42}$$

$$\begin{aligned} & \{ \{ (\lambda^+ + \lambda^-) - (x_m^{M-} - x_m^+) / (x_m^{M+} - x_m^{L-}) \\ & - (x_m^{M+} - x_m^-) / (x_m^{M-} - x_m^{L+}) \} / \{ (\lambda^+ - \lambda^-) \\ & + (x_m^{M+} - x_m^-) / (x_m^{M-} - x_m^{L+}) - (x_m^{M-} - x_m^+) / (x_m^{M+} - x_m^{L-}) \} \} \\ & \leq \alpha_2 \in [0, 1] \quad \forall m, \end{aligned} \tag{43}$$

$$c_i^{D-} \leq c_i^+ \leq c_i^{H+}; c_i^{D-} \leq c_i^- \leq c_i^{H+} \quad \forall i \tag{44}$$

$$x_m^{L-} \leq x_m^+ \leq x_m^{M+}; x_m^{L-} \leq x_m^- \leq x_m^{M+} \quad \forall m, \tag{45}$$

$$c_i^- \leq c_i^+ \quad \forall i, \tag{46}$$

$$x_m^- \leq x_m^+ \quad \forall m, \tag{47}$$

$$0 \leq \lambda^+ \leq 1; \quad 0 \leq \lambda^- \leq 1, \tag{48}$$

$$\lambda^- \leq \lambda^+, \tag{49}$$

where  $\mu_{E_i^\pm}^-(c_i^\pm)$  and  $\mu_{F_m^\pm}^-(x_m^\pm)$  are the imprecise membership functions, defining the grey fuzzy goals of PCA ( $E_i^\pm$ ) and dischargers ( $F_m^\pm$ ). In constraints (42) and (43),  $\alpha_1$  and  $\alpha_2$  are optimistic thresholds assumed equal to zero. The optimistic thresholds  $\alpha_1$  and  $\alpha_2$  may be interpreted as the grades of acceptability of the premises “ $\mu_{E_i^\pm}^-(c_i^\pm)$  is less than  $\lambda^\pm$ ” and “ $\mu_{F_m^\pm}^-(x_m^\pm)$  is less than  $\lambda^\pm$ ”, respectively. Therefore, when  $\alpha_1 = \alpha_2 = 0$ , the mean values of both the intervals become equal to the mean value of  $\lambda^\pm$ , and thus a conservative optimal solution or solution at no optimism level for constraints (12) and (13) is obtained, whose deterministic equivalents are expressed through constraints (40)–(43). Similarly, as the values of  $\alpha_1$  and  $\alpha_2$  increase towards unity a more optimistic optimal solution is obtained. The decision-maker selects the values of  $\alpha_1$  and  $\alpha_2$  equal to zero when the water-quality management issues in the river system are too critical and important, otherwise some optimistic strategy can be considered by choosing values of  $\alpha_1$  and  $\alpha_2$  close to unity. A linear relationship,  $c_i^\pm = f(x_m^\pm)$  [analogous to Eq. (17)], is developed, considering the method proposed by Fujiwara et al. (1987, 1988), and Sasikumar and Mujumdar (1998), based on the Streeter and Phelps equations (Streeter and Phelps, 1925), to express the DO deficit,  $c_i^\pm$  as a function of the levels of fractional removal,  $x_m^\pm$ . All  $c_i^\pm$  terms are expressed in terms of  $x_m^\pm$ , which results in  $x_m^\pm$  as the only decision variables in the optimization model (37)–(49). For further verification it should be noted that a detailed description of the transport model used in the present hypothetical case study and the linear expressions [same as Eq. (17)] relating DO-deficit ( $c_i^\pm$ ) and BOD removal ( $x_m^\pm$ ) are given in Sasikumar (1998) and Mujumdar and Sasikumar (2002).

In the modified GFWLAM, objectives (37) and (38) are maximized, and objective (39) is minimized individually in three separate subproblems along with constraints (40)–(49) to obtain the maximum and minimum possible values of  $\lambda^+, \lambda^-$  and  $[(\lambda^+ - \lambda^-)/(\lambda^+ + \lambda^-)]$  (i. e., ideal points  $Q_1, Q_2, Q_3$  and worst possible values  $q_1, q_2, q_3$  of the fuzzy multiobjective optimization technique as mentioned in the appendix), respectively. As discussed earlier, another objective of the river water-quality management is to

Table 2  
Details of imprecise membership functions\*

River Reach (1)	Check points (2)	$c^D_l$ (mg/L) (3)		$c^H_l$ (mg/L) (4)		$x^L_m$ (5)		$x^M_m$ (6)	
		–	+	–	+	–	+	–	+
R <sub>1</sub>	1–2		(0.00)		(3.00)		(0.30)		(0.85)
		0.00	0.00	2.70	3.20	0.25	0.35	0.80	0.90
R <sub>2</sub>	3–6		(0.10)		(3.00)		(0.30)		(0.85)
		0.00	0.10	2.70	3.20	0.25	0.35	0.80	0.90
R <sub>3</sub>	7–11		(0.20)		(3.50)		(0.35)		(0.85)
		0.17	0.22	3.30	3.70	0.30	0.40	0.80	0.90
R <sub>4</sub>	12–18		(0.20)		(3.50)		(0.35)		(0.85)
		0.17	0.22	3.30	3.70	0.30	0.40	0.80	0.90

( ) : deterministic case, ‘–’ : lower bound, ‘+’ : upper bound, ‘\*’: source of data (Sasikumar, 1998; Mujumdar and Sasikumar, 2002).

permit more flexibility (i.e., more width of the interval) in the optimal fractional removal level ( $\hat{x}_m^\pm$ ). Thus maximization of the grey degree of  $x_m^\pm$  [i.e.,  $Gd(x_m^\pm) = (x_m^+ - x_m^-)/0.5(x_m^+ + x_m^-)$ ] is considered as another objective along with objectives (37)–(39). The maximum and minimum values of the grey degree of  $x_m^\pm$  are determined from the three subproblems, which are considered as the ideal (i.e.,  $Q_{4m}$ ) and worst (i.e.,  $q_{4m}$ ) values, where subscript “ $m$ ” denotes the discharger. To represent the objective of maximizing the value of  $Gd(x_m^\pm)$  a non-decreasing linear membership function, taking  $Q_{4m}$  and  $q_{4m}$  as the reference point and reservation level, respectively, is considered as

$$\mu_{4m}[Gd(x_m^\pm)] = \begin{cases} 1 & Gd(x_m^\pm) > Q_{4m} \\ [Gd(x_m^\pm) - q_{4m}] / [Q_{4m} - q_{4m}] & q_{4m} \leq Gd(x_m^\pm) \leq Q_{4m} \\ 0 & Gd(x_m^\pm) < q_{4m} \end{cases} \quad \forall m, \quad (50)$$

Similarly, maximization of objectives (37) and (38) is achieved by using non-decreasing linear membership functions. Objective (39) is a minimization type, and is addressed in the model by the non-increasing linear membership function, given by

$$\mu_3[(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)] = \begin{cases} 1 & [(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)] < q_3, \\ [Q_3 - \{(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)\}] / [Q_3 - q_3]; & q_3 \leq [(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)] \leq Q_3, \\ 0 & [(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)] > Q_3, \end{cases} \quad (51)$$

where  $Q_3$  and  $q_3$  are the possible individual maximum and minimum values, respectively, of the third objective (39). The fuzzy decision concept with the minimum operator (i.e., “logical and”) is applied to aggregate the membership functions of objectives (37)–(39) along with four other objectives for maximizing  $Gd(x_m^\pm)$  for the four dischargers. The solution of the resulting problem gives a Pareto-

optimal solution of the fuzzy multiobjective optimization problem for fractional removal levels of BOD (i.e.,  $\{\hat{X}^\pm = \{\hat{x}_m^\pm\}\}$ ).

The water-quality management problem with the same hypothetical river system is solved by using a fuzzy goal programming technique [given in the appendix], which confirms the consistency of solutions.

The results obtained by applying the modified GFWLAM to the hypothetical river system are described in the following section.

5. Results and discussion

The results obtained from the modified GFWLAM are summarized in Tables 3–5, and facilitate a comparison between the deterministic case, where the membership parameters are deterministic numbers, and the grey uncertain case, where the membership parameters are uncertain.

Table 3 shows the optimal fractional removal levels of the pollutants by different dischargers (i.e.,  $\hat{X}^\pm = \{\hat{x}_m^\pm\}$ ) for the grey uncertain case (with average grey degree of 31.30%, as given in Table 2 for  $c_{jl}^{D\pm}, c_{jl}^{H\pm}, x_{mn}^{L\pm}$ , and  $x_{mn}^{M\pm}$ ) obtained from the modified GFWLAM using the fuzzy multiobjective optimization technique. In Table 3, columns 1–3 show the results obtained from subproblems 1–3, i.e., maximization of  $\lambda^+$ , maximization of  $\lambda^-$  and minimization

of the width of  $\lambda^\pm$  (equivalent to minimizing  $[(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)]$ ), respectively. The payoff matrix of the fuzzy multiobjective optimization is formed, picking the minimum and maximum values of  $\lambda^+$  [objective (37)],  $\lambda^-$  [objective (38)],  $(\lambda^+ - \lambda^-) / (\lambda^+ + \lambda^-)$  [objective (39)] and  $Gd(x_1^\pm), \dots, Gd(x_4^\pm)$  from columns 1–3; rows 6, 5, 8, and 9–12, respectively, as the procedure mentioned in the

Table 3  
Details of fractional removal levels using fuzzy multiobjective optimization technique (when  $\alpha_1$  and  $\alpha_2 = 0$ )

Row No.		Subproblem-1 (Max. $\lambda^+$ ) (1)	Subproblem-2 (Max. $\lambda^-$ ) (2)	Subproblem-3 [Min. ( $\lambda^+ - \lambda^-$ )/( $\lambda^+ + \lambda^-$ )] (3)	Modified GFWLAM (4)	Deterministic model (FWLAM) (5)	GFWLAM (6)
1	$x_1^\pm$	[0.4845, 0.7751]	[0.5955, 0.5964]	[0.6060, 0.6531]	[0.5198, 0.6693]	[0.6150, 0.6150]	[0.5970, 0.6410]
2	$x_2^\pm$	[0.4682, 0.7987]	[0.5967, 0.5970]	[0.4301, 0.6578]	[0.5089, 0.6711]	[0.6150, 0.6150]	[0.5970, 0.6410]
3	$x_3^\pm$	[0.5190, 0.7951]	[0.6124, 0.6126]	[0.5517, 0.5679]	[0.5446, 0.6810]	[0.6360, 0.6360]	[0.6120, 0.6700]
4	$x_4^\pm$	[0.5172, 0.7970]	[0.6121, 0.6127]	[0.6324, 0.6517]	[0.5402, 0.6786]	[0.6360, 0.6360]	[0.6120, 0.6700]
5	$\lambda^-$	0.0006	0.3121	0.1052	0.1966	0.4277	0.3126
6	$\lambda^+$	0.9592	0.3618	0.1052	0.5528	0.4277	0.5745
7	Gd( $\lambda^\pm$ )	—	—	—	0.9506	0.0000	0.5903
8	( $\lambda^+ - \lambda^-$ )/ ( $\lambda^+ + \lambda^-$ )	0.9987	0.0737	0.0000	—	—	—
9	Gd( $x_1^\pm$ )	0.4615	0.0015	0.0748	0.2514	0.0000	0.0711
10	Gd( $x_2^\pm$ )	0.5217	0.0004	0.4186	0.2750	0.0000	0.0711
11	Gd( $x_3^\pm$ )	0.4202	0.0003	0.0288	0.2225	0.0000	0.0905
12	Gd( $x_4^\pm$ )	0.4259	0.0010	0.0300	0.2272	0.0000	0.0905
13	Avg. Gd( $X^\pm$ )	—	—	—	0.2440	0.0000	0.0812

Table 4  
Details of DO-deficit at the checkpoints

Checkpoints (1)	DO-deficit (mg/L)		Gd( $c_1^\pm$ ) from modified GFWLAM (4)	Gd( $c_1^\pm$ ) from GFWLAM (5)
	Lower bound ( $c_1^\pm$ ) (2)	Upper bound ( $c_1^\pm$ ) (3)		
1	0.0543	0.0688	0.2355	0.0678
2	0.0796	0.1050	0.2756	0.0909
3	0.1466	0.1718	0.1584	0.0516
4	0.2611	0.3449	0.2763	0.0797
5	0.3653	0.5022	0.3157	0.0952
6	0.4593	0.6443	0.3353	0.1002
7	0.5439	0.7277	0.2892	0.0861
8	0.7020	0.9631	0.3136	0.1006
9	0.8452	1.1762	0.3276	0.1084
10	0.9750	1.3695	0.3366	0.1137
11	1.0916	1.5435	0.3430	0.1167
12	1.1380	1.5852	0.3285	0.1118
13	1.2863	1.8053	0.3358	0.1167
14	1.4191	2.0027	0.3411	0.1206
15	1.5386	2.1805	0.3452	0.1232
16	1.6452	2.3392	0.3483	0.1253
17	1.7401	2.4806	0.3508	0.1270
18	1.8236	2.6051	0.3529	0.1284

appendix. For example, columns 1–3, row 5 show the values of  $\lambda^-$  obtained from sub problems 1–3, respectively. The maximum value of  $\lambda^-$  (i.e.,  $Q_2 = 0.3121$ ) is obtained from sub problem 2 and the minimum value (i.e.,  $q_2 = 0.0006$ ) is obtained from sub problem 1. The fuzzy set requirements of all the objectives are quantified by eliciting linear membership functions considering the elements in the payoff matrix as membership parameters as discussed in the appendix-fuzzy multiobjective optimization”. Column 4, rows 1–6 show the optimal fractional

removal levels of the pollutants by different dischargers (i.e.,  $\hat{X}^\pm = \{\hat{x}_m^\pm\}$ ) and corresponding  $\hat{\lambda}^\pm$  values. To evaluate the quality of input or output uncertain information, a measure of “Grey degree” is used [Eq. (2)]. The decrease in the grey degree of the optimal value of the objective function indicates increasing effectiveness of the grey model and decreasing system uncertainty. In column 4, row 7 the grey degree of  $\hat{\lambda}^\pm$  (i.e., 0.9506) shows the quality of the optimal solutions ( $\hat{\lambda}^\pm = [0.1966, 0.5528]$ ; where the WMV ( $\lambda_m$ ) = 0.3747, width ( $\lambda_w$ ) = 0.3562 and grey degree of  $\hat{\lambda}^\pm = (0.3562/0.3747) = 0.9506$ , i.e., 95.06%). In column 5 the deterministic case is presented, for which the average value of the grey degree of input parameters is zero. The resulting values of  $\hat{\lambda}^\pm$  and  $\hat{X}^\pm$  in this case show the deterministic solutions, indicated by the zero values of grey degrees in rows 7 and 9–12. The value of  $\hat{\lambda}^\pm$  in the deterministic case, [0.4277, 0.4277] lies in the closed interval as obtained in the grey uncertain case, i.e., [0.1966, 0.5528], as the values of all membership parameters (i.e.,  $c_{jl}^{D\pm}, c_{jl}^{H\pm}, x_{mm}^{L\pm}$ , and  $x_{mm}^{M\pm}$ ) for the deterministic case lie within the ranges of intervals given in Table 2 for the grey uncertain case. Next in column 6, the grey uncertain case is presented as obtained from GFWLAM (Karmakar and Mujumdar, 2004a), for which the average value of the grey degree of input parameters (i.e., 31.304%) is the same as in the modified GFWLAM given in Table 2. Comparing the results shown in columns 4 and 6, rows 1–4, it can be concluded that the widths of the optimal fractional removal levels of BOD ( $\hat{X}^\pm$ ) are more than those of GFWLAM, which allows a wider choice to the decision-maker in decision-making. The same observation can be made more clearly by comparing the Gd( $\hat{X}^\pm$ ) values shown in rows 9–12 of columns 4 and 6. The value of Gd( $\hat{\lambda}^\pm$ ) (given in row 7 of column 4) is, however, more than the value obtained from GFWLAM (row 7, column 6), which

Table 5  
Comparison of solutions obtained by applying fuzzy multiobjective and fuzzy goal programming techniques

Row no.	Fuzzy multiobjective optimization model			Fuzzy goal programming (minimax approach)		
	$\alpha_1$ and $\alpha_2 = 0$ (1)	$\alpha_1$ and $\alpha_2 = 0.25$ (2)	$\alpha_1$ and $\alpha_2 = 0.5$ (3)	$\alpha_1$ and $\alpha_2 = 0$ (4)	$\alpha_1$ and $\alpha_2 = 0.25$ (5)	$\alpha_1$ and $\alpha_2 = 0.5$ (6)
1 $\lambda^\pm$	[0.1966, 0.5528]	[0.2206, 0.5741]	[0.2530, 0.6054]	[0.2066, 0.5064]	[0.2259, 0.6044]	[0.2612, 0.6153]
2 Gd $[\lambda^\pm]$	0.9506	0.8896	0.8211	0.8411	0.9117	0.8081
3 $x_1^\pm$	[0.5198, 0.6693]	[0.5945, 0.6561]	[0.3867, 0.6353]	[0.5268, 0.6652]	[0.5594, 0.6482]	[0.6278, 0.6299]
4 $x_2^\pm$	[0.5089, 0.6711]	[0.4889, 0.6550]	[0.6030, 0.6353]	[0.5302, 0.6656]	[0.5383, 0.6531]	[0.5753, 0.6302]
5 $x_3^\pm$	[0.5446, 0.6810]	[0.5890, 0.6672]	[0.5715, 0.6481]	[0.5367, 0.6757]	[0.5575, 0.6634]	[0.5390, 0.6431]
6 $x_4^\pm$	[0.5402, 0.6786]	[0.5441, 0.6663]	[0.5973, 0.6460]	[0.5357, 0.6747]	[0.5291, 0.6618]	[0.5662, 0.6431]
7 Avg. Gd $[X^\pm]$	0.2440	0.1789	0.1856	0.2294	0.1840	0.0994
8 Avg. Gd $[C^\pm]$	0.3116	0.2441	0.1969	0.2808	0.2195	0.1055

indicates more uncertainty in the system compared to that resulting from the GFWLAM solution. The result obtained from the modified GFWLAM is more useful to the decision-makers as it gives a wider range in the interval valued optimal fractional removal levels of the pollutants than GFWLAM, although at the cost of increasing uncertainty.

Table 4 gives the DO-deficit values for all 18 checkpoints resulting from the fractional removal levels of BOD listed in Table 3 (column 4, rows 1–4). The relationships given in Eq. (18) and (19) are used to obtain the lower and upper bounds of the water-quality indicators ( $c_l^-$  and  $c_l^+$ ) from the upper and lower bounds of the fractional removal level of pollutant ( $x_m^+$  and  $x_m^-$ ), respectively. Columns 4 and 5 of the Table 4 compare the grey degrees of concentrations of DO-deficit obtained from the modified GFWLAM and GFWLAM.

In Table 5, columns 1–3 show the optimal solutions obtained for  $\alpha_1 = 0, \alpha_2 = 0$ ;  $\alpha_1 = 0.25, \alpha_2 = 0.25$ ; and  $\alpha_1 = 0.50, \alpha_2 = 0.50$ ; respectively. As the values of  $\alpha_1$  and  $\alpha_2$  increase to unity, constraints (12) and (13), whose deterministic equivalents are expressed through constraints (40)–(43), approach their limiting values and result in more optimistic optimal solutions. Comparing values of  $\hat{\lambda}^\pm$  in row 1 of columns 1–3, it is seen that as the values of  $\alpha_1$  and  $\alpha_2$  increase, the values of both the bounds of  $\lambda^\pm$  increase, indicating an increased level of optimistic optimal solutions. The optimal ranges of fractional removal levels of BOD (i.e.,  $\hat{X}^\pm$ ) for  $\alpha_1 = 0, \alpha_2 = 0$ ;  $\alpha_1 = 0.25, \alpha_2 = 0.25$ ; and  $\alpha_1 = 0.50, \alpha_2 = 0.50$  are given in rows 3–6 of columns 1–3, respectively. Columns 4–6 show the solutions obtained from the fuzzy goal programming technique (minimax approach), which confirms the consistency of the results given in columns 1–3. Similarly, comparing values of  $\hat{\lambda}^\pm$  in row 1 of columns 4–6, it is seen that as the values of  $\alpha_1$  and  $\alpha_2$  increase the values of both the bounds of  $\lambda^\pm$  increase, indicating the increased level of optimistic optimal solutions. The optimal ranges of fractional removal levels of BOD (i.e.,  $\hat{X}^\pm$ ) obtained by applying the fuzzy goal programming technique for  $\alpha_1 = 0, \alpha_2 = 0$ ;  $\alpha_1 = 0.25,$

$\alpha_2 = 0.25$ ; and  $\alpha_1 = 0.50, \alpha_2 = 0.50$  are given in rows 3–6 of column 4–6, respectively.

Rows 1–4 of column 5 in Table 3 present the optimal fractional removal levels of BOD for dischargers 1–4, i.e.,  $x_1 \approx 0.62, x_2 \approx 0.62, x_3 \approx 0.64,$  and  $x_4 \approx 0.64$ , fixed by the decision-maker as obtained from the deterministic water-quality management model, FWLAM, developed earlier by Sasikumar and Mujumdar (1998), which gives the optimal goal fulfillment level as  $\lambda = 0.4277$  (rows 5–6 in column 5). As the values of fractional removal levels are crisp or deterministic (i.e., white numbers), the decision-maker would not get any flexibility on adjusting these values in the final decision scheme of pollutant removal considering the technical and economic feasibility of these pollutant treatment levels. In column 4, the optimal values of the lower and upper bounds of  $\lambda^\pm$  (rows 5–6) correspond to different distributions of optimal fractional removal levels of BOD (rows 1–4). Therefore, the decision-maker gets a range of optimal solutions in the flexible decision space. For example, the interval valued optimal fractional removal of BOD for discharger 1 ( $\hat{x}_1^\pm$ ) is [0.5198, 0.6693]  $\approx$  [0.52, 0.67], which indicates that any value of BOD removal between 52% and 67% may be fixed for discharger 1, which ensures a  $\lambda^\pm$ -value within [0.1966, 0.5528]. Similarly, for other dischargers, i.e.,  $\hat{x}_2^\pm \approx$  [0.51, 0.67];  $\hat{x}_3^\pm \approx$  [0.54, 0.68]; and  $\hat{x}_4^\pm \approx$  [0.54, 0.68], wider ranges of optimal fractional removal levels are obtained than those given by the previously developed GFWLAM, presented in rows 1–4, column 6 (i.e.,  $\hat{x}_1^\pm \approx$  [0.60, 0.64];  $\hat{x}_2^\pm \approx$  [0.60, 0.64];  $\hat{x}_3^\pm \approx$  [0.61, 0.67]; and  $\hat{x}_4^\pm \approx$  [0.61, 0.67]). This indicates that the decision-maker gets a more flexible optimal solution space within which the final decision for fractional removal can be fixed. Selection of particular values from the optimal interval valued fractional removal levels of BOD ( $\hat{x}_m^\pm$ ) for implementation in the field ensures a  $\lambda^\pm$ -value within [0.1966, 0.5528]. From the model formulation it can be concluded that planning for an upper bound of  $\hat{\lambda}^\pm$ , i.e.,  $\hat{\lambda}^+$  represents an “optimistic strategy”; whereas, planning for a lower bound of  $\hat{\lambda}^\pm$ , i.e.,  $\hat{\lambda}^-$  represents a “conservative strategy”.

## 6. Concluding remarks

This paper presents a modified grey fuzzy waste load allocation model (modified GFWLAM) for water-quality management of a river system. A previously developed fuzzy waste load allocation model (FWLAM) is extended to address uncertainty involved in fixing the membership functions for conflicting goals of the pollution control agency (PCA) and dischargers considering the membership parameters as interval grey numbers. The paper demonstrates modelling aspects of uncertain membership functions and shows the usefulness of solutions with a simplified hypothetical river system in which only the BOD point sources are considered as pollutants and the DO-deficit at a finite number of checkpoints is used as the water-quality indicator. In a majority of waste load allocation models, the measure of performance is related to the overall cost of pollution control, including the waste treatment cost. Incorporating the cost functions directly into the optimization models poses difficulties due to uncertainty, lack of adequate data, and non-linearity associated with the cost functions. The FWLAM has the advantage that the cost functions are eliminated while the fuzzy goals of the dischargers are incorporated through membership functions.

The membership functions themselves are subjective statements of the perceptions of the decision-makers. The values of the parameters in the membership functions depend on the responses of a decision-maker to water-quality and fractional removal levels. In practical situations, different water-quality standards for surface water intakes are used for different uses, which results in an uncertainty in the membership parameters for the goals of PCA. To address uncertainty in these bounds, the membership functions themselves are treated as fuzzy in the fuzzy optimization model. Consideration of imprecise membership parameters in the fuzzy optimization model imparts flexibility in decision-making as the fractional removal levels are obtained as interval grey numbers. The modified GFWLAM gives a new methodology to get a satisfactory deterministic equivalent of a grey fuzzy optimization problem, using the concept of acceptability index for a meaningful ranking between two partially or fully overlapping intervals. Although the solutions obtained from the present model provide more flexibility than those given by the previously developed GFWLAM (Karmakar and Mujumdar, 2004a), the application of the modified GFWLAM is limited to grey fuzzy goals expressed by linear imprecise membership functions, whereas GFWLAM has the capability to solve the grey fuzzy optimization model with monotonic non-linear imprecise membership functions with  $\alpha_{jl}$  and  $\beta_{jmn} \neq 1$ , in Eqs. (8) and (9).

The model will be more useful with more realistic solutions if it is applied to a critical case study where all dischargers play a critical role in influencing the value of the model performance measure,  $\lambda^\pm$ . A limitation of the

model presented in this paper, however, is that in situations where multiple solutions exist, the optimal values obtained from the deterministic model may not lie within the optimal interval values obtained from the modified GFWLAM, for some solutions. The present model is not limited in application to any particular pollutant or water-quality indicator in the river system. Given appropriate transfer functions for spatial and temporal distribution of the pollutants in the water body, it can be used for water-quality management of any general water system. The methodology described in the formulation of the modified GFWLAM to get a satisfactory deterministic equivalent of a grey fuzzy optimization model is valid only for linear imprecise membership functions. Addressing the uncertainty in membership parameters with non-linear imprecise membership functions should provide a new direction for future research. A new grey fuzzy stochastic programming technique (Huang et al., 1995; Huang and Loucks, 2000) can be developed to get the optimal values for fractional removal in the form of interval grey numbers addressing also the uncertainty due to randomness in variables such as streamflow, effluent flow, reaeration coefficient, deoxygenation coefficient, and temperature.

## Appendix

### A.1. Acceptability index

Sengupta et al. (2001) introduced an extended order relation between the intervals  $a^\pm = [a^-, a^+]$  and  $b^\pm = [b^-, b^+]$  on the real line  $\mathcal{R}$ . A premise  $a^\pm (<) b^\pm$  is formed to imply that  $a^\pm$  is inferior to  $b^\pm$ . Here, the term “inferior to” (“superior to”) is analogous to “less than” (“greater than”). Let  $I$  be the set of all closed intervals on the real line  $\mathcal{R}$ . The acceptability index ( $\mathcal{A}$ ) is defined as  $\mathcal{A} : I \times I \rightarrow [0, \infty)$  such that

$$\mathcal{A}[a^\pm (<) b^\pm] = [m(b^\pm) - m(a^\pm)] / [w(b^\pm) + w(a^\pm)], \quad (\text{A.1})$$

where  $[w(b^\pm) + w(a^\pm)] \neq 0$ ;  $w(a^\pm)$  is the half-width of  $a^\pm = 1/2(a^+ - a^-)$ ;  $m(a^\pm)$  is the mean of  $a^\pm = 1/2(a^- + a^+)$  which is same as WMV in the grey systems (Huang et al., 1995). Notations are similarly defined for the interval grey number  $b^\pm$ . The grade of acceptability of  $a^\pm (<) b^\pm$  may be classified and interpreted further on the basis of the comparative position of the mean and half-width of interval  $b^\pm$  with respect to those of interval  $a^\pm$  as follows:

$$\mathcal{A}[a^\pm (<) b^\pm] = \begin{cases} = 0 & \text{if } m(a^\pm) = m(b^\pm), \\ > 0, < 1 & \text{if } m(a^\pm) < m(b^\pm) \text{ and } a^+ > b^-, \\ \geq 1 & \text{if } m(a^\pm) < m(b^\pm) \text{ and } a^+ \leq b^-. \end{cases} \quad (\text{A.2})$$

If  $\mathcal{A}[a^\pm (<) b^\pm] = 0$ , then the premise “ $a^\pm$  is inferior to  $b^\pm$ ” is not accepted. If  $0 < \mathcal{A}[a^\pm (<) b^\pm] < 1$ , then the premise  $[a^\pm (<) b^\pm]$  is accepted with different grades ranging from zero to one. If  $\mathcal{A}[a^\pm (<) b^\pm] \geq 1$ , the interpreter is absolutely satisfied with the premise  $[a^\pm (<) b^\pm]$ .

Similar to the grey degree of grey systems theory (Huang et al., 1995) the acceptability index is a means of quantitatively comparing the uncertainty involved in the interval grey numbers. Let us consider an interval  $d^\pm$  which is inferior to the equi-centered intervals  $b_1^\pm$  and  $b_2^\pm$  [i.e.,  $m(b_1^\pm) = m(b_2^\pm)$ ]. Then, as compared to  $d^\pm$ , the grade of acceptability of superiority of the less uncertain interval is higher than the grade of acceptability of superiority of the more uncertain interval. Symbolically, If  $A [d^\pm(<)b_1^\pm] > 0$ ,  $A [d^\pm(<)b_2^\pm] > 0$  and  $A [b_1^\pm(<)b_2^\pm] = 0$ , but  $b_1^\pm \neq b_2^\pm$  as  $w(b_1^\pm) \neq w(b_2^\pm)$ ; then,

$$A [d^\pm(<)b_1^\pm] > A [d^\pm(<)b_2^\pm] \text{ iff } w(b_1^\pm) < w(b_2^\pm), \tag{A.3}$$

and

$$A [d^\pm(<)b_1^\pm] < A [d^\pm(<)b_2^\pm] \text{ iff } w(b_1^\pm) > w(b_2^\pm). \tag{A.4}$$

Let us consider an interval inequality relation  $a^\pm x \geq b^\pm$ , where  $x$  is a deterministic variable. Keeping in view the properties of acceptability index, a satisfactory deterministic equivalent form of the interval inequality relation  $a^\pm x \geq b^\pm$ , is proposed as

$$a^\pm x \geq b^\pm \Rightarrow \{a^- x \geq b^- \text{ and } A [a^\pm x(<)b^\pm] \leq \alpha \in [0, 1]\}, \tag{A.5}$$

where  $\alpha$  is interpreted as an optimistic threshold assumed and fixed by the decision-maker. At  $\alpha = 0$ , the solutions at no optimism level are obtained. Therefore, the position (i.e., position of mean) of an interval compared to that of another reference interval results in whether the former is superior or inferior to the latter. On the other hand, the width of a superior (inferior) interval compared to that of the reference interval specifies the grade to which the decision-maker is satisfied with the superiority (inferiority) of the former compared to the latter. Similarly, a satisfactory deterministic equivalent form of the interval inequality relation  $a^\pm x \leq b^\pm$  is proposed as

$$a^\pm x \leq b^\pm \Rightarrow \{a^+ x \leq b^+ \text{ and } A [a^\pm x(>)b^\pm] \leq \alpha \in [0, 1]\}, \tag{A.6}$$

where the symbol ( $>$ ) indicates “superior to”, which is analogous to “greater than”.

### A.2. Fuzzy multiobjective optimization

The fuzzy multiobjective optimization technique was first introduced by Sakawa (1984). In general the multi-objective programming problem is represented as the following vector maximization problem (Sakawa, 1998):

$$\max_{x \in X} \{f_1(x), \dots, f_k(x)\}^T \tag{A.7}$$

$$\text{Subject to } g_h(x) \leq 0; x \in X | x \in R^n; h = 1, \dots, s, \tag{A.8}$$

where  $f_1(x), \dots, f_k(x)$  are  $k$  fuzzy objective functions of the decision vector  $x$ ;  $g_1(x), \dots, g_s(x)$  are  $s$  inequality constraints; and  $X$  is the feasible set of constrained decisions and a subset of decision space  $R^n$ . Here, for simplicity, assume that all  $f_i(x)$ ,  $i = 1, \dots, k$ , are convex and differenti-

able, and the constrained set  $X$  is convex and compact. Fundamental to the multiobjective programming problems is the Pareto-optimal or non-inferior or efficient solution.  $x^*$  is said to be a Pareto-optimal solution if and only if there does not exist another  $x \in X$ , such that,  $f_i(x) \geq f_i(x^*)$  for all  $i$  and  $f_i(x) \neq f_i(x^*)$  for at least one  $i$ .

The fuzzy set requirements in the objective functions can be quantified by defining membership functions  $\mu_i[f_i(x)]$  for all  $i = 1, \dots, k$ . For example, in a maximization problem, a strictly monotonic non-decreasing function with respect to  $f_i(x)$  is assumed as follows

$$\mu_i[f_i(x)] = \begin{cases} 1 & f_1(x) > Q_i, \\ d_i(x) & q_i \leq f_1(x) \leq Q_i, \\ 0 & f_1(x) < q_i, \end{cases} \tag{A.9}$$

where  $Q_i$  and  $q_i$  represent the values of  $f_i(x)$  such that the grade of the membership function  $\mu_i[f_i(x)]$  is 1 (maximum) and 0 (minimum), respectively. The intermediate function values are expressed by a strictly monotonic non-decreasing function  $d_i(x)$  with respect to  $f_i(x)$ . Sakawa (1984) suggested calculation of the individual maximum ( $Q_i$ ) and minimum ( $q_i$ ) of each objective function,  $f_i(x)$ , under the given constraints by solving  $k$  number of sub-problems. Let  $x^1$  be a solution, then,  $Q_i = f_i(x^1) | \text{Max}\{\mu_i[f_i(x)] | x \in X\}$ . Similarly,  $q_i = f_i(x^i) | \text{Min}\{\mu_i[f_i(x)] | x \in X\}$ . The vector  $Q = (Q_1, Q_2, \dots, Q_k)$  is called the “ideal point” and  $q = (q_1, q_2, \dots, q_k)$  is called the “worst possible value”. It is clear that the inequalities  $q_i \leq f_i(x) \leq Q_i$  hold for each  $x$  in  $X$ . The “payoff matrix” of a  $k$ -objective problem is defined as

$$\begin{matrix} & x^1 & x^2 & \dots & \dots & x^k \\ \begin{matrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_k \end{matrix} & \begin{bmatrix} Q_1 & f_1(x^2) & \dots & \dots & f_1(x^k) \\ f_2(x^1) & Q_2 & \dots & \dots & f_2(x^k) \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ f_k(x^1) & f_k(x^2) & \dots & \dots & Q_k \end{bmatrix} \end{matrix} \tag{A.10}$$

The intermediate function values are expressed by a strictly monotonic non-decreasing function  $d_i(x)$  with respect to  $f_i(x)$ .  $d_i(x)$  may be linear, exponential, hyperbolic, hyperbolic inverse, and piecewise linear. For example, if  $d_i(x)$  is considered as a linear non-decreasing function, then

$$\mu_i[f_i(x)] = d_i(x) = [f_i(x) - q_i] / [Q_i - q_i]. \tag{A.11}$$

Having determined the membership functions for each of the objective functions, Sakawa (1984) suggested to use the fuzzy decision concept (Bellman and Zadeh, 1970) to obtain the aggregated decision considering all the objectives:

$$\text{Min}_{i=1, \dots, k} (\mu_1[f_1(x)], \mu_2[f_2(x)], \dots, \mu_k[f_k(x)]), \tag{A.12}$$

where  $\mu_1[f_1(x)], \dots, \mu_k[f_k(x)]$  are  $k$  membership functions for fuzzy objectives 1 to  $k$ . The resulting problem to be

solved is equivalent to solving the following problem:

$$\text{Max } \Lambda, \quad (\text{A.13})$$

$$\text{Subject to } \mu_i[f_i(x)] \geq \Lambda, \quad \forall i = 1, \dots, k, \quad (\text{A.14})$$

$$g_h(x) \leq 0, \quad \forall h = 1, \dots, s, \quad (\text{A.15})$$

$$x \in X | x \in R^n, \quad (\text{A.16})$$

where capital lambda ( $\Lambda \in [0, 1]$ ) is the degree of goal fulfillment level; and  $g_1(x), \dots, g_h(x), \dots, g_s(x)$  are  $s$  inequality constraints. As the value of the  $\Lambda$  increases, the objectives  $f_1(x), \dots, f_k(x)$  are fulfilled more. The solution  $x^*$  is a unique solution to the problem given in (A.13)–(A.16), and thus  $x^*$  is a Pareto-optimal solution to the multiobjective optimization problem.

### A.3. Fuzzy goal programming

The fuzzy goal programming technique for multiobjective programming problems presented by Pal and Moitra (2003) is extended by applying the minimax approach proposed by Sakawa et al. (1987). The multiobjective optimization problem presented in the previous subsection may be formulated as a fuzzy goal programming problem and written as:

$$\text{Min}[\text{Max}_{x \in X} \sum_{i=1}^k w_i^- d_i^-], \quad (\text{A.17})$$

$$\text{Subject to } \mu_i[f_i(x)] + d_i^- - d_i^+ = 1, \quad \forall i = 1, \dots, k, \quad (\text{A.18})$$

$$d_i^-, d_i^+ \geq 0 \text{ with } d_i^- d_i^+ = 0, \quad \forall i = 1, \dots, k, \quad (\text{A.19})$$

$$g_h(x) \leq 0, \quad \forall h = 1, \dots, s, \quad (\text{A.20})$$

$$x \in X | x \in R^n, \quad (\text{A.21})$$

where  $d_i^- (\geq 0)$  and  $d_i^+ (\geq 0)$  represent the under- and over-deviational variables, respectively, from the aspired levels of the respective fuzzy goals and where the values of  $w_i^-$  are determined as (Pal and Moitra, 2003)

$$w_i^- = 1/(Q_i - q_i) \forall i = 1, \dots, k. \quad (\text{A.22})$$

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