

ENTRAINMENT IN VAN DER POL'S OSCILLATOR IN THE PRESENCE OF NOISE

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Abstract—The response of Van der Pol's oscillator to a combination of harmonic and white noise excitations is considered. The harmonic excitation frequency is taken to be in the neighbourhood of the system limit cycle frequency. The effect of addition of noise on the entrainment behaviour is investigated using a combination of methods of stochastic averaging and equivalent non-linearization. Results based on the gaussian closure technique are also obtained and the theoretical solutions are compared with digital simulations.

1. INTRODUCTION

An important source of non-linearity in engineering systems is the presence of a self-excitation mechanism. This results in one or more periodic equilibrium states called limit cycles. In the study of self-excited systems possessing one stable limit cycle, the Van der Pol oscillator can be regarded as a touchstone. The behaviour of this system under deterministic and random excitations has been a subject of extensive study in the past. The effect of broad band noise on the limit cycle behaviour has been studied by several authors [1–5]. It has been shown that the interaction between limit cycle and the external noise produces bimodal probability distribution for the displacement and velocity processes [5]. This in turn invalidates the assumption gaussianness of linearization techniques in the response analysis. Under a harmonic excitation Van der Pol's oscillator exhibits the well known phenomenon of frequency entrainment. The forming and breaking of entrained oscillations is associated with a series of bifurcations and this has been investigated notably by Cartwright [6], Minorsky [7], Dewan [8] and Holmes and Rand [9]. If in addition to harmonic excitation the system is perturbed by a random noise, the response will have several interesting features arising out of the interaction between the output components due to the basic limit cycle, the harmonic excitation and the random noise. In the past, Stratonovich [2] has studied this problem and presented a number of analytical results based on various approximations. In particular, he has investigated the breaking of entrained oscillation in terms of the stability of the response phase process. The method of gaussian closure has been employed by the present authors [10] to find response in the primary resonance region. The acceptability of the approximate solution has been verified with the help of a stochastic stability analysis. In the present paper, the response of Van der Pol's oscillator to combined periodic and white noise excitation is analysed using the stochastic averaging method. It is shown that the averaged equations can be solved exactly for a specific choice of system parameters. A general approximate solution is however possible for all parameter values through the method of equivalent non-linearization.

2. ANALYSIS

The system under study is

$$\begin{aligned}\ddot{x} - \varepsilon \dot{x}(1 - 4x^2) + x &= Q \cos \lambda t + W(t) \\ \langle W(t_1) W(t_2) \rangle &= 2D\delta(t_1 - t_2).\end{aligned}\quad (1)$$

Here a dot represents derivative with respect to time t , $W(t)$ is a gaussian white noise process, $\langle \cdot \rangle$ denotes the expectation operator and $\delta(\cdot)$ is Dirac's delta function. The

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parameters ε , Q and D are assumed to be small compared to unity. It may be noted that for the unforced system the amplitude and frequency of the limit cycle oscillation are both equal to unity. In the absence of noise the response can be expected to consist of two periodic components, one due to the limit cycle of the system and the other due to the external harmonic excitation. However, if λ is in the neighbourhood of the limit cycle frequency, the response at the limit cycle frequency is suppressed and the system oscillates only at the harmonic forcing frequency. This is the well known phenomenon of frequency entrainment [7]. The interest in the present work is to study the effect of addition of the noise $W(t)$ on the system response. In their earlier work the present authors [10] assumed the steady state solution of equation (1) in the form

$$x(t) = R \cos(\lambda t - \chi) + Z(t) \quad (2)$$

where the first term represented the mean periodic response and $Z(t)$ the stationary random component of the response. Under the closure assumption that $Z(t)$ is a gaussian random process it was shown that

$$\begin{aligned} R^2[\Delta^2 + (1 - 4\sigma^2 - R^2)^2] &= Q/(\varepsilon\lambda)^2 \\ \chi &= \tan^{-1}[\Delta/(1 - 4\sigma^2 - R^2)] \\ \Delta &= (1 - \lambda^2)/(\varepsilon\lambda) \\ \sigma^2 &= 0.125\{(1 - 2R^2) + \sqrt{[1 - 2R^2 - 16(D/\varepsilon)]}\} \end{aligned} \quad (3)$$

where $\sigma^2 = \langle Z^2 \rangle$. Further, this solution was deemed to be acceptable only if it satisfied a stochastic stability criterion. It is to be noted that the above solution has several drawbacks. Firstly the solution assumed in equation (2) is not valid in the limit $Q \rightarrow 0$. This is because, for the case of $Q = 0$, the excitation consists of only white noise and as noted already the linearization methods such as the gaussian closure technique are not suited to handle this case. Also, the type of steady state response assumed in equation (2), namely, that a periodic mean plus a stationary random component can also be questioned. In view of this, it is of interest to investigate whether the stochastic averaging technique can avoid these limitations. For this purpose, equation (1) is re-written as

$$\ddot{x} + \lambda^2 x = -\tilde{\Delta}x + \varepsilon\dot{x}(1 - 4x^2) + Q \cos \lambda t + W(t) \quad (4)$$

where $\tilde{\Delta} = 1 - \lambda^2$ is a detuning parameter. The solution of this equation is taken in the form

$$\begin{aligned} x(t) &= A \cos(\lambda t + \psi) \\ \dot{x}(t) &= -A\lambda \sin(\lambda t + \psi). \end{aligned} \quad (5)$$

Here A and ψ are slowly varying random processes. A pair of simplified equations for A and ψ can now be obtained based on the method of stochastic averaging [11]. This leads to

$$\begin{aligned} \dot{A} &= 0.5\varepsilon A(1 - A^2) + 0.5(Q/\lambda)\sin \psi + (0.5D/A) + \sqrt{D} W_1(t) \\ \dot{\psi} &= 0.5(\tilde{\Delta}/\lambda) - 0.5[Q/(A\lambda)]\cos \psi + (\sqrt{D}/A) W_2(t). \end{aligned} \quad (6, 7)$$

Here $W_1(t)$ and $W_2(t)$ are independent gaussian white noise processes with unit strength. From the above equations, it can be observed that the equation governing the amplitude process is coupled to that of the phase process. Thus in order to obtain an approximate solution of equation (1), one has to solve the two-dimensional Fokker-Planck-Kolmogorov (FPK) equation governing the joint probability density function (pdf) of A and ψ . This, in general, is not possible and hence further approximations would be necessary. It may be mentioned here that the above pair of equations for A and ψ has earlier been studied by Stratonovich [2]. For this purpose, he has adopted different procedures based on physical arguments. In one case he has linearized A and ψ around their respective deterministic values and computed the autocorrection of the response process. In another approximation he has considered equation (7) and replaced the terms containing A by the

corresponding deterministic values. In the present case this would amount to setting $A = 1$ in equation (7) which leads to

$$\dot{\psi} = 0.5(\tilde{\Delta}/\lambda) - 0.5(Q/\lambda)\cos \psi + \sqrt{D} W_2(t). \tag{8}$$

Thus the equation of ψ gets uncoupled from that of A which in turn enables the determination of the stationary pdf of ψ . In the present investigation, however, an alternative approach is adopted. Here an approximate solution of equations (6) and (7) is obtained based on the equivalent non-linearization technique. This method consists of optimally replacing the given non-linear stochastic equation by an equivalent non-linear system so that the resulting equation can be solved exactly [12]. In order to apply this technique it is advantageous to first carry out the stochastic averaging of equation (4) in cartesian co-ordinates. For this purpose, the solution is taken in the form

$$\begin{aligned} x(t) &= Z_1 \cos \lambda t + Z_2 \sin \lambda t \\ \dot{x}(t) &= -Z_1 \lambda \sin \lambda t + Z_2 \lambda \cos \lambda t. \end{aligned} \tag{9}$$

Here Z_1 and Z_2 are slowly varying random processes. In the transformed co-ordinates (Z_1, Z_2) , equation (4) now reads

$$\begin{aligned} \dot{Z}_1 &= -(\sin \lambda t/\lambda) \{ -\tilde{\Delta}(Z_1 \cos \lambda t + Z_2 \sin \lambda t) + \varepsilon(-Z_1 \lambda \sin \lambda t + Z_2 \lambda \cos \lambda t) \\ &\quad \times [1 - 4(Z_1 \cos \lambda t + Z_1 \sin \lambda t)^2] + Q \cos \lambda t + W(t) \} \\ \dot{Z}_2 &= (\cos \lambda t/\lambda) \{ -\tilde{\Delta}(Z_1 \cos \lambda t + Z_2 \sin \lambda t) + \varepsilon(-Z_1 \lambda \sin \lambda t + Z_2 \lambda \cos \lambda t) \\ &\quad \times [1 - 4(Z_1 \cos \lambda t + Z_2 \sin \lambda t)^2] + Q \cos \lambda t + W(t) \}. \end{aligned} \tag{10}$$

This pair of stochastic equations can be simplified using the stochastic averaging method. This leads to

$$\begin{aligned} \dot{Z}_1 &= P_1(Z_1, Z_2) + \sqrt{D_1} W_1(t) \\ \dot{Z}_2 &= P_2(Z_1, Z_2) + \sqrt{D_1} W_2(t) \\ P_1 &= 0.5\varepsilon(1 - Z_1^2 - Z_2^2)Z_1 + (0.5\tilde{\Delta}Z_2/\lambda) \\ P_2 &= 0.5\varepsilon(1 - Z_1^2 - Z_2^2)Z_2 + (0.5\tilde{\Delta}Z_1/\lambda) + (0.5Q/\lambda). \end{aligned} \tag{11}$$

Here W_1 and W_2 are independent white noise processes with unit strength and $D_1 = (D/\lambda^2)$. The FPK equation governing the joint pdf of Z_1 and Z_2 is given by

$$\begin{aligned} \frac{\partial p}{\partial t} &= -\frac{\partial G_1}{\partial Z_1} - \frac{\partial G_2}{\partial Z_2} \\ G_1 &= P_1(Z_1, Z_2) - 0.5D_1 \frac{\partial p}{\partial Z_1} \\ G_2 &= P_2(Z_1, Z_2) - 0.5D_1 \frac{\partial p}{\partial Z_2}. \end{aligned} \tag{12}$$

Here, G_1 and G_2 are the so-called probability currents. For an FPK equation of the above form, the stationary state probability currents, in general, are not constants. However, for certain conditions on P_1 and P_2 called the potential conditions [13], the steady state G_1 and G_2 are identically equal to zero. These conditions require that P_1 and P_2 are obtained as gradients of a potential U , that is,

$$P_1 = \frac{\partial U}{\partial Z_1}; \quad P_2 = \frac{\partial U}{\partial Z_2}. \tag{13}$$

The necessary and sufficient condition for the existence of U is the potential condition

$$\frac{\partial P_1}{\partial Z_2} - \frac{\partial P_2}{\partial Z_1} = 0. \tag{14}$$

For the present problem one finds

$$\frac{\partial P_1}{\partial Z_2} - \frac{\partial P_2}{\partial Z_1} = \tilde{\Delta}/\lambda. \tag{15}$$

At this stage, one has to consider two cases.

Case (i) $\tilde{\Delta} = 0$

This is the special case in which the harmonic forcing frequency is equal to the frequency of limit cycle oscillation. For this case the condition of equation (14) is satisfied and accordingly equation (12) admits the following exact stationary solution

$$p(Z_1, Z_2) = N_1 \exp [(0.25\epsilon/D_1)\{2(Z_1^2 + Z_2^2) - (Z_1^2 + Z_2^2)^2\} + \{QZ_2/(\lambda D_1)\}]. \tag{16}$$

Here N_1 is the normalization constant. Thus $\tilde{\Delta} = 0$ represents the special case in which solution of equation (1) can be obtained within the framework of the stochastic averaging method. It may be noted that the above solution has earlier been obtained by Stratonovich [2] using a similar procedure.

Case (ii) $\tilde{\Delta} \neq 0$

For this case the condition of equation (14) is not satisfied and thus the exact stationary joint pdf of Z_1 and Z_2 is no longer obtainable. Thus any further attempt to analyse equation (12) would require additional approximations. Here the method of equivalent non-linearization is employed. In this method, equation (11) is replaced by

$$\begin{aligned} \dot{Z}_1 &= 0.5\epsilon Z_1(1 - Z_1^2 - Z_2^2) + k_1 + \sqrt{D_1} W_1(t) \\ \dot{Z}_2 &= 0.5\epsilon Z_2(1 - Z_1^2 - Z_2^2) + (0.5Q/\lambda) - k_2 + \sqrt{D_1} W_2(t). \end{aligned} \tag{17}$$

Based on the minimum mean square error criterion the equivalent parameters k_1 and k_2 are found to be

$$\begin{aligned} k_1 &= 0.5\tilde{\Delta}\langle Z_2 \rangle/\lambda \\ k_2 &= 0.5\tilde{\Delta}\langle Z_1 \rangle/\lambda. \end{aligned} \tag{18}$$

The new set of equations (17) satisfy the condition of equation (14) and hence the stationary pdf is given by

$$\begin{aligned} p(Z_1, Z_2) &= N_2 \exp [(0.25\epsilon/D_1)\{2(Z_1^2 + Z_2^2) - (Z_1^2 + Z_2^2)^2\} \\ &\quad + \{QZ_2/(\lambda D_1)\} + (2/D_1)(k_1 Z_1 - k_2 Z_2)] \end{aligned} \tag{19}$$

where N_2 is the normalization constant. The equivalent parameters k_1 and k_2 have to be determined by solving a pair of non-linear equations obtained by combining equations (18) and (19). It can be noted that in equation (19) if $\tilde{\Delta}$ is set to zero, one gets back equation (16) and if both $\tilde{\Delta}$ and Q are set to zero, the pdf is identical to the first order stochastic averaging solution for the case when only noise acts on the system [5]. It also follows from equations (5), (9) and (19) that

$$\begin{aligned} p(x, \dot{x}, t) &= \lambda N_2 \exp \{ (0.25\epsilon/D_1)[2(x^2 + \dot{x}^2/\lambda^2) - (x^2 + \dot{x}^2/\lambda^2)^2] \\ &\quad + [Q/(\lambda D_1)][x \sin \lambda t + (\dot{x}/\lambda) \cos \lambda t] \\ &\quad + (2k_1/D_1)[x \cos \lambda t - (\dot{x}/\lambda) \sin \lambda t] \\ &\quad - (2k_2/D_1)[x \sin \lambda t + (\dot{x}/\lambda) \cos \lambda t] \} \\ p(A, \psi) &= N_2 A \exp [(0.25\epsilon/D_1)(2A^2 - A^4) - [QA \sin \psi/(\lambda D_1)] \\ &\quad + A(2/D_1)(k_1 \cos \psi - k_2 \sin \psi)]. \end{aligned} \tag{20, 21}$$

Thus it may be observed that although Z_1 and Z_2 reach stationarity, the response process $x(t)$ is still non-stationary. Further, it can also be noted that in this approximation the process $[x(t) - \langle x \rangle]$ is non-stationary.

3. NUMERICAL RESULTS AND DISCUSSION

Numerical results are obtained based on the stochastic averaging solution given by equation (16), the combined averaging and non-linearization solution of equation (19) and the gaussian closure solution given by equation (3). The results are presented for different values of the detuning parameter $\Delta = (1 - \lambda^2)/(\epsilon\lambda)$, noise parameter $N = 16(D/\epsilon)$ and harmonic excitation level $H = Q^2/(\epsilon\lambda)^2$. As noted earlier the solution of equation (1) within the framework of the method of stochastic averaging is obtainable only for the special case of $\Delta = 0$. This solution is compared with the corresponding gaussian closure approximation in Figs 1 and 2. For $\Delta \neq 0$, the solution based on the combined averaging and non-linearization is compared in Fig. 3 with the closure solution. Figure 3a also shows the mean response for the case of $N = 0$, which corresponds to the response when no noise acts on the system. The mean and the variance of $x(t)$ as per equation (9) is given by

$$\begin{aligned} \langle x \rangle &= \langle Z_1 \rangle \cos \lambda t + \langle Z_2 \rangle \sin \lambda t \\ \sigma_x^2(t) &= [\langle Z_1^2 \rangle - \langle Z_1 \rangle^2] \cos^2 \lambda t + [\langle Z_2^2 \rangle - \langle Z_2 \rangle^2] \sin^2 \lambda t \\ &\quad + [\langle Z_1 Z_2 \rangle - \langle Z_1 \rangle \langle Z_2 \rangle] \sin 2\lambda t. \end{aligned} \tag{22}$$

Thus, in the averaging and the combined averaging and non-linearization approximations mean response is periodic with amplitude $R^2 = \langle Z_1 \rangle^2 + \langle Z_2 \rangle^2$ and the response variance is time varying. It must be noted that the response variance shown in Figs 1-3 are the temporal average over $(0, 2\pi/\lambda)$ of the variance given in equation (22). From the numerical results presented it can be observed that both the closure and averaging solutions show qualitatively identical behaviour. The addition of noise for a given level of harmonic excitation is seen to reduce the mean response amplitude and increase the variance. At resonance the periodic term controls the response leading to higher mean amplitudes and lower variance levels. However, as the external frequency is changed the noise effects become more important as the mean reduces and the variance increases. The scope of these

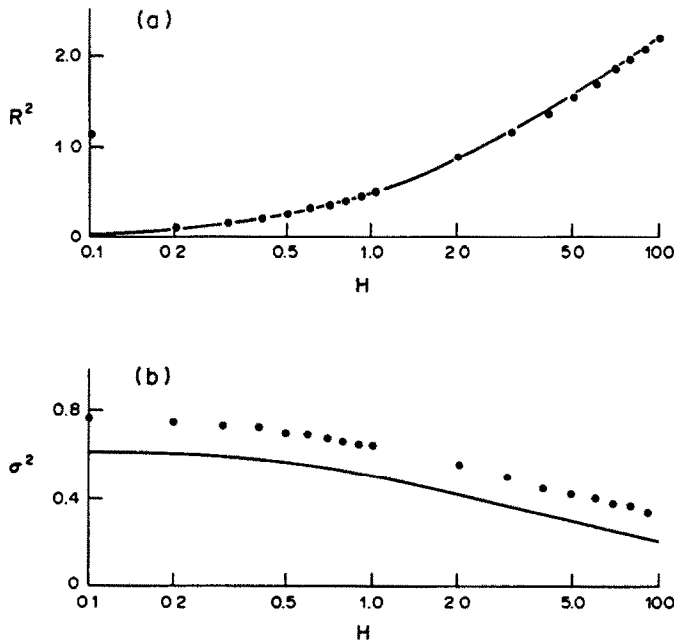


Fig. 1. Steady state moments (a) mean (b) variance, — stable gaussian closure solution, ● stochastic averaging. $\Delta = 0, \epsilon = 0.1, N = 16$.

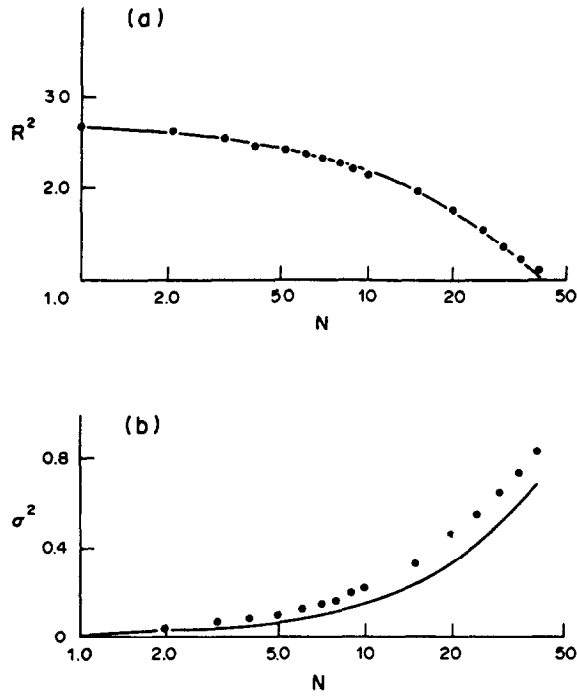


Fig. 2. Steady state moments (a) mean (b) variance, — stable gaussian closure solution, ● stochastic averaging. $\Delta = 0$, $\epsilon = 0.1$, $H = 8.0$.

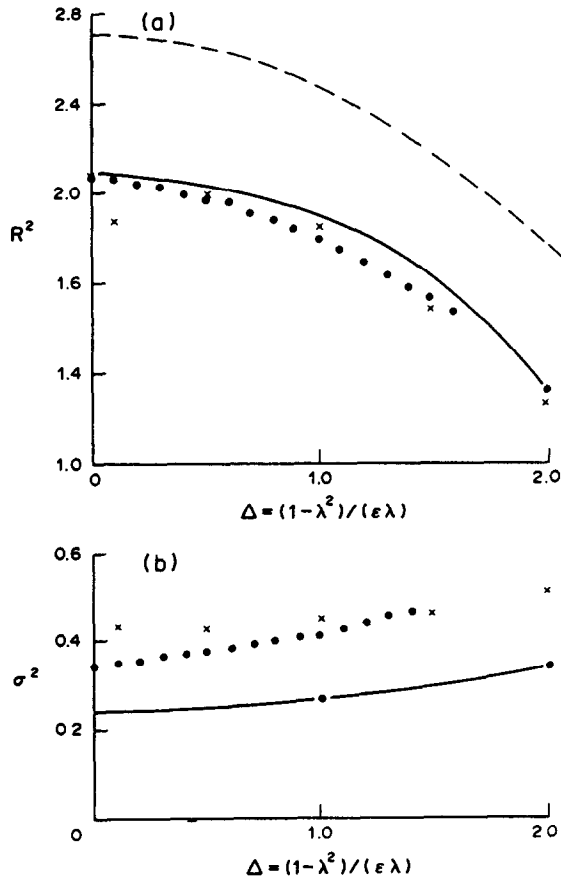


Fig. 3. Steady state moments (a) mean (b) variance, — stable gaussian closure solution, ● combined averaging and non-linearization. $\epsilon = 0.1$, $H = 8.0$, $N = 16.0$, × simulation, ---- deterministic solution for $N = 0$.

approximations has further been examined through numerical simulations. For this purpose, the time variable in equation (1) is transformed to $\tau = t/(2\pi)$ to get

$$x'' - 2\pi\epsilon x'(1 - 4x^2) + 4\pi^2 x = 4\pi^2 Q \cos 2\pi\lambda\tau + W(\tau)$$

$$\langle W(\tau_1)W(\tau_2) \rangle = 16\pi^3 D\delta(\tau_2 - \tau_1). \quad (23)$$

This equation is solved by a fourth order Runge–Kutta algorithm for 100 samples of digitally simulated white noise process. The time histories of ensemble average and variance for a length of 50 cycles are obtained. The amplitude of the mean in the last cycle is shown in Fig. 3(a). The mean of the ensemble variance in the last two cycles is taken as the estimate of the stationary variance. This variance is presented in Fig. 3(b). It may be observed that qualitatively the theoretical predictions and the simulations compare very well. The closure approximation is found to underestimate the response, while the estimation of the mean response is fairly accurate. On the other hand the solution based on combined averaging and non-linearization shows a better comparison with the simulated results.

4. SUMMARY AND CONCLUSIONS

The behaviour of Van der Pol's oscillator under periodic excitation is well known. The effect of adding noise onto the excitation has been studied in this paper. For this purpose, a combination of methods of stochastic averaging and equivalent non-linearization has been employed. It may be noted that the FPK equation corresponding to equation (1) has time varying drift coefficients due to the presence of a periodic term in the excitation. Thus a stationary solution for this equation does not exist. Hence the method of equivalent non-linearization as developed by Caughey [11] which depends on the existence of exact stationary solutions of FPK equations is not directly applicable in analysing equation (1). On the other hand, equation (1) can be readily handled using the stochastic averaging technique. In many random vibration problems the method leads to a one-dimensional approximation to the response amplitude and significantly simplifies the solution procedure. In the present problem, however, the simplified equation for response amplitude is coupled with the equation for the phase and a solution for the joint pdf of amplitude and phase is not possible in general. Thus one would see that within the individual frameworks of methods of stochastic averaging and equivalent non-linearization it is not possible to obtain an approximate solution of equation (1). On the other hand, by combining these two techniques one can gain the advantages of both these methods. Thus in the present study the pair of simplified equations for response variables obtained using the stochastic averaging technique is further analysed using the method of equivalent non-linearization. This leads to a non-gaussian estimate for the joint pdf of x and \dot{x} . In contrast with the linearization solution this approximation is valid even in the limit of $Q \rightarrow 0$. The response moments obtained using this method are observed to show better comparison with the simulation results than the gaussian closure solution and point towards the usefulness of this approach.

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