Linear Programming

Simplex Method - I

Introduction and Objectives

Simplex method is the most popular method used for the solution of *Linear Programming Problems* (LPP).

Objectives

- To discuss the motivation of simplex method
- To discuss Simplex algorithm
- To demonstrate the construction of simplex tableau

Motivation of Simplex method

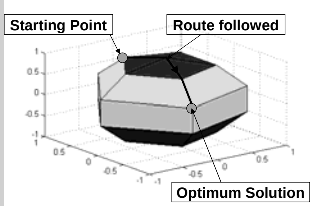
- Solution of a LPP, if exists, lies at one of the vertices of the feasible region.
- All the basic solutions can be investigated one-by-one to pick up the optimal solution.
- For 10 equations with 15 variables there exists ${}^{15}C_{10}$= 3003 basic feasible solutions!
- Too large number to investigate one-by-one.
- This can be overcome by simplex method

Simplex Method: Concept in 3D case

- In 3D, a feasible region (i.e., volume) is bounded by several surfaces
- Each vertex (a basic feasible solution) of this volume is connected to the three other adjacent vertices by a straight line to each, being intersection of two surfaces.
- *Simplex algorithm* helps to move from one vertex to another adjacent vertex which is closest to the optimal solution among all other adjacent vertices.
- Thus, it follows the shortest route to reach the optimal solution from the starting point.

Simplex Method: Concept in 3D case

Pictorial representation



Thus there is no need to investigate all the basic feasible solutions.

Sequence of basic feasible solutions on the shortest route is generated by *simplex algorithm*

General procedure of simplex method

Simplex method involves following steps

1. General form of given LPP is transformed to its *canonical form* (refer Lecture notes 1).
2. Find a basic feasible solution of the LPP (there should exist at least one).
3. Move to an adjacent basic feasible solution which is closest to the optimal solution among all other adjacent vertices.
4. Repeat until optimum solution is achieved.

Step three involves '*Simplex Algorithm'*

Simplex Algorithm

Let us consider the following LPP

$$\begin{aligned}
\text{Maximize} \quad & Z = 4x_1 - x_2 + 2x_3 \\
\text{subject to} \quad & 2x_1 + x_2 + 2x_3 \le 6 \\
& x_1 - 4x_2 + 2x_3 \le 0 \\
& 5x_1 - 2x_2 - 2x_3 \le 4 \\
& x_1, x_2, x_3 \ge 0
\end{aligned}$$

Simplex Algorithm … contd.

LPP is transformed to its standard form

$$\begin{aligned}
\text{Maximize} \quad & -4x_1 + x_2 - 2x_3 + Z = 0 \\
\text{subject to} \quad & 2x_1 + x_2 + 2x_3 + x_4 = 6 \\
& x_1 - 4x_2 + 2x_3 + x_5 = 0 \\
& 5x_1 - 2x_2 - 2x_3 + x_6 = 4 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \ge 0
\end{aligned}$$

Note that x_4, x_5 and x_6 are slack variables

Simplex Algorithm … contd.

Set of equations, including the objective function is transformed to canonical form

$$\begin{aligned}
-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z &= 0 \\
2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 &= 6 \\
x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 &= 0 \\
5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 &= 4
\end{aligned}$$

Basic feasible solution of above canonical form is $x_4 = 6$, $x_5 = 0$, $x_6 = 4$, $x_1 = x_2 = x_3 = 0$ and $Z = 0$

x_4, x_5, x_6 : Basic Variables; x_1, x_2, x_3 : Nonbasic Variables

Simplex Algorithm … contd.

Symbolized form (for ease of discussion)

$$\begin{array}{llllllll}
(Z) & c_1x_1 & +c_2x_2 & +c_3x_3 + & c_4x_4 & + c_5x_5 + & c_6x_6 & +Z = b \\
(x_4) & c_{41}x_1 & +c_{42}x_2 & +c_{43}x_3 + & c_{44}x_4 & + c_{45}x_5 + & c_{46}x_6 & = b_4 \\
(x_5) & c_{51}x_1 & +c_{52}x_2 & +c_{53}x_3 + & c_{54}x_4 & + c_{55}x_5 + & c_{56}x_6 & = b_5 \\
(x_6) & c_{61}x_1 & +c_{62}x_2 & +c_{63}x_3 + & c_{64}x_4 & + c_{65}x_5 + & c_{66}x_6 & = b_6
\end{array}$$

- The left-most column is known as ***basis*** as this is consisting of basic variables
- The coefficients in the first row ($c_1, \ldots, c_6$) are known as ***cost coefficients***.
- Other subscript notations are self explanatory

Simplex Algorithm … contd.

This completes first step of algorithm. After completing each step (iteration) of algorithm, following three points are to be examined:

1. Is there any possibility of further improvement?
2. Which nonbasic variable is to be entered into the basis?
3. Which basic variable is to be exited from the basis?

Simplex Algorithm … contd.

Is there any possibility of further improvement?

If any one of the cost coefficients is negative further improvement is possible.

Which nonbasic variable is to be entered?

Entering variable is decided such that the unit change of this variable should have maximum effect on the objective function. Thus the variable having the coefficient which is minimum among all cost coefficients is to be entered, i.e., x_s is to be entered if cost coefficient c_s is minimum.

Simplex Algorithm … contd.

Which basic variable is to be exited?

After deciding the entering variable x_s, x_r (from the set of basic variables) is decided to be the exiting variable if $\frac{b_r}{c_{rs}}$ is minimum for all possible r, provided c_{rs} is positive.

c_{rs} is considered as pivotal element to obtain the next canonical form.

$$\begin{aligned}
-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z &= 0\\
2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 &= 6\\
x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 &= 0\\
5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 &= 4
\end{aligned}$$

Simplex Algorithm … contd.

Entering variable

c_1 is minimum (- 4), thus, x_1 is the entering variable for the next step of calculation.

Exiting variable

r may take any value from 4, 5 and 6. It is found that $\frac{b_4}{c_{41}} = \frac{6}{2} = 3$, $\frac{b_5}{c_{51}} = \frac{0}{1} = 0$ $\frac{b_6}{c_{61}} = \frac{4}{5} = 0.8$. As $\frac{b_5}{c_{51}}$ is minimum, r is 5. Thus, x_5 is to be exited.

c_{51} (= 1) is considered as pivotal element and x_5 is replaced by x_1 in the basis.

Thus a new canonical form is obtained through pivotal operation, which was explained in first class.

Simplex Algorithm … contd.

Pivotal operation as a refresher

- Pivotal row is transformed by dividing it with the pivotal element. In this case, pivotal element is 1.
- For other rows: Let the coefficient of the element in the pivotal column of a particular row be "l". Let the pivotal element me "m". Then the pivotal row is multiplied by 'l / m' and then subtracted from that row to be transformed. This operation ensures that the coefficients of the element in the pivotal column of that row becomes zero, e.g., *Z row*: l = -4 , m = 1. So, pivotal row is multiplied by l / m = -4 / 1 = -4, obtaining $-4x_1 + 16x_2 - 8x_3 + 0x_4 - 4x_5 + 0x_6 = 0$. This is subtracted from *Z row*, obtaining, $0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z = 0$

  The other two rows are also suitably transformed.

Simplex Algorithm … contd.

After the pivotal operation, the canonical form obtained as follows

$$\begin{array}{llr}
(Z) & 0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z & = 0\\
(x_4) & 0x_1 + 9x_2 - 2x_3 + 1x_4 - 2x_5 + 0x_6 & = 6\\
(x_1) & 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 & = 0\\
(x_6) & 0x_1 + 18x_2 - 12x_3 - 0x_4 - 5x_5 + 1x_6 & = 4
\end{array}$$

The basic solution of above canonical form is $x_1 = 0$, $x_4 = 6$, $x_6 = 4$, $x_2 = x_3 = x_5 = 0$ and $Z = 0$.

Note that cost coefficient c_2 is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

$$\begin{array}{llr}
(Z) & 0x_1 - 15x_2 + 6x_3 + 0x_4 + 4x_5 + 0x_6 + Z & = 0\\
(x_4) & 0x_1 + 9x_2 - 2x_3 + 1x_4 - 2x_5 + 0x_6 & = 6\\
(x_1) & 1x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 & = 0\\
(x_6) & 0x_1 + 18x_2 - 12x_3 - 0x_4 - 5x_5 + 1x_6 & = 4
\end{array}$$

Simplex Algorithm … contd.

Entering variable

c_2 is minimum (- 15), thus, x_2 is the entering variable for the next step of calculation.

Exiting variable

r may take any value from 4, 1 and 6. However, c_{12} is negative (- 4). Thus, r may be either 4 or 6. It is found that $\frac{b_4}{c_{42}} = \frac{6}{9} = 0.667$ and $\frac{b_6}{c_{62}} = \frac{4}{18} = 0.222$.

As $\frac{b_6}{c_{62}}$ is minimum, r is 6. Thus x_6 is to be exited. c_{62} (= 18) is considered as pivotal element and x_6 is to be replaced by x_2 in the basis.

Thus another canonical form is obtained.

Simplex Algorithm … contd.

The canonical form obtained after third iteration

$$\begin{array}{llr}
(Z) & 0x_1 + 0x_2 - 4x_3 + 0x_4 - \frac{1}{6}x_5 + \frac{5}{6}x_6 + Z & = \frac{10}{3}\\
(x_4) & 0x_1 + 0x_2 + 4x_3 + 1x_4 + \frac{1}{2}x_5 - \frac{1}{2}x_6 & = 4\\
(x_1) & 1x_1 + 0x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{1}{9}x_5 + \frac{2}{9}x_6 & = \frac{8}{9}\\
(x_2) & 0x_1 + 1x_2 - \frac{2}{3}x_3 + 0x_4 - \frac{5}{18}x_5 + \frac{1}{18}x_6 & = \frac{2}{9}
\end{array}$$

The basic solution of above canonical form is $x_1 = 8/9$, $x_2 = 2/9$, $x_4 = 4$, $x_3 = x_5 = x_6 = 0$ and $Z = 10/3$.

Simplex Algorithm … contd.

Note that cost coefficient c_3 is negative. Thus optimum solution is not yet achieved. Further improvement is possible.

Entering variable (Following the similar procedure)
x_3 is the entering variable for the next step of calculation.

Exiting variable (Following the similar procedure)
x_4 is the exiting variable. Thus c_{43} (= 4) is the pivotal element and x_4 is to be replaced by x_3 in the basis.

Thus another canonical form is obtained.

Simplex Algorithm … contd.

The canonical form obtained after fourth iteration

$$(Z) \quad 0x_1 + 0x_2 + 0x_3 + 1x_4 + \frac{1}{3}x_5 + \frac{1}{3}x_6 + Z = \frac{22}{3}$$

$$(x_3) \quad 0x_1 + 0x_2 + 1x_3 + \frac{1}{4}x_4 + \frac{1}{8}x_5 - \frac{1}{8}x_6 = 1$$

$$(x_1) \quad 1x_1 + 0x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{1}{36}x_5 + \frac{5}{36}x_6 = \frac{14}{9}$$

$$(x_2) \quad 0x_1 + 1x_2 + 0x_3 + \frac{1}{6}x_4 - \frac{7}{36}x_5 - \frac{1}{36}x_6 = \frac{8}{9}$$

The basic solution of above canonical form is
$x_1 = 14/9$, $x_2 = 8/9$, $x_3 = 1$, $x_4 = x_5 = x_6 = 0$ and $Z = 22/3$.

Simplex Algorithm … contd.

Note that all the cost coefficients are nonnegative. Thus the optimum solution is achieved.

Optimum solution is

$$Z = \frac{22}{3} = 7.333$$

$$x_1 = \frac{14}{9} = 1.556$$

$$x_2 = \frac{8}{9} = 0.889$$

$$x_3 = 1$$

Construction of *Simplex Tableau: General notes*

Calculations shown till now can be presented in a tabular form, known as simplex tableau

After preparing the canonical form of the given LPP, first simplex tableau is constructed.

After completing each simplex tableau (iteration), few steps (somewhat mechanical and easy to remember) are followed.

Logically, these steps are exactly similar to the procedure described earlier.

Construction of *Simplex Tableau: Basic steps*

Check for optimum solution:

1. Investigate whether all the elements (coefficients of the variables headed by that column) in the first row (i.e., Z row) are nonnegative or not. If all such coefficients are nonnegative, optimum solution is obtained and no need of further iterations.
   If any element in this row is negative follow next steps to obtain the simplex tableau for next iteration.

Construction of *Simplex Tableau: Basic steps*

Operations to obtain next simplex tableau:

2. Identify the entering variable (described earlier) and mark that column as *Pivotal Column*.
3. Identify the exiting variable from the basis as described earlier and mark that row as *Pivotal Row*.
4. Mark the coefficient at the intersection of *Pivotal Row* and *Pivotal Column* as *Pivotal Element*.

Construction of *Simplex Tableau: Basic steps*

Operations to obtained simplex tableau…contd.:

5. In the basis, replace the exiting variable by entering variable.
6. Divide all the elements in the pivotal row by pivotal element.
7. For any other row, identify the elementary operation such that the coefficient in the pivotal column, in that row, becomes zero. Apply the same operation for all other elements in that row and change the coefficients.

   Follow similar procedure for all other rows.

Construction of *Simplex Tableau: example*

Consider the same problem discussed before.
Canonical form of this LPP is

$$\begin{aligned}
-4x_1 + x_2 - 2x_3 + 0x_4 + 0x_5 + 0x_6 + Z &= 0 \\
2x_1 + x_2 + 2x_3 + 1x_4 + 0x_5 + 0x_6 &= 6 \\
x_1 - 4x_2 + 2x_3 + 0x_4 + 1x_5 + 0x_6 &= 0 \\
5x_1 - 2x_2 - 2x_3 + 0x_4 + 0x_5 + 1x_6 &= 4
\end{aligned}$$

Construction of *Simplex Tableau: example*

Corresponding simplex tableau

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
1	Z	1	-4	1	-2	0	0	0	0	--
	x_4	0	2	1	2	1	0	0	6	3
	x_5	0	1	-4	2	0	1	0	0	0
	x_6	0	5	-2	-2	0	0	1	4	$\frac{4}{5}$

Pivotal Row
Pivotal Column
Pivotal Element

Construction of *Simplex Tableau: example*

Successive iterations

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
2	Z	1	0	-15	6	0	4	0	0	--
	x_4	0	0	9	-2	1	-2	0	6	1/3
	x_1	0	1	-4	2	0	1	0	0	--
	x_6	0	0	18	-12	0	-5	1	4	2/9

Construction of *Simplex Tableau: example*

Successive iterations…contd.

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
3	Z	1	0	0	-4	0	$-\frac{1}{6}$	$\frac{5}{6}$	$\frac{10}{3}$	--
	x_4	0	0	0	4	1	$\frac{1}{2}$	$-\frac{1}{2}$	4	1
	x_1	0	1	0	$-\frac{2}{3}$	0	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{8}{9}$	--
	x_2	0	0	1	$-\frac{2}{3}$	0	$-\frac{5}{18}$	$\frac{1}{18}$	$\frac{2}{9}$	--

Final Tableau

Iteration	Basis	Z	Variables x_1	x_2	x_3	x_4	x_5	x_6	b_r	$\frac{b_r}{c_{rs}}$
4	Z	1	0	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{22}{3}$	
	x_3	0	0	0	1	$\frac{1}{4}$	$\frac{1}{8}$	$-\frac{1}{8}$	1	
	x_1	0	1	0	0	$\frac{1}{6}$	$-\frac{1}{36}$	$\frac{2}{9}$	$\frac{14}{9}$	
	x_2	0	0	1	0	$\frac{1}{6}$	$-\frac{7}{36}$	$-\frac{1}{36}$	$\frac{8}{9}$	

Optimum value of Z

All the coefficients are nonnegative. Thus optimum solution is achieved.

Value of x_3
Value of x_1
Value of x_2

Final results from *Simplex Tableau*

All the elements in the first row (i.e., Z row), at iteration 4, are nonnegative. Thus, optimum solution is achieved.

Optimum solution is

$$Z = \frac{22}{3} = 7.333$$
$$x_1 = \frac{14}{9} = 1.556$$
$$x_2 = \frac{8}{9} = 0.889$$
$$x_3 = 1$$

Construction of *Simplex Tableau: A note*

It can be noted that at any iteration the following two points must be satisfied:

1. All the basic variables (other than Z) have a coefficient of zero in the Z row.
2. Coefficients of basic variables in other rows constitute a unit matrix.

Violation of any of these points at any iteration indicates a wrong calculation. However, reverse is not true.

Thank You