Linear Programming

Simplex method - II

Objectives

Objectives

- To discuss the *Big-M* method
- Discussion on different types of LPP solutions in the context of Simplex method
- Discussion on maximization verses minimization problems

Big-M Method

Simplex method for LP problem with 'greater- than-equal- to' (≥) and 'equality' (=) constraints needs a modified approach. This is known as *Big-M* method.

- The LPP is transformed to its standard form by incorporating a large coefficient M

Transformation of LPP for Big-M method

1. One 'artificial variable' is added to each of the 'greater-than-equal-to' (≥) and equality (=) constraints to ensure an initial basic feasible solution.
2. Artificial variables are 'penalized' in the objective function by introducing a large negative (positive) coefficient for maximization (minimization) problem.
3. Cost coefficients, which are supposed to be placed in the Z-row in the initial simplex tableau, are transformed by 'pivotal operation' considering the column of artificial variable as 'pivotal column' and the row of the artificial variable as 'pivotal row'.
4. If there are more than one artificial variables, step 3 is repeated for all the artificial variables one by one.

Example

Consider the following problem

$$\begin{aligned}
\text{Maximize} \quad & Z = 3x_1 + 5x_2 \\
\text{subject to} \quad & x_1 + x_2 \ge 2 \\
& x_2 \le 6 \\
& 3x_1 + 2x_2 = 18 \\
& x_1, x_2 \ge 0
\end{aligned}$$

Example

- After incorporating the artificial variables

$$\begin{aligned}
\text{Maximize} \quad & Z = 3x_1 + 5x_2 - Ma_1 - Ma_2 \\
\text{subject to} \quad & x_1 + x_2 - x_3 + a_1 = 2 \\
& x_2 + x_4 = 6 \\
& 3x_1 + 2x_2 + a_2 = 18 \\
& x_1, x_2 \ge 0
\end{aligned}$$

where x_3 is surplus variable, x_4 is slack variable and a_1 and a_2 are the artificial variables

Transformation of cost coefficients

Considering the objective function and the first constraint

$$Z - 3x_1 - 5x_2 + Ma_1 + Ma_2 = 0 \qquad (E_1)$$

$$x_1 + x_2 - x_3 + a_1 = 2 \qquad (E_2)$$

Pivotal Row

Pivotal Column

By the pivotal operation $E_1 - M \times E_2$ the cost coefficients are modified as

$$Z - (3 + M)x_1 - (5 + M)x_2 + Mx_3 + 0a_1 + Ma_2 = -2M$$

Transformation of cost coefficients

Considering the modified objective function and the third constraint

$$Z - (3 + M)x_1 - (5 + M)x_2 + Mx_3 + 0a_1 + Ma_2 = -2M \qquad (E_3)$$

$$3x_1 + 2x_2 + a_2 = 18 \qquad (E_4)$$

Pivotal Row

Pivotal Column

By the pivotal operation $E_3 - M \times E_4$ the cost coefficients are modified as

$$Z - (3 + 4M)x_1 - (5 + 3M)x_2 + Mx_3 + 0a_1 + 0a_2 = -20M$$

Construction of *Simplex Tableau*

Corresponding simplex tableau

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
1	Z	1	$-3-4M$	$-5-3M$	M	0	0	0	$-20M$	--
	a_1	0	1	1	-1	0	1	0	2	2
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	3	2	0	0	0	1	18	6

Pivotal row, pivotal column and pivotal elements are shown as earlier

Simplex Tableau…contd.

Successive simplex tableaus are as follows:

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
2	Z	1	0	$-2+M$	$-3-3M$	0	$3+4M$	0	$6-12M$	--
	x_1	0	1	1	-1	0	1	0	2	--
	x_4	0	0	1	0	1	0	0	6	--
	a_2	0	0	-1	3	0	-3	1	12	4

Simplex Tableau…contd.

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
3	Z	1	0	-3	0	0	M	$1+M$	18	--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--

Simplex Tableau…contd.

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	a_1	a_2	b_r	$\frac{b_r}{c_{rs}}$
4	Z	1	0	0	0	3	M	$1+M$	36	--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	--
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	--

Optimality has reached. Optimal solution is Z = 36 with $x_1 = 2$ and $x_2 = 6$

Simplex method: 'Unbounded', 'Multiple' and 'Infeasible' solutions

Unbounded solution

- If at any iteration no departing variable can be found corresponding to entering variable, the value of the objective function can be increased indefinitely, i.e., the solution is unbounded.

Simplex method: 'Unbounded', 'Multiple' and 'Infeasible' solutions

Multiple (infinite) solutions

- If in the final tableau, one of the non-basic variables has a coefficient 0 in the Z-row, it indicates that an alternative solution exists.
- This non-basic variable can be incorporated in the basis to obtain another optimal solution.
- Once two such optimal solutions are obtained, infinite number of optimal solutions can be obtained by taking a weighted sum of the two optimal solutions.

Simplex method: Example of Multiple (indefinite) solutions

Consider the following problem

$$
\begin{aligned}
\text{Maximize} \quad & Z = 3x_1 + 2x_2 \\
\text{subject to} \quad & x_1 + x_2 \geq 2 \\
& x_2 \leq 6 \\
& 3x_1 + 2x_2 = 18 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- Only modification, compared to earlier problem, is that the coefficient of x_2 is changed from 5 to 2 in the objective function.
- Thus the slope of the objective function and that of third constraint are now same, which leads to multiple solutions

Simplex method: Example of Multiple (indefinite) solutions

Following similar procedure as described earlier, final simplex tableau for the problem is as follows:

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
3	Z	1	0	0	0	0	M	$1+M$	18	--
	x_1	0	1	$\frac{2}{3}$	0	0	0	$\frac{1}{3}$	6	9
	x_4	0	0	1	0	1	0	0	6	6
	x_3	0	0	$-\frac{1}{3}$	1	0	-1	$\frac{1}{3}$	4	--

Coefficient of non-basic variable x_2 is zero

Simplex method: Example of Multiple (indefinite) solutions

As there is no negative coefficient in the Z-row optimal solution is reached.

Optimal solution is $Z = 18$ with $x_1 = 6$ and $x_2 = 0$

However, the coefficient of non-basic variable x_2 is zero in the *Z-row*

Another solution is possible by incorporating x_2 in the basis

Based on the $\frac{b_r}{c_{rs}}$, x_4 will be the exiting variable

Simplex method: Example of Multiple (indefinite) solutions

Iteration	Basis	Z	Variables						b_r	$\frac{b_r}{c_{rs}}$
			x_1	x_2	x_3	x_4	a_1	a_2		
4	Z	1	0	0	0	0	M	$1+M$	18	--
	x_1	0	1	0	0	$-\frac{2}{3}$	0	$\frac{1}{3}$	2	--
	x_2	0	0	1	0	1	0	0	6	6
	x_3	0	0	0	1	$\frac{1}{3}$	-1	$\frac{1}{3}$	6	18

Coefficient of non-basic variable x_4 is zero

If one more similar step is performed, previous simplex tableau will be obtained back

So, another optimal solution is $Z = 18$ with $x_1 = 2$ and $x_2 = 6$

Simplex method: Example of Multiple (indefinite) solutions

Thus, two sets of solutions are: $\begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$ and $\begin{Bmatrix} 2 \\ 6 \end{Bmatrix}$

Other optimal solutions will be obtained as $\beta\begin{Bmatrix} 6 \\ 0 \end{Bmatrix} + (1-\beta)\begin{Bmatrix} 2 \\ 6 \end{Bmatrix}$ where $\beta \in [0,1]$

For example, let β = 0.4, corresponding solution is $\begin{Bmatrix} 3.6 \\ 3.6 \end{Bmatrix}$

Note that values of the objective function are not changed for different sets of solution; for all the cases Z = 18.

Simplex method: 'Unbounded', 'Multiple' and 'Infeasible' solutions

Infeasible solution

- If in the final tableau, at least one of the artificial variables still exists in the basis, the solution is indefinite.

Minimization versus maximization problems

- Simplex method is described based on the standard form of LP problems, i.e., objective function is of maximization type
- However, if the objective function is of minimization type, simplex method may still be applied with a small modification

Minimization versus maximization problems

The required modification can be done in either of following two ways.

1. The objective function is multiplied by -1 so as to keep the problem identical and 'minimization' problem becomes 'maximization'. This is because minimizing a function is equivalent to the maximization of its negative
2. While selecting the entering nonbasic variable, the variable having the maximum coefficient among all the cost coefficients is to be entered. In such cases, optimal solution would be determined from the tableau having all the cost coefficients as non-positive (≤ 0)

Minimization versus maximization problems

- One difficulty, that remains in the minimization problem, is that it consists of the constraints with 'greater-than-equal-to' ($\geq$) sign. For example, minimize the price (to compete in the market), however, the profit should cross a minimum threshold. Whenever the goal is to minimize some objective, lower bounded requirements play the leading role. Constraints with 'greater-than-equal-to' ($\geq$) sign are obvious in practical situations.
- To deal with the constraints with 'greater-than-equal-to' ($\geq$) and sign, *Big-M* method is to be followed as explained earlier.

LP: Elementary Transformations

More often than not, the LP model originally constructed does not satisfy the characteristics of a standard form or a canonical form. The following elementary operations enable one to transform an LP model into any desirable form.

1. Maximization of a function $f(x)$ is equal to the minimization of its negative counterpart, that is, Max $f(x) = \text{Min}[-f(x)]$.
2. Constraints of the $\geq$ type can be converted to the $\leq$ type by multiplying by -1 on both sides of the inequality.
3. An equation can be replaced by *two* inequalities of the opposite sign. For example, an equation $g(x) = b$ can be substituted by $g(x) \leq b$ and $g(x) \geq b$.
4. An inequality involving an absolute expression can be replaced by *two* inequalities without an absolute sign. For example, $|g(x)| \leq b$ can be replaced by $g(x) \leq b$ *and* $g(x) \geq -b$.
5. If a decision variable x is unrestricted-in-sign (i.e., it can be positive, zero, or negative), then it can be replaced by the difference of *two* nonnegative decision variables; $x = x^+ - x^-$, where $x^+ \geq 0$ and $x^- \geq 0$.
6. To transform an inequality into an equation, a nonnegative variable can be added or subtracted.

Assumptions in LP Models

- Proportionality assumption

  This implies that the contribution of the jth decision variable to the effectiveness measure, $c_j x_j$, and its usage of the various resources, $a_{ij} x_j$, are directly proportional to the value of the decision variable.
- Additivity assumption

  This assumption means that, at a given level of activity (x_1, x_2,..., x_n), the total usage of resources and contribution to the overall measure of effectiveness are equal to the sum of the corresponding quantities generated by each activity conducted by itself.
- Divisibility assumption

  Activity units can be divided into any fractional level, so that non integer values for the decision variables are permissible.
- Deterministic assumption

  All parameters in the model are known constants without uncertainty.

CE 210 Systems Techniques in Water Resources & Environmental Engg

Assignment - 2

1. Solving the following linear programming problems by Graphical Method and identify in-active constraints, if any. Also show the feasible region and optimal point on the plot.

(a)

$$\begin{aligned} &\textit{Maximize } Z = 5a + 8b \\ &\textit{subject to} \\ &40a + 30b \le 480 \\ &24a + 32b \le 480 \\ &20a + 24b \le 480 \\ &a \ge 0; b \ge 0 \end{aligned}$$

(b)

$$\begin{aligned} &\textit{Minimize } Z = x + 1.5y \\ &\textit{subject to} \\ &0.15x + 0.60y \ge 3000 \\ &0.25x + 0.30y \ge 2500 \\ &0.60x + 0.10y \ge 2000 \\ &x \ge 0; y \ge 0 \end{aligned}$$

(c)

$$\begin{aligned} &\textit{Maximize } Z = x_1 + 2x_2 \\ &\textit{subject to} \\ &2x_1 + 3x_2 \le 12 \\ &5x_1 + 2x_2 \le 15 \\ &x_1 \ge 0; x_2 \ge 0 \end{aligned}$$

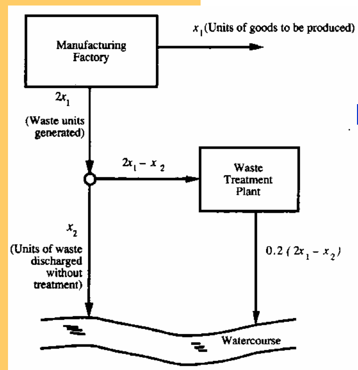
(d)

$$\begin{aligned} &\textit{Minimize } Z = 4x + 6y \\ &\textit{subject to} \\ &2x + 5y \ge 10 \\ &3x + 2y \ge 6 \\ &x \ge 0; y \ge 0 \end{aligned}$$

(e) In problem 1(d), if the objective function is changed to $Z=x+8y$, will there be a different optimal solution ?

2. Consider a system composed of a manufacturing factory and a waste treatment plant owned by the manufacturer. The manufacturing plant produces finished goods that sell for a unit price of Rs 10,000. However, the finished goods cost Rs 3,000 per unit to produce. In the manufacturing process two units of waste are generated for each unit of finished goods produced. In addition to deciding how many units of goods to produce, the plant manager must also decide how much waste will be discharged into a river without treatment so that the total net benefit to the company can be maximised and the water quality requirement of the water course is met. The treatment plant has a maximum capacity of treating ten units of waste with 80% waste removal efficiency at a treatment cost of Rs 600 per unit of waste. There is also an effluent tax imposed on the waste discharged to the receiving water body (Rs 2,000 for each unit of waste discharged). The water pollution control authority has set an upper limit of four units on the amount of waste the company may discharge. Formulate an LP model clearly specifying the decision variables. Objective function and constraints and solve it is using both graphical method as well as simplex method.

LP Problem



$$\begin{aligned} &\textit{Maximize } Z = 5x_1 - x_2 \\ &\textit{subject to} \\ &2x_1 - x_2 \le 10 \\ &0.4x_1 + 0.8x_2 \le 4 \\ &2x_1 - x_2 \ge 0 \\ &x_1 \ge 0; x_2 \ge 0 \end{aligned}$$