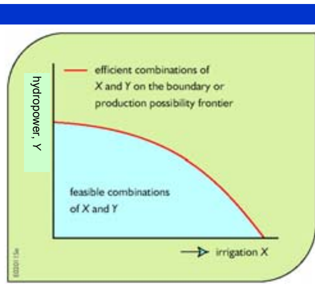
Multi Objective Optimization

Multiple Objectives in Water Resources

- Maximizing aggregated net benefits is a common objective
- Other objectives include
  - Water Quality, Regional Development, Resource Utilization, Social issues
- Conflicting objectives
  - Irrigation, Hydropower, Recreation
- Multi objective optimization
  - Trade-off analysis

Multi Objective Analysis



Multi Objective Analysis

- Methods of multi-criteria or multi-objective analyses are not designed to identify the best solution, but only to provide information on the tradeoffs between given sets of quantitative performance criteria.
- **Dominance**
  - A plan X dominates all others if it results in an equal or superior value for all objectives, and at least one objective value is strictly superior to those of each other plan.
- **Noninferior solution**
  - A noninferior solution is one in which no increase in any objective is possible without a simultaneous decrease in atleast one of the other objectives

Pareto Optimal Solutions

- Non-inferior, efficient, or non-dminated solutions are often called Pareto optimal because they satisfy the conditions proposed by Italian economist Pareto, that
  - to improve the value of any single objective, one should have to accept a diminishment of at least one other objective.

Multi Objective Analysis

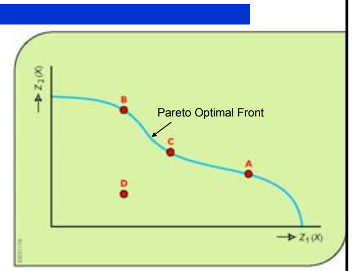
Four discrete plans along with a continuous efficiency frontier associated with two objectives, Z_1 and Z_2.

A pair-wise comparison of plans or objectives may not identify all the non-dominated plans.

All objectives should be considered before declaring a plan inferior.

Plans A, B and C are non-inferior solutions

Plan D is inferior solution



Multi Objective Analysis

- Vector Maximization
  - Let X be a vector of decision variables, $X=\{x_1, x_2, x_3, \ldots, x_n\}$
  - $Z_j(X)$ denote p objectives, j = 1, 2, 3, …, p, each of which is to be maximized.
  - **Multi Objective Optimization Problem**
    Maximize $[Z_1(X), Z_2(X), \ldots, Z_p(X)]$
    Subjected to $g_i(X)=b_i$ for i=1,2,….,m constraints.

Multi Objective Analysis

- Essential steps in Multi Objective Analysis
  - **Plan Formulation**
    - Aimed at generating the noninferior set of solutions (or set of technologically efficient solutions)
  - **Plan Selection**
    - Selecting the best compromise solution
    - Methods
      - **Weighting Method**
      - **Constraint Method**

Weighting Method

In the Weighting Method, the objective function (in the vector form) is converted to a scalar by expressing it as weighted sum of the various objectives by associating relative weight to each objective function.

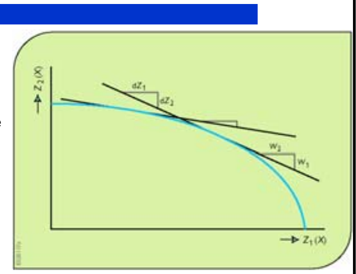
Maximize $Z = w_1 Z_1(X) + w_2 Z_2(X) + \ldots + w_p Z_p(X)$

Subject to $g_i(X)=b_i$ for i=1,2,….,m constraints.

Relative weights, w_j, reflect the trade-off or the marginal rate of transformation of pairs of objective functions. Weights imply value judgments.
These weights are varied systematically and solution is obtained for each set.
Solution obtained for a set of weights gives one generated set of noninferior or efficient solutions are plans.
Major limitation of weighting approach is that it cannot generate the complete set of efficient plans unless the Pareto front is strictly convex.

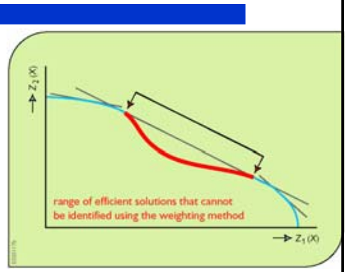
Weighting Method

The efficiency frontier between two objectives, $Z_1(X)$ and $Z_2(X)$, showing the reduction in one objective, say $Z_1(X)$, as the relative weight, w_2, associated with the other objective, increases.



Weighting Method

An efficiency frontier that cannot be completely identified in its convex region using the weighting method when objectives are being maximized. Similarly for concave regions when objectives are being minimized.



Constraint Method

In the Constraint Method, one objective is maximized with lower bounds on all the other objectives.

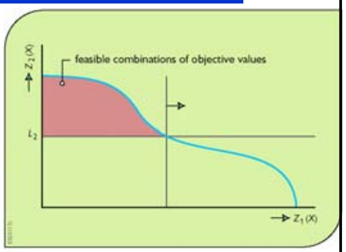
Maximize $Z = Z_j(X)$

Subject to $g_i(X)=b_i$ for i=1,2,….,m constraints,
and $Z_k(X) \geq L_k$ for all k not equal to j.

Any set of feasible values of L_k resulting in a solution with binding constraints gives an efficient alternative solution (noninferior solution).
If it is solved with LP, sensitivity analysis helps to do the trade-off.
Dual variable values of the binding constraints with L_k on the RHS are the marginal rates of transformation of the objectives $Z_j(X)$ and $Z_k(X)$.

Constraint Method

The constraint method for identifying the efficiency frontier by maximizing $Z_1(X)$ while constraining $Z_2(X)$ to be no less than L_2.



Multi Objective Optimization - Example

- Following two objectives are to be maximized subject to constraints

  Maximize $Z_1(X)=5x_1 - 4x_2$ and $Z_2(X) = -2x_1 + 8x_2$

  Subject to
  $-x_1 + x_2 \leq 6$
  $x_1 \leq 12$
  $x_1 + x_2 \leq 16$
  $x_2 \leq 8$
  x_1 & $x_2 \geq 0$
- Generate a Pareto Front of noninferior solutions.
- Plot efficient combinations of Z_1 and Z_2.
- Maximize $Z_1(X)$ and $Z_2(X)$ using the weighting method, given the weights for Z_1 and Z_2 are 1 and 2 respectively.
- Solve it using constraint method to generate efficient solutions.

Decision Space

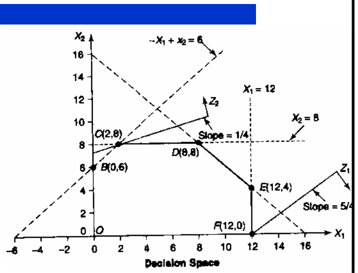
Feasible region: OBCDEF

Lines for Z_1 and Z_2 will be parallel to the lines shown in the figure.

Line of Maximum Z_1 passes through F(12,0).

Line of Maximum Z_2 passes through C(2,8).

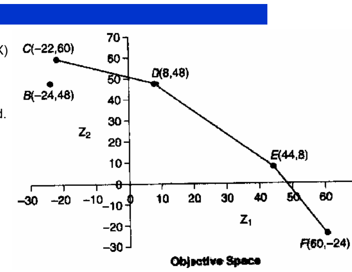
Noninferior set of solutions are represented by CDEF



Objective Space

Evaluating the values of $Z_1(X)$ and $Z_2(X)$ at C, D, E, and F, the line of efficient combinations of Z_1 and Z_2 in the objective space is plotted.

Line CDEF in the figure is called Pareto Front.



Weighting Method

Maximize $Z = w_1Z_1 + w_2Z_2$

with $w_1 = 1$ and $w_2 = 2$

$$Z = Z_1(X) + 2Z_2(X) = (5X_1 - 4X_2) + 2(-2X_1 + 8X_2) = X_1 + 12X_2$$

Objective is to maximize $Z = X_1 + 12X_2$, subject to the given constraints.

Decision Space:

The Z line $Z=X_1+12X_2$ has slope -1/12 in the decision space and Z has maximum value of 104 at D(8,8).

Objective Space:

Objective function $Z=Z_1+2Z_2$ has slope -1/2 in the objective space and Z has maximum value at D(8,48). $Z=8+2(48)=104$.

Constraint Method

Maximize $Z_1 = 5X_1 - 4X_2$

subject to
$$-X_1 + 4X_2 \geq L_2$$
$$-X_1 + X_2 \leq 6$$
$$-X_1 \leq 12$$
$$X_1 + X_2 \leq 16$$
$$X_2 \leq 8$$
$$X_1, X_2 \geq 0$$

Methods for selecting the most desirable non-dominated plan

- Satisficing
- Lexicography
- Indifference Analysis
- Goal Attainment
- Goal Programming
- Interactive Methods
- Plan Simulation and Evaluation

Multi Objective Optimization Exercise Problem

- A reservoir is planned both for gravity and lift irrigation through withdrawals from its storage. The total storage available for both the uses is limited to 5 units each year. It is decided to limit the gravity irrigation withdrawals in a year to 4 units. If X_1 is the allocation of water to gravity irrigation and X_2 is the allocation for lift irrigation, two objectives are planned to be maximized and are expressed as

  Maximize $Z_1(X)=3x_1 - 2x_2$ and $Z_2(X) = - x_1 + 4x_2$
- Generate a Pareto Front of noninferior solutions by plotting Decision space and Objective space.
- Formulate multi objective optimization model using weighting approach with w_1 and w_2 as weights for gravity and lift irrigation respectively.
- Solve it, if (i) w_1=1 and w_2=2 (ii) w_1=2 and w_2=1
- Formulate the problem using constraint method

  Solution: (i) X_1=0, X_2=5; (ii) X_1=4, X_2=0 to 1;

Thank You