Available online at www.sciencedirect.com
ScienceDirect
Probabilistic Engineering Mechanics 24 (2009) 60–74
PROBABILISTIC ENGINEERING MECHANICS
www.elsevier.com/locate/probengmech

A Kalman filter based strategy for linear structural system identification based on multiple static and dynamic test data

R. Tipireddy[a], H.A. Nasrellah[b,c], C.S. Manohar[c,*]

[a] GE India Technology Centre, Bangalore, India
[b] Al zaeim Al azhari University, Khartoum, Sudan
[c] Department of Civil Engineering, Indian Institute of Science, Bangalore 560 012, India

Received 29 March 2007; received in revised form 3 January 2008; accepted 18 January 2008
Available online 3 February 2008

Abstract

The problem of identification of stiffness, mass and damping properties of linear structural systems, based on multiple sets of measurement data originating from static and dynamic tests is considered. A strategy, within the framework of Kalman filter based dynamic state estimation, is proposed to tackle this problem. The static tests consists of measurement of response of the structure to slowly moving loads, and to static loads whose magnitude are varied incrementally; the dynamic tests involve measurement of a few elements of the frequency response function (FRF) matrix. These measurements are taken to be contaminated by additive Gaussian noise. An artificial independent variable τ, that simultaneously parameterizes the point of application of the moving load, the magnitude of the incrementally varied static load and the driving frequency in the FRFs, is introduced. The state vector is taken to consist of system parameters to be identified. The fact that these parameters are independent of the variable τ is taken to constitute the set of 'process' equations. The measurement equations are derived based on the mechanics of the problem and, quantities, such as displacements and/or strains, are taken to be measured. A recursive algorithm that employs a linearization strategy based on Neumann's expansion of structural static and dynamic stiffness matrices, and, which provides posterior estimates of the mean and covariance of the unknown system parameters, is developed. The satisfactory performance of the proposed approach is illustrated by considering the problem of the identification of the dynamic properties of an inhomogeneous beam and the axial rigidities of members of a truss structure.

Keywords: System identification; Dynamic state estimation; Kalman filter

1. Introduction

The problem of structural system identification lies at the heart of condition assessment of existing structures and in developing structural health monitoring strategies. This class of problems constitutes inverse problems, in which properties of the structure need to be estimated based on noisy data for then applied forces and a limited set of response measurements. These problems are closely associated with problems of finite element (FE) model updating [9,23] and structural damage detection using response data [5]. These problems have received wide attention in the broader context of engineering dynamical systems [8,26,20]. One of the important mathematical tools that form the basis of the development of structural system identification methods is the Kalman filter [19,18,3,13]. The Kalman filter and its variants have been widely used in the development of structural system identification strategies for both linear and nonlinear dynamical systems [35,15,17,10,32]. The Kalman filter provides the exact solution to the problem of state estimation when process and measurement equations are linear and noises are additive and Gaussian. When these conditions are not met, one can develop suboptimal strategies based on linearization or transformation methods [3,27], or, alternatively, employ Monte Carlo simulation strategies to solve the problem [7]. The application of the latter class of approaches to structural engineering problems has been recently attempted by a few authors. Thus, Ching et al. [4], have applied a stochastic simulation based filtering technique, namely, the sequential importance sampling based method as developed by Doucet et al. [6], and an extended Kalman filter

* Corresponding author. Tel.: +91 80 2293 3121; fax: +91 80 2360 0404.
E-mail address: manohar@civil.iisc.ernet.in (C.S. Manohar).

0266-8920/$ - see front matter
doi:10.1016/j.probengmech.2008.01.001

(EKF) for identifying parameters of three different classes of dynamical systems. Manohar and Roy [21] have applied three simulation-based filtering strategies to the problem of system parameter identification in two typical nonlinear oscillators, namely, the Duffing Coulomb oscillators. The filters that these authors have investigated included: the density based Monte Carlo filter as developed by Tanizaki [33], the Bayesian bootstrap algorithm due to Gordon et al. [12] and the sequential importance sampling based method as developed by Doucet et al. [6]. The application of the Rao–Blackwell theorem in conjunction with a time domain substructuring scheme to identify localized nonlinearities has been investigated by Sajeeb et al. [28]. Namdeo and Manohar [22] have developed a bank of self-learning particle filters for the identification of parameters of nonlinear systems. The recent work by Ghosh et al. [11] outlines a particle filtering procedure to deal with nonlinear measurement models and (or) additive /multiplicative non-Gaussian noises.

Problems of system identification when structural response to only static data is available have been considered by several authors. We cite here a few representative studies. Hoshiya and Sutoh [16] consider problems of system identification in the context of geotechnical engineering problems and combine the finite element method with a Kalman filter-weighted local iteration procedure. Banan etal [1,2] consider the problem of estimating elastic constants of a structural model based on measured displacements under known static loads. These authors minimize an index of discrepancy between experimental analytical model predictions on nodal forces or displacements at measured sites. Sanayei and Saletnik [29] discuss the relative advantages of using strain measurements over measuring displacements, in the context of system identification using static test data. These authors have developed a method that optimizes a quadratic performance error formed using difference between the analytical and measured strains. In a subsequent paper these authors [30] have discussed the accuracy of the system identification *vis-à-vis* the presence of noise in measurements and in the selection of measurement locations. The performance of three alternative identification schemes based on the measurement of strains and displacements has been investigated by Sanayei et al. [31]. This study is based on experiments conducted on a steel frame and demonstrates the successful updating of parameters of the computational model. Yeo et al. [34] illustrate the use of regularization methods in context of damage detection in structures using static test data. Hjelstad and Shin [14] discuss a parameter grouping scheme that enables the identification of the location of structural damage. Paola and Bilello [25] note that a variation in the bending stiffness of a linear elastic beam can be modeled as a superimposed curvature depending on the variation of the flexural rigidity and the applied bending moment. An integral equation based formulation is subsequently proposed for damage detection in such beams. Nejad et al. [24] employ a nonlinear optimization scheme to detect changes in the elastic properties of structures based on static test data. The objective function here is defined in terms of load vectors of damaged and undamaged structures. These authors also propose schemes for the selection of the measurement and driving points that enhance the performance of the damage detection algorithm.

In a condition assessment study of existing engineering structures, it is typically possible to measure the structural response to both statically or dynamically applied loads. Thus, for instance, in the context of the condition assessment of railway bridges, it is possible to measure the structural response when a wagon formation with a known weight distribution can be made to roll across the bridge in a quasi-static manner: this leads to the measurement of influence lines for response quantities such as the strains and displacements at various points on the bridge. Similarly, by parking a wagon on the bridge at a set of specific positions, and, by varying the payload of the wagon, it is possible to measure the response as a function of incrementally varied static load. By allowing a relatively light vehicle to run on the bridge at various velocities the dynamic response of the bridge could be measured. Alternatively, the frequency response functions (FRFs) of the bridge structure can be measured by using modal shakers or an automated sledge hammer. The collection of vibration signatures when operating trains pass the bridge provides data on dynamics of the bridge–train interacting system. The free vibration decay that follows the exit of the trains provides useful data for the estimation of the modal characteristics of the bridge. These data would invariably be spatially incomplete and corrupted by measurement noise. One of the challenges that have to be faced in the problem of the condition assessment of existing bridges, in this context, lies in the ability to handle and assimilate a large amount of noisy measurements in the problem of the identification of system parameters. Specifically, it needs to be appreciated that a part of the data originates from static structural behavior, in which case, the governing equilibrium equations are algebraic in nature; and, conversely, the data from vibration behavior are associated with a set of differential equations in time. The motivation for the present study lies in these considerations and we propose a strategy to assimilate data from diverse testing procedures in a unified manner within the framework of dynamic state estimation procedures. To achieve this, we introduce an artificial independent variable related to the problem on hand and ‘sequence’ the estimation procedure into a recursive format. The process equations consists of statements that the structural parameters are invariant with respect to the independent variable and the measurement equations are formulated based on the governing equations of equilibrium. A Neumann’s expansion of the structural dynamic/static stiffness matrices is further carried out to linearize the measurement equations with respect to the structural parameters of interest. Furthermore, the benefits of adopting a global iteration strategy are also demonstrated. Illustrative examples on the identification of the properties of an inhomogeneous beam (involving 32 system parameters) and the identification of the axial stiffness of elements in a truss structure (involving 25 system parameters) using noisy synthetic data are presented and the methods are shown to perform satisfactorily.

2. Measurement models based on finite element analysis

We illustrate the strategy adopted in this study via typical examples as shown in Fig. 1.

Fig. 1a and b respectively show an inhomogeneous beam subjected to the action of a set of slowly moving loads and a set of static loads whose magnitude can be varied incrementally. The slowly moving loads can be interpreted as constant static loads whose point of application is varied incrementally so that they do not induce any dynamic action in the structure. Fig. 1c illustrates the scheme for the measurement of elements of the FRF matrix. Similar problems for the case of a truss structure are shown in Fig. 1d and e. These structures are taken to behave linearly under the action of the applied loads. We consider the finite element (FE) model for a linear structure with a $N \times N$ reduced global stiffness matrix K, mass matrix M and damping matrix C. Let θ be the $d \times 1$ vector of system parameters that we are interested in determining. The elements of θ are taken to collectively contain parameters related to the mass, damping and stiffness properties of the structure. If the FE model is made up of N_e elements, the structural matrices can be written as

$$K = \sum_{e=1}^{N_e} \bar{A}_e^t K_e \bar{A}_e; \qquad M = \sum_{e=1}^{N_e} \bar{A}_e^t M_e \bar{A}_e; \qquad C = \sum_{e=1}^{N_e} \bar{A}_e^t C_e \bar{A}_e \tag{1}$$

where the subscript e denotes the structural matrices for the eth element in the global coordinate system, the superscript t denotes the matrix transpose, $\bar{A}_e$ denotes the $N \times n_e$ nodal connectivity matrix for the eth element that relates u_e, the $n_e \times 1$ nodal displacement vector of the eth element, with the $N \times 1$ global displacement vector u through the relation $u_e = \bar{A}_e u$. We assume that it is possible to measure the strain response and (or) the displacement response of the structure at a set of points. The problem in hand consists of estimating elements of θ based on these measurements. The study allows for the measurements being noisy and for the underlying mathematical models being inaccurate. We outline the proposed strategy for structural system identification by considering the case of measurements under slowly moving loads, incrementally varied static loads, and FRF measurements separately and also consider the case when all these data are available simultaneously.

2.1. Measurement model for structure under moving loads

The equation of equilibrium can be written in terms of the structural flexibility matrix $S = K^{-1}$ as $u = SF$ where F is the vector of equivalent nodal forces. If we now consider the structure to be acted upon by a static force P whose point of application is varied incrementally (see Fig. 1a), the equilibrium equation can be written as

$$u(a) = S(\theta)F(a) + \tilde{\xi}(a). \tag{2}$$

Here $\tilde{\xi}(a)$ denotes a random process evolving in a that represents the effect of unmodeled aspects of structural mechanical behavior. The source of this term, for instance, could be originating from idealizations made in the form of constitutive laws (*e.g.*, assumption of isotropy, homogeneity, and linearity), postulated displacement fields (*e.g.*, Euler–Bernoulli's beam hypothesis) or in modeling of joint flexibility and boundary conditions. If we measure only a $q \times 1$ subset of $u(a)$, the model for the measurement could be written as

$$y_j(a,\theta) = \sum_{i=1}^{N} S_{ji}(\theta)F_i(a) + \tilde{\xi}_j(a) + \bar{\xi}_j(a); \quad j = 1, \dots, q. \tag{3}$$

Here $\bar{\xi}(a)$ is a $q \times 1$ vector random process evolving in the parameter a that models the measurement noise associated with the measurement of $\left\{y_j(a,\theta)\right\}_{j=1}^{q}$. The range of values taken by a is given by 0 to $L = L_B + L_V$(see Fig. 1a).

In addition to displacements, if a set of strain components, denoted by $\{\varepsilon_i\}_{i=1}^{s}$ are also measured, a model for these measurements can also be obtained in terms of the system flexibility matrix. For the purpose of illustration we consider the truss structure shown in Fig. 1b and consider the eth element as in Fig. 2. Let ε_e denote the axial strain in this element measured in the element local coordinate system. The element stiffness matrix in the global coordinate system, K_e, is related to the matrix in the local coordinate system through the transformation matrix T_e via the relation $K_e = T_e^t K_e^{(l)} T_e$ where

$$K_e^{(l)} = \frac{A_e E_e}{L_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad T_e = \begin{bmatrix} \cos\phi & \sin\phi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \end{bmatrix}. \tag{4}$$

The element displacement vector in global coordinates given by $u_e^{(g)}$ is related to the global displacement vector u through the relation $u_e^{(g)} = \bar{A}_e u$. The element displacement vector in local coordinates is thus given by $u_e^{(l)} = T_e u_e^{(g)} = T_e \bar{A}_e u$. Furthermore, the axial strain is related to the vector $u_e^{(l)}$ through the relation $\varepsilon_e = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix} \left\{u_e^{(l)}\right\}$. Thus the measurement equation for the strain here can be written as

$$\varepsilon_e(a,\theta) = B_e T_e \bar{A}_e u + \tilde{\mu}(a). \tag{5}$$

Here $\tilde{\mu}(a)$ is a random process that models the measurement noise associated with the strain measurement and $B_e = \begin{bmatrix} \frac{-1}{L} & \frac{1}{L} \end{bmatrix}$. By combining Eq. (2) with Eq. (5) we can write

$$\varepsilon_e(a,\theta) = B_e T_e \bar{A}_e \left\{S(\theta)F(a) + \tilde{\xi}(a)\right\} + \tilde{\mu}_e(a). \tag{6}$$

This form of equation can be could be derived for any given strain component in a given structure. We rewrite Eqs. (3) and (6) as

$$y_j(a,\theta) = \sum_{i=1}^{N} S_{ji}(\theta)F_i(a) + \xi_j(a); \quad j = 1, 2, \dots, q$$

$$\varepsilon_e(a,\theta) = B_e T_e \bar{A}_e S(\theta)\, F(a) + \mu_e(a); \quad e = 1, 2, \dots, s \tag{7}$$

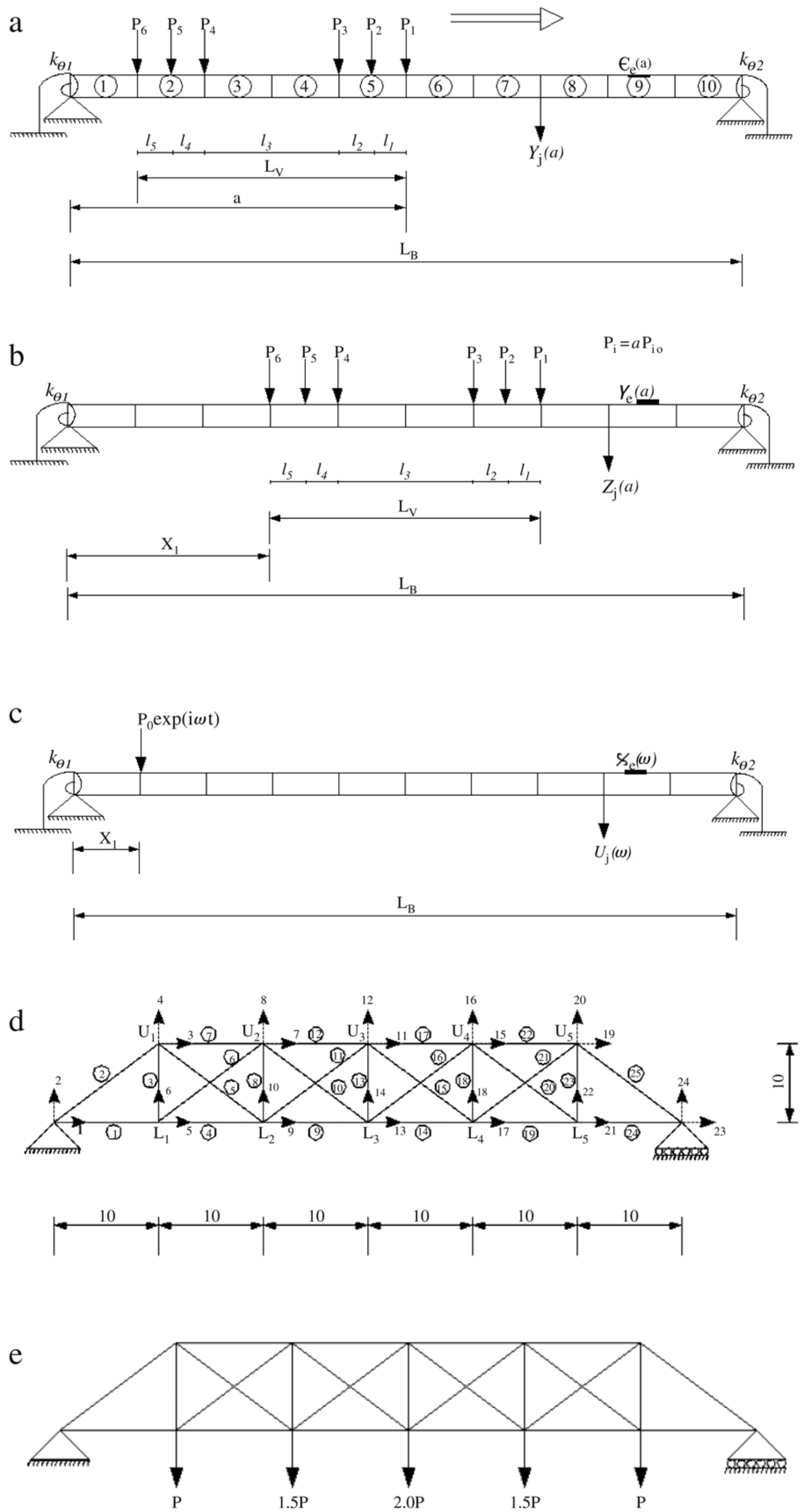

Fig. 1. Example structures considered in the study; (a) beam subjected to quasi-static moving loads; (b) beam subjected to incrementally varied static loads; (c) beam subjected to harmonic excitation; (d) truss structure under study; (e) loading configuration for incrementally varied static load; all dimensions in m.

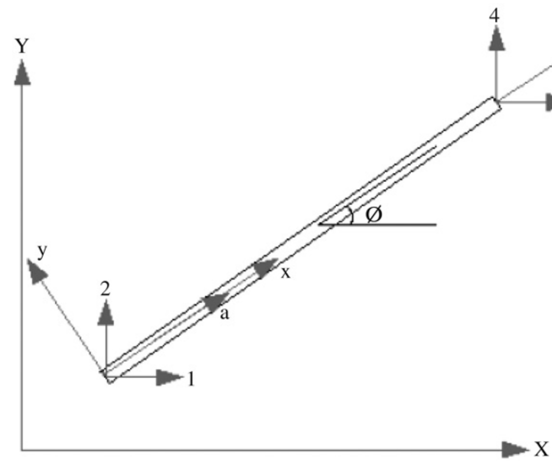

Fig. 2. Axially deforming truss element.

where $\xi_j(a) = \tilde{\xi}_j(a) + \bar{\xi}_j(a)$ and $\mu_e(a) = B_e T_e \bar{A}_e \tilde{\xi}(a) + \tilde{\mu}_e(a)$.

2.2. Measurement model for structure under incrementally varied static load

The above formulation has been presented for the case of structure subjected to a static load whose point of application is incrementally varied. Similar measurement equations can also be obtained if the points of application of the loads are held fixed and their magnitude varied in increments; see Fig. 1b and e. If α is the parameter that characterizes the load increments, the measurement equations here would be identical to Eq. (7) except that the parameter a is replaced by α and, accordingly, we get the equations

$$z_j(\alpha,\theta) = \sum_{i=1}^{N} S_{ji}(\theta) F_i(\alpha) + \varsigma_j(\alpha); \quad j = 1, 2, \ldots, q$$
$$\gamma_e(\alpha,\theta) = B_e T_e \bar{A}_e S(\theta) F(\alpha) + \rho_e(\alpha); \quad e = 1, 2, \ldots, r \quad (8)$$

where z and γ represent, respectively, the displacement and strain components.

2.3. Measurement model for FRFs

The equation of dynamic equilibrium in the steady state, expressed in the frequency domain, is given by

$$U(\omega,\theta) = \Xi(\omega,\theta) F(\omega) + \tilde{\vartheta}(\omega) \quad (9)$$

where $\Xi(\omega,\theta) = \left[-\omega^2 M + \mathrm{i}\omega C + K\right]^{-1}$ is the structural receptance matrix and $\tilde{\vartheta}(\omega)$ is a $N \times 1$ complex valued vector random process evolving in parameter ω which represents the effect of mathematical inaccuracies in the formulation of the above model. By writing $U = U^R + \mathrm{i}U^I$ and separating the real and imaginary parts, we can rewrite the above equations as

$$\begin{Bmatrix} U^R(\omega) \\ U^I(\omega) \end{Bmatrix} = \begin{bmatrix} K - \omega^2 M & -\omega C \\ \omega C & K - \omega^2 M \end{bmatrix}^{-1} \begin{Bmatrix} F^R(\omega) \\ F^I(\omega) \end{Bmatrix} + \begin{Bmatrix} \tilde{\vartheta}^R(\omega) \\ \tilde{\vartheta}^I(\omega) \end{Bmatrix}. \quad (10)$$

This equation is further written in the form

$$\begin{aligned} U^R(\omega,\theta) &= \Xi^R(\omega,\theta) F^R(\omega) - \Xi^I(\omega,\theta) F^I(\omega) + \tilde{\vartheta}^R(\omega) \\ U^I(\omega,\theta) &= \Xi^R(\omega,\theta) F^I(\omega) + \Xi^I(\omega,\theta) F^R(\omega) + \tilde{\vartheta}^I(\omega). \end{aligned} \quad (11)$$

Here the superscripts R and I denote, respectively, the real and imaginary parts. As before, if we now assume that we measure only a $q \times 1$ subset of displacement dofs, the above equation can be recast as

$$\begin{aligned} V_i^R(\omega,\theta) &= \sum_{l=1}^{N} \left[\Xi_l^R(\omega,\theta) F_l^R(\omega) - \Xi_l^I(\omega,\theta) F_l^I(\omega)\right] + \tilde{\vartheta}_i^R(\omega) + \varphi_i^R(\omega) \\ V_i^I(\omega,\theta) &= \sum_{l=1}^{N} \left[\Xi_l^R(\omega,\theta) F_l^I(\omega) + \Xi_l^I(\omega,\theta) F_l^R(\omega)\right] + \tilde{\vartheta}_i^I(\omega) + \varphi_i^I(\omega); \quad i = 1, 2, \ldots q. \end{aligned} \quad (12)$$

Here $\varphi_i^R(\omega)$ and $\varphi_i^I(\omega)$ are random processes that account for noise in measuring $V_i^R(\omega)$ and $V_i^I(\omega)$ respectively. In the further work we write $\vartheta_i^R(\omega) = \tilde{\vartheta}_i^R(\omega) + \phi_i^R(\omega)$ and $\vartheta_i^I(\omega) = \tilde{\vartheta}_i^I(\omega) + \phi_i^I(\omega)$. In formulating the above equations, it is assumed that we are measuring the displacement response. However if quantities such as mobility or accelerance are measured, the function $\Xi(\omega)$ can be replaced by $j\omega\Xi(\omega)$ or $-\omega^2\Xi(\omega)$ respectively, and the model for measurement would be similar to that provided in Eq. (12). Following the steps outlined in Section 2.1 it is also possible to derive the expressions for real and imaginary parts of FRFs associated with the strain measurements and these can be shown to be given by

$$\begin{aligned} \chi_e^R(\omega,\theta) &= B_e T_e \bar{A}_e \left[\Xi^R(\omega,\theta) F^R(\omega) - \Xi^I(\omega,\theta) F^I(\omega)\right] + \lambda_e^R(\omega); \quad e = 1, 2, \ldots, r \\ \chi_e^I(\omega,\theta) &= B_e T_e \bar{A}_e \left[\Xi^R(\omega,\theta) F^I(\omega) + \Xi^I(\omega,\theta) F^R(\omega)\right] + \lambda_e^I(\omega); \quad e = 1, 2, \ldots, r. \end{aligned} \quad (13)$$

Here $\lambda_e^R(\omega)$ and $\lambda_e^I(\omega)$ are two random processes which represent the combined effect of measurement noise and modeling errors: see Eq. (7).

3. Identification problem

The identification problem on hand consists of determining the $d \times 1$ vector θ of system parameters based on the information contained in measurements as in Eqs. (7), (8), (12) and (13). The objective of the present study is to develop a strategy to enable the application of dynamic state estimation methods to solve this problem. To achieve this, we first need to address two issues: firstly, the definition of an independent variable, and, secondly, the formulation of a process equation. When measurements are available from only one source of experimentation, namely, testing under moving loads or testing under incrementally varied static loads, the parameters a or α could serve as the independent variable. Similarly, if measurements are available only on FRFs, the parameter ω can serve as the independent variable. In each of these situations the process equation could be formulated by stating the fact that the variables $\{\theta\}_{i=1}^{d}$ are independent of the relevant independent variable. However, when results from more than one test are available, it is not obvious which parameter would serve as the independent variable. To address this difficulty we introduce a dummy independent variable τ such that $0 \le \tau \le 1$ and parameterize a, α and ω in terms of τ as

$$a = \frac{\tau}{L}; \qquad \alpha = (\alpha_{\max} - \alpha_{\min})\,\tau + \alpha_{\min}$$
$$\text{and} \quad \omega = (\omega_{\min} - \omega_{\max})\,\tau + \omega_{\min} \tag{14}$$

so that, as τ is varied between 0 and 1, the parameters a, α, and ω vary across their respective range of values of 0 to L, $\alpha_{\min}$ to $\alpha_{\max}$, and $\omega_{\min}$ to $\omega_{\max}$. The process equation states now the fact that θ is independent of τ and, accordingly, we get the equations

$$\frac{\mathrm{d}\theta_i}{\mathrm{d}\tau} = 0; \quad i = 1, 2, \ldots d; \; \theta_i(0) = \theta_{i0}. \tag{15}$$

It may be noted that the process equation is free from noise terms since, the statement that θ is independent of τ, does not contain any modeling approximations. To develop the identification procedure, we discretize τ using $\tau_k = k\Delta\tau; k = 0, 1, 2, \ldots, n_\tau$ so that $n_\tau \Delta\tau = 1$. This automatically implies that the parameters a, α and ω are also discretized into n_τ states across their respective range of values. The process equation and the measurement equations can now be written in the discretized form respectively as follows:

$$\theta_{i,k+1} = \theta_{i,k}; \quad i = 1, 2, \ldots, d; \; k = 1, 2, \ldots n_\tau \tag{16}$$

$$y_j\left(\frac{\tau_k}{L}, \theta_k\right) = \sum_{i=1}^{N} S_{ij}(\theta_k) F_i\left(\frac{\tau_k}{L}\right) + \xi_j\left(\frac{\tau_k}{L}\right); \quad j = 1, 2, \ldots, q$$

$$\varepsilon_e\left(\frac{\tau_k}{L}, \theta_k\right) = B_e T_e \bar{A}_e S(\theta_k)\, F\left(\frac{\tau_k}{L}\right) + \mu_e\left(\frac{\tau_k}{L}\right); \quad e = 1, 2, \ldots, r$$

$$\begin{aligned} z_j\,[(\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}, \theta_k] &= \sum_{i=1}^{N} S_{ij}(\theta_k) F_i\,((\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}) \\ &\quad + \varsigma_j\,((\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}); \quad j = 1, 2, \ldots, q \end{aligned}$$

$$\begin{aligned} \gamma_e\,[(\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}, \theta_k] &= B_e T_e \bar{A}_e S(\theta_k)\, F\,((\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}) \\ &\quad + \rho_e\,((\alpha_{\max} - \alpha_{\min})\,\tau_k + \alpha_{\min}); \quad e = 1, 2, \ldots, r \end{aligned}$$

$$\begin{aligned} V_i^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) &= \sum_{l=1}^{N} \Big[\Xi_l^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k)\, F_l^R\,(\omega) \\ &\quad - \Xi_l^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F_l^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \Big] \\ &\quad + \vartheta_i^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \end{aligned}$$

$$\begin{aligned} V_i^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) &= \sum_{l=1}^{N} \Big[\Xi_l^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k)\, F_l^I\,(\omega) \\ &\quad + \Xi_l^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F_l^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \Big] \\ &\quad + \vartheta_i^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \quad i = 1, 2, \ldots q \end{aligned}$$

$$\begin{aligned} \chi_e^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) &= B_e T_e \bar{A}_e [\Xi^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \\ &\quad - \Xi^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min})] \\ &\quad + \lambda_e^R((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}); \quad e = 1, 2, \ldots, r \end{aligned}$$

$$\begin{aligned} \chi_e^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) &= B_e T_e A_e [\Xi^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}) \\ &\quad + \Xi^I\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k) \\ &\quad \times F^R\,((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min})] \\ &\quad + \lambda_e^I((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}); \quad e = 1, 2, \ldots, r. \end{aligned} \tag{17}$$

The above equations are in a form that is amenable for solution via the dynamic state estimation methods with Eq. (16) representing the set of process equations and Eq. (17) the measurement equations. Each of the random vector quantities $\xi_j, \varsigma_j, \mu_e, \rho_e, \vartheta_i^R, \vartheta_i^I, \lambda_i^R$ and λ_i^I evolve in τ_k and are taken to have zero mean and Gaussian distribution. For distinct values of τ_k, these variables are taken to be independent; however, for a given value of τ_k, these variables are, in general, correlated and the associated covariance matrix is taken to be known. Clearly, the process equations are linear and noise free and the measurement equations are nonlinear in the state variables $\{\theta_i\}_{i=1}^{d}$. The problem in hand consists of determining the posterior probability density function $p\left(\theta_k | y_{1:k}, \varepsilon_{1:k}, z_{1:k}, \gamma_{1:k}, V_{1:k}^R, V_{1:k}^I, \chi_{1:k}^R, \chi_{1:k}^I\right)$ where the subscript 1:k denotes the measurements 'up to' $\tau = \tau_k$. As is well known, in problems of dynamic state estimation, when process and measurement equations are linear in the state variables, and, when the noises are additive and Gaussian, the Kalman filter provides the exact solution. Clearly, in the present context we do not have this advantage, and, therefore, we need to resort to alternative strategies. One possibility is to employ Monte Carlo simulation based filtering strategies, and, the other is to linearize the measurement equations and apply the Kalman filter in an iterative manner. We explore in this study the second alternative.

$$\Psi(\theta) = \Psi_0(\theta_0) + \Delta\Psi(\Delta\theta)$$
$$\Psi_0 = \begin{bmatrix} \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \left(K_{e0} - \omega^2 M_{e0}\right) T_e \bar{A}_e & -\omega \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t C_{e0} T_e \bar{A}_e \\ \omega \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t C_{e0} T_e \bar{A}_e & \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \left(K_{e0} - \omega^2 M_{e0}\right) T_e \bar{A}_e \end{bmatrix}$$
$$\Delta\Psi = \begin{bmatrix} \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \left(\Delta K_{e0} - \omega^2 \Delta M_{e0}\right) T_e \bar{A}_e & -\omega \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \Delta C_{e0} T_e \bar{A}_e \\ \omega \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \Delta C_{e0} T_e \bar{A}_e & \sum_{s=1}^{N_e} \bar{A}_e^t T_e^t \left(\Delta K_{e0} - \omega^2 \Delta M_{e0}\right) T_e \bar{A}_e \end{bmatrix}$$

Box I.

4. Solution by linearization, Kalman filtering and global iterations

The nonlinearity in the measurement equation (17) arises due to the nonlinear dependence of the flexibility matrix $S(\theta)$ and FRF matrix $\Xi(\omega, \theta)$ on θ. With a view to developing an approximate strategy to solve the state estimation problem, we linearize $S(\theta)$ and $\Xi(\omega, \theta)$ around an initial guess θ_0 on θ so that the resulting problem becomes amenable for solution via Kalman filtering. To achieve this, we note that $S(\theta) = K^{-1}(\theta)$ and $\Xi(\omega, \theta) = \left[-\omega^2 M + \mathrm{i}\omega C + K\right]^{-1}$ with the structural matrices M, K and C assembled as in Eq. (1). In Eq. (1) it has been assumed that the matrices T_e and $\bar{A}_e$ do not depend upon the system parameter vector θ. With an initial guess $\theta = \theta_0$, we can write $K_e^{(i)} = K_{e0}^{(l)} + \Delta K_e^{(l)}$ which leads to

$$K(\theta) = K_0(\theta_0) + \Delta K(\Delta\theta)$$
$$K_0 = \sum_{e=1}^{N_e} \bar{A}_e^t T_e^t K_{e0}^{(l)} T_e \bar{A}_e$$
$$\Delta K = \sum_{e=1}^{N_e} \bar{A}_e^t T_e^t \Delta K_{e0}^{(l)} T_e \bar{A}_e. \quad (18)$$

Now, using the Neumann expansion, we can write

$$S(\theta) = K^{-1}(\theta) = [K_0 + \Delta K]^{-1}$$
$$\approx K_0^{-1}(\theta_0) - K_0^{-1}(\theta_0)\, \Delta K(\Delta\theta)\, K_0^{-1}(\theta_0). \quad (19)$$

Similar expansion for the matrix (see Eq. (10))

$$\Psi(\omega) = \begin{bmatrix} K - \omega^2 M & -\omega C \\ \omega C & K - \omega^2 M \end{bmatrix} \quad (20)$$

leads to

$$\begin{bmatrix} \Xi^R & -\Xi^I \\ \Xi^I & \Xi^R \end{bmatrix} = \Psi^{-1}(\omega, \theta) = [\Psi_0 + \Delta\Psi]^{-1}$$
$$\approx \Psi_0^{-1}(\omega, \theta_0) - \Psi_0^{-1}(\omega, \theta_0)\Delta\Psi(\omega, \Delta\theta)\, \Psi_0^{-1}(\omega, \theta_0). \quad (21)$$

For the values of $\Psi(\theta)$, Ψ_0 and $\Delta\Psi$, see Box I.

We further introduce the notations

$$\begin{bmatrix} \Xi_0^R & -\Xi_0^I \\ \Xi_0^I & \Xi_0^R \end{bmatrix} = \Psi_0^{-1}(\omega, \theta_0)$$

$$\begin{bmatrix} \Delta\Xi_0^R & -\Delta\Xi_0^I \\ \Delta\Xi_0^I & \Delta\Xi_0^R \end{bmatrix}$$
$$= -\Psi_0^{-1}(\omega, \theta_0)\Delta\Psi(\omega, \Delta\theta)\, \Psi_0^{-1}(\omega, \theta_0). \quad (22)$$

It may be noted that ΔK and $\Delta\Psi$ in these equations depend linearly on $\Delta\theta$. Consequently, $S(\theta)$ and $\Xi(\omega, \theta)$, as given by the above equations, become linear in $\Delta\theta$ while remaining nonlinear in θ_0. With this assumption in place, the process equations are given by

$$\Delta\theta_{i,k+1} = \Delta\theta_{i,k}; \quad i = 1, 2, \ldots, d; \; k = 0, 1, 2, \ldots n_\tau \quad (23)$$

and, similarly, the measurement equations are obtained for $k = 1, 2, \ldots, n_\tau$ as

$$\tilde{y}_{jk} = y_j\left(\frac{\tau_k}{L}, \theta_k\right) - \sum_{i=1}^{N} K_{0ij}^{-1}(\theta_{0k}) F_i\left(\frac{\tau_k}{L}\right)$$
$$= -\sum_{i=1}^{N} K_{0ij}^{-1}(\theta_{0k}) \Delta K_{0ij}(\Delta\theta_{0k}) K_{0ij}^{-1}(\theta_{0k}) F_i\left(\frac{\tau_k}{L}\right)$$
$$+ \xi_j\left(\frac{\tau_k}{L}\right); \quad j = 1, 2, \ldots, q$$
$$\tilde{\varepsilon}_{ek} = \varepsilon_e\left(\frac{\tau_k}{L}, \theta_k\right) - B_e T_e \bar{A}_e K_0^{-1}(\theta_k)\, F\left(\frac{\tau_k}{L}\right)$$
$$= -B_e T_e \bar{A}_e K_0^{-1}(\theta_k)\, \Delta K(\Delta\theta_k)\, K_0^{-1}(\theta_k)\, F\left(\frac{\tau_k}{L}\right)$$
$$+ \mu_e\left(\frac{\tau_k}{L}\right); \quad e = 1, 2, \ldots, r$$
$$\tilde{z}_{ik} = z_j\left[(\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}, \theta_k\right]$$
$$- \sum_{i=1}^{N} K_{0ij}^{-1}(\theta_{0k}) F_i\left((\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}\right)$$
$$= -\sum_{i=1}^{N} K_{0ij}^{-1}(\theta_{0k}) \Delta K_{0ij}(\Delta\theta_{0k}) K_{0ij}^{-1}(\theta_{0k})$$
$$\times F_i\left((\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}\right)$$
$$+ \varsigma_j\left((\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}\right); \quad j = 1, 2, \ldots, q$$
$$\tilde{\gamma}_{ek} = \gamma_e\left[(\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}, \theta_k\right]$$
$$- B_e T_e \bar{A}_e K_0^{-1}(\theta_k)$$
$$\times F\left((\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}\right)$$
$$= -B_e T_e \bar{A}_e K_0^{-1}(\theta_k)\, \Delta K(\Delta\theta_k)\, K_0^{-1}(\theta_k)$$
$$\times F\left((\alpha_{\max} - \alpha_{\min})\tau_k + \alpha_{\min}\right)$$

$$
\begin{aligned}
&+ \rho_e \left((\alpha_{\max} - \alpha_{\min})\, \tau_k + \alpha_{\min} \right); \quad e = 1, 2, \ldots, r \\
\tilde{V}_i^R(\tau_k) &= V_i^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&- \sum_{l=1}^{N} \Big[\Xi_{0l}^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^R(\omega) - \Xi_{0l}^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \Big] \\
&= \sum_{l=1}^{N} \Big[\Delta\Xi_l^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^R(\omega) - \Delta\Xi_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \Big] \\
&+ \vartheta_i^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \\
\tilde{V}_i^I(\tau_k) &= V_i^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&- \sum_{l=1}^{N} \Big[\Xi_{0l}^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^R(\omega) + \Xi_{0l}^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \Big] \\
&= \sum_{l=1}^{N} \Big[\Delta\Xi_l^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^R(\omega) + \Delta\Xi_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F_l^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \Big] \\
&+ \vartheta_i^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right); \quad i = 1, 2, \ldots q \\
\tilde{\chi}_e^R &\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&- B_e T_e \bar{A}_e [\Xi_0^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \\
&- \Xi_0^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right)] \\
&= B_e T_e \bar{A}_e [\Delta\Xi^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \\
&- \Delta\Xi^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right)] \\
&+ \lambda_e^R\left((\omega_{\max} - \omega_{\min})\tau_k\right); \quad e = 1, 2, \ldots, r \qquad (24) \\
\tilde{\chi}_e^I &\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&- B_e T_e \bar{A}_e [\Xi_0^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \\
&+ \Xi_0^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right)] \\
&= B_e T_e \bar{A}_e [\Delta\Xi^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}, \theta_k\right) \\
&\times F^R\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right) \\
&- \Delta\Xi^I\left((\omega_{\max} - \omega_{\min})\tau_k, +\omega_{\min}, \theta_k\right) \\
&\times F^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right)] \\
&+ \lambda_e^I\left((\omega_{\max} - \omega_{\min})\tau_k + \omega_{\min}\right); \quad e = 1, 2, \ldots, r.
\end{aligned}
$$

Here $\tilde{y}_{ik}, \tilde{\varepsilon}_{ik}, \tilde{z}_{ik}, \tilde{\gamma}_{ik}, \tilde{V}_{ik}^R, \tilde{V}_{ik}^I, \tilde{\chi}_{ek}^R$, and $\tilde{\chi}_{ek}^I$ represent the so called pseudo-measurements. In the above equations both the process equations and the measurement equations have become linear in the new state variables $\{\Delta\theta_i\}_{i=1}^d$, and, consequently, the state estimation problem is now amenable for solution via the Kalman filter method. The above equations can be cast in the canonical format as

$$
\begin{aligned}
&\Delta\theta_{k+1} = I\Delta\theta_k; \quad k = 0, 1, 2, \ldots, n_\tau \\
&y_k = H_k \Delta\theta_k + \upsilon_k; \quad k = 1, 2, \ldots, n_\tau.
\end{aligned} \tag{25}
$$

Here I is the $d \times d$ identity matrix and dimension of y_k, H_k and υ_k depend upon the details of measurements available. Apart from the guess to be made on the initial state of the state vector $\Delta\theta$ and its covariance at $k = 0$, the implementation of the Kalman filter also requires a guess to be made on θ_0. With these initial guesses in place, the filter provides the estimate of the expected value and covariance of $\Delta\theta$ conditioned on the measurements made. To enhance the performance of the identification procedure, an additional step involving a global iteration is also employed in the present study. Here, the guess on θ_0 is updated at the end of a given cycle of filtering and is used as the starting guess for the next cycle of filtering. This global iteration loop is repeated till a satisfactory convergence on the expected value of $\Delta\theta$ is obtained. We close this section with the following remarks:

1. The above formulation has been presented for the case in which the structural displacement and strain responses to quasi-static moving loads, incrementally varied static loads and dynamic loads have been measured. When one or more of these measurements are not available the relevant equation in the measurement equations can be eliminated. The process equation would remain unchanged.
2. The formulation has been presented with the assumption that the variables a, α and ω have been discretized into equal number of divisions. It is often possible in practice that this may not be true. This variation can be incorporated into the proposed formulation by modifying the linearized measurement models by inserting zeros in the relevant rows of H_k matrix in Eq. (25). This aspect would be illustrated later in this paper through an example.

5. Numerical results

For the purpose of illustration of the procedure formulated in the previous section we consider two example structures as shown in Fig. 1. Fig. 1a–c show a one-span beam structure with span $L_B = 20$ m and with ten sections of equal length and with varying flexural rigidity. The support condition of the beam is taken to lie between states of being completely fixed and hinged and this is reflected by the inclusion of two rotary springs at the two ends. The flexural rigidity of the beam sections $\{EI_i\}_{i=1}^{10}$, the spring constants at the ends $K_{\theta 1}$ and $K_{\theta 2}$, and the mass per unit length of the elements $\{m_i\}_{i=1}^{10}$ are taken to be the unknown system parameters to be identified. Furthermore, the structure damping matrix is constructed as

$$C = \sum_{e=1}^{N_e} p_e \bar{A}_e^t M_e \bar{A}_e. \tag{26}$$

If p_e is independent of e, then the above model implies a mass-proportional damping matrix for the entire structure; if this is not so, then, damping matrix would be non-proportional in nature and the damping matrix is parameterized in terms of the variables $\{p_e\}_{e=1}^{N_e=10.}$. Thus the vector θ here consists of 32 elements given by $\theta = \left\{\{EI_i\}_{i=1}^{10}, \{m_i\}_{i=1}^{10}, \{p\}_{i=1}^{10}, K_{\theta 1}, K_{\theta 2}\right\}$. The beam is taken to be acted upon by quasi-static moving loads (Fig. 1a) with $P_i = 300$ kN, $i = 1, 2, \ldots, 6$, $L_v = 8$ mm, $l_1 = 1$ m, $l_2 = 1$ m, $l_3 = 4$ m, $l_4 = 1$ m, and $l_5 = 1$ m. Thus the parameter a takes value from $a = 0$ to $a = 28$ m. For the case of incrementally varied static loads (Fig. 1b), it is assumed that $P_{i0=}$300 N, $l_1 = 1$ m, $l_2 = 1$ m, $l_3 = 4$ m, $l_4 = 1$ m, and $l_5 = 1$ m. Here we take α to vary from $\alpha = 1$ to 4. For the case of dynamic load (Fig. 1c) it is assumed that $x_1 = 2$ m and driving frequency is varied from 5.0 to 900.0 rad/s at 1500 equidistant points. In all the loading cases it is assumed that the displacement response at $x_l = 2l\ m; l = 1, 2, \ldots 9$. are measured. Measurements on rotations at the support are also made but are not used in the identification steps: instead, these data are used at a later stage to check the performance of identification process. Synthetic measurements on the beam response to the three loading scenarios are generated by using FEM and these responses are seeded by samples of noise processes to represent measurement noise and modeling errors. The various noise processes are taken to have zero mean and standard deviations in the range of 1%–4% of maxima of respective response quantities in the absence of noise. In the illustrative examples the noise components are taken to be uncorrelated: it may however be noted that any information that might be available on correlations could be very well incorporated into the analysis.

In the second example, the 25 member truss structure shown in Fig. 1d and e is considered. Here Young's moduli of the 25 members are taken to be the parameters to be identified. The area of cross section of the members is taken to be 0.15600E−02 m^2. The loading scenario is taken to consist of two cases: in the first case, a load P = 10 kN traverses the truss on its lower chord in a quasistatic manner, and, in the second case, static loads, as shown in Fig. 1e, are varied incrementally over range of $P = 5$–17 kN in 61 equal steps. In the first loading case the variable a is discretized into 61 values distributed equally over the truss span of 60 m. Measurements are assumed to be made on axial strains of each of the truss member and also on transverse displacement at a few points on the lower chord, namely, L_1, L_2, L_3, L_4 and L_5 (see Fig. 1d). For the case of the incrementally varied static loads the standard deviation of the noise processes in the member strains are taken to be 6.34, 4.24, 1.83, 5.03, 2.85, 5.36, 0.19, 6.66, 1.31, 7.60, 0.54, 6.65, 1.28, 7.62, 0.27, 4.96, 2.72, 5.44, 1.89, 4.90, 4.49, 1.68, 0.47, 0.39, 1.40 μ strain. Similarly, for the case of quasi-static moving loads these quantities are taken to be 0.89, 0.59, 0.46, 0.63, 0.44, 0.65, 0.20, 0.76, 0.33, 0.85, 0.22, 0.75, 0.31, 0.86, 0.20, 0.62, 0.42, 0.67, 0.45, 0.69, 0.63, 0.27, 0.23, 0.19, 0.22. μ strain. The standard deviation of the noise in displacement responses at L_1, L_2, L_3, L_4, and L_5 are taken to be respectively 1.0E−04× (0.4343, 0.6748, 0.7974, 0.6566, 0.4146) m for both the loading scenarios. All the noise processes are assumed to be mutually uncorrelated. The performance of the identification algorithm is examined by using synthetically generated measurements on strains and displacements by solving the underlying finite element model. The reference values of the Young's modulus for the members are taken to be respectively given by $E = 2.0\text{E}{+}11\times$ [0.85, 0.90, 1.15, 1.10, 0.85, 0.90, 1.15, 1.10, 0.85, 0.90, 1.15, 1.10, 0.85, 0.90, 1.15, 1.10, 0.85, 0.90, 1.15, 1.10, 0.85, 0.85, 0.90, 1.15, 1.10] N/m^2.

5.1. Studies on the beam structure

The identification problem is solved for the following cases: case 1: the beam response to the action of moving loads (Fig. 1a) is assumed to be available, case 2: the beam response to the action of incrementally varied static loads (Fig. 1b) is assumed to be available, case 3: the beam response to the action of unit harmonic load (Fig. 1c) is assumed to be available, case 4: the responses to action of moving loads (Fig. 1a) and incrementally varied static load (Fig. 1b) are available, and case 5: the responses to the action of moving loads (Fig. 1a), incrementally varied static loads and under the action of unit harmonic load are available.

Tables 1 and 2, respectively, show the expected value and standard deviation of the system parameters after the measurements have been assimilated. Table 3 summarizes the estimates of the first six undamped natural frequencies of the system from the known numerical model and from the identified model. The system parameters here have been parameterized in the form

$$\begin{aligned} EI_i &= EI_i^0(1+\tilde{\gamma}_i); \quad i = 1, 2, \ldots, 10 \\ m_i &= m_i^0(1+\tilde{\alpha}_i); \quad i = 1, 2, \ldots, 10 \\ p_i &= p_i^0(1+\tilde{\beta}_i); \quad i = 1, 2, \ldots, 10 \\ K_{\theta 1} &= K_{\theta 1}^0(1+\tilde{\gamma}_{11}) \\ K_{\theta 2} &= K_{\theta 2}^0(1+\tilde{\gamma}_{12}) \end{aligned} \tag{27}$$

and the results of the identification studies have been reported in terms of the parameters $\tilde{\alpha}, \tilde{\beta}$, and $\tilde{\gamma}$. In the numerical work it is assumed that $EI_i^0 = 1.0\text{E}{+}11$ N m^2, $m_i^0 =$ 4625 kg/m, $p^0 = 1.3507\text{E}{+}003$ N s/m/kg for all values of i and $K_{\theta 1}^0 = 4.0\text{E}{+}11$ N m/rad, and $K_{\theta 2}^0 = 4.0\text{E}{+}11$ N m/rad. The standard deviation of the noise in the nine displacement terms are taken to be as follows: response under moving loads: 1.0E−5× (0.0191 0.0557 0.0928 0.1138 0.1293 0.1167 0.0944 0.0544 0.0193) m; response under incrementally varied static loads: 1.0E−5× (0.0729 0.2167 0.3642 0.4711 0.5113 0.4656 0.3568 0.2101 0.0707) m, and, response under unit harmonic load: 1.0E−10× (0.1347 0.2536 0.3040 0.3097 0.2748 0.2223 0.1427 0.0754 0.0232) m. Clearly, for cases 1, 2 and 4, only the stiffness properties of the system could be identified while, for cases 3 and 5, it is possible to identify the mass,

Table 1
Studies on the beam structure shown in Fig. 1a–c; results on the expected values of the system parameters after the measurements have been assimilated

Element	Reference values			Case (1)	Case (2)	Case (3)			Case (4)	Case (5)		
	α	β	γ	γ	γ	α	β	γ	γ	α	β	γ
1	1.0000	0.8000	0.9100	0.9008	0.9084	1.0416	0.7643	0.9021	0.9010	1.0416	0.7659	0.9021
2	0.9600	1.0000	0.9300	0.9285	0.9126	0.9465	1.0112	0.9258	0.9267	0.9463	1.0108	0.9257
3	0.9400	0.9500	0.9700	0.9709	0.9772	0.9415	0.9374	0.9678	0.9705	0.9417	0.9377	0.9678
4	0.9300	0.8800	1.0000	0.9990	1.0022	0.9293	0.8818	1.0026	1.0007	0.9287	0.8815	1.0026
5	1.0000	0.9600	1.0000	1.0017	0.9968	0.9855	0.9513	0.9996	1.0008	0.9856	0.9512	0.9995
6	0.9100	1.0000	0.9400	0.9348	0.9449	0.9184	1.0037	0.9357	0.9348	0.9182	1.0038	0.9356
7	0.8900	0.9200	1.0000	0.9994	0.9936	0.8686	0.9182	1.0018	1.0004	0.8681	0.9182	1.0017
8	1.0000	1.0000	0.9900	0.9861	1.0013	1.0134	0.9999	0.9911	0.9858	1.0132	0.9994	0.9910
9	0.9500	1.0000	0.9500	0.9514	0.9408	0.9382	1.0043	0.9474	0.9510	0.9385	1.0045	0.9473
10	1.0000	0.9000	0.9600	0.9591	0.9628	1.0221	0.8897	0.9592	0.9589	1.0207	0.8888	0.9592
$k_{\theta 1}$	1.08			1.0748	1.0446			1.0715	1.0740			1.0717
$k_{\theta 2}$	1.05			1.0499	1.0322			1.0475	1.0504			1.0472

Table 2
Studies on the beam structure shown in Fig. 1a–c; results on the standard deviation of the system parameters after the measurements have been assimilated

Element no.	Case (1)	Case (2)	Case (3)			Case (4)	Case (5)		
	1.0e−3	1.0e−3	1.0e−3	1.0e−3	1.0e−4	1.0e−3	1.0e−3	1.0e−3	1e−4
	γ	γ	α	β	γ	γ	α	β	γ
1	0.1948	0.2169	0.5869	0.2894	0.1338	0.1442	0.5867	0.2875	0.1313
2	0.2867	0.6086	0.1946	0.1069	0.0544	0.2527	0.1945	0.1065	0.0490
3	0.4180	0.9165	0.1486	0.0876	0.1072	0.3892	0.1485	0.0873	0.1068
4	0.3701	0.4570	0.1478	0.0923	0.1927	0.2713	0.1473	0.0921	0.1917
5	0.3821	0.4048	0.1633	0.1110	0.2510	0.2957	0.1632	0.1108	0.2483
6	0.3831	0.4046	0.1919	0.1351	0.2585	0.2954	0.1918	0.1348	0.2563
7	0.3708	0.4540	0.2357	0.1605	0.2119	0.2692	0.2355	0.1602	0.2106
8	0.4159	0.9156	0.3018	0.2050	0.1780	0.3869	0.3017	0.2046	0.1752
9	0.2887	0.6075	0.5028	0.3320	0.1168	0.2534	0.5027	0.3316	0.1141
10	0.1932	0.2140	1.8110	1.1626	0.1203	0.1421	1.8102	1.1613	0.1196
$k_{\theta 1}$	0.4886	0.7940			0.2962	0.4053			0.2872
$k_{\theta 2}$	0.4871	0.7927			0.3285	0.4035			0.3241

damping and stiffness properties. This needs to be borne in mind while interpreting results given in Tables 1 and 2. Fig. 3 shows the plots of percentage errors in determination of system parameters. It is observed from this figure that the identification procedure generally performs better for stiffness parameters than for damping and mass parameters with the poorest performance being for damping properties for elements at the ends of the beam. The rotary springs at the two ends of the beam are identified satisfactorily for all the cases. In any case, the highest error in all the cases is found to be less than 5%. The measures of errors shown in Fig. 3 are expected to depend upon the details of the noise processes, selection of points of driving and measurements, and relative completeness of measurements and further studies are required to clarify these dependencies and, especially, the one with respect to the placement of driving and sensing. The estimates of the system natural frequencies (Table 3) from alternative cases of identification show satisfactory match with the corresponding reference values. To get an idea on how the estimated structure predicts the structural responses that have not been included in identification procedure, we have shown in Figs. 4–6 the plots of estimated rotation at $x = 20$ m along with the corresponding measurements from the reference structure. The mutual agreement from the two sets of results is observed to be satisfactory. Fig. 7 illustrates the evolution of expected value and standard deviation of one the system parameters (mass per unit length for element 1 for case 5) with respect to the independent variable τ. The estimate of the expected value is observed to approach the reference value as the assimilation of the measurement data progresses.

Table 3
Natural frequencies of the beam structure shown in Fig. 1a

Mode	Reference (rad/s)	Case (3) (rad/s)	Case (5) (rad/s)
1	251.28	250.67	251.30
2	697.23	695.98	697.28
3	1352.31	1351.12	1352.27
4	2265.78	2259.81	2265.90
5	3384.93	3381.47	3385.13
6	4717.16	4725.00	4717.40

5.2. *Studies on the truss structure*

In these studies axial strains and transverse displacements at a few nodes are measured under the action of quasi-static moving loads and incrementally varied static loads. Consequently, the focus here is on estimating static stiffness characteristics and, accordingly, the Young's moduli of the 25

Table 4
Studies on the truss structure shown in Fig. 1d and e; results on the expected values of the system parameters after the measurements have been assimilated

Element no.	Reference value	Case (1)	Case (2)	Case (3)	Case (4)
	10e11 N/m^2	10e11 N/m^2	10e11 N/m^2	10e11 N/m^2	10e11 N/m^2
1	1.7	1.65570	1.655313	1.65683	1.66261
2	1.8	1.76116	1.764855	1.77453	1.77330
3	2.3	2.26559	2.280273	2.29481	2.28362
4	2.2	2.18756	2.178025	2.17170	2.19794
5	1.7	1.64235	1.617774	1.60353	1.65082
6	1.8	1.79566	1.786872	1.78474	1.81866
7	2.3	2.24714	2.226546	2.27677	2.22454
8	2.2	2.17005	2.184549	2.18969	2.19129
9	1.7	1.62487	1.601980	1.61286	1.63443
10	1.8	1.78256	1.776963	1.78035	1.78502
11	2.3	2.22153	2.240927	2.23553	2.24994
12	2.2	2.17579	2.170359	2.16925	2.18938
13	1.7	1.66630	1.573046	1.56950	1.62296
14	1.8	1.78746	1.782681	1.77947	1.80113
15	2.3	2.26181	2.239440	2.21516	2.23517
16	2.2	2.18222	2.185361	2.18689	2.19990
17	1.7	1.66998	1.687667	1.68909	1.70115
18	1.8	1.77862	1.766943	1.76393	1.77549
19	2.3	2.25698	2.249946	2.24678	2.27105
20	2.2	2.18227	2.185153	2.18634	2.19410
21	1.7	1.66661	1.658105	1.64643	1.67437
22	1.7	1.64540	1.668449	1.69380	1.68984
23	1.8	1.77789	1.809138	1.79956	1.80822
24	2.3	2.30184	2.315315	2.30793	2.31692
25	2.2	2.18033	2.154755	2.14808	2.17429

members are taken as system parameters to be identified. The identification problem here is solved for the following cases: case 1: the truss response to the action of a quasi-static moving load applied on the lower chord of the truss (Fig. 1d) is assumed to be available, case 2: the truss response to the action of incrementally varied static loads (Fig. 1e) is assumed to be available, case 3: the responses to action of the moving load as in Case 1 and incrementally varied static load (Fig. 1e) are available, and case 4: the responses to action of moving load as in Case 1 and incrementally varied static load (Fig. 1e) are available; the range of parameter a is discretized into 60 equally spaced values while the range of α is discretized into 30 equally spaced values.

Tables 4 and 5 summarize the results of the estimation of the 25 system parameters. Fig. 8 shows the percentage errors in the system parameters as a function of the element number. The highest error is observed to be about 8% for the cases when measurement data from quasistatic moving load test and incrementally varied static load test are used separately in the identification process (Cases 1 and 2). When the results from the tests are pooled together, the error is observed to reduce to about 6%. Fig. 9a and b show the estimates of some of the response quantities, namely, displacement at U2 on the top chord and the reaction at the right support, which have been computed using the expected values of the identified system parameters. These estimates have also been compared with the corresponding measurements and are seen to display satisfactory mutual agreement. These measured data, it may be noted, have not been used in the system identification procedure.

6. Discussion and conclusions

A systematic framework that employs Kalman filtering and Neumann expansion for the structural static and (or) dynamic stiffness matrices has been proposed to identify mass, stiffness and damping parameters of linear structural systems. Measurement data are taken to be available from multiple static and (or) dynamic load tests. In static tests the point of application of the load and (or) the magnitude of the load are incrementally varied and measurements are made on displacements and (or) strains at a set of points on the structure. In dynamic tests the FRFs associated with these response variables are taken to be measured. The proposed method has been shown to perform satisfactorily in the two numerical examples considered. Some of the features of the proposed procedure are as follows:

1. The independent variable is artificial in nature which helps to assimilate the measurements from more than one testing episode in a pseudo-sequential manner. The specific sequence in which the data can be arranged itself is not unique but this does not pose any conceptual difficulty in the implementation of the identification method. The identification procedure here is implemented essentially in an off-line manner.
2. The process equations are statements of fact that the system parameters to be identified are independent of point of application, magnitude and frequency of the applied loads. Consequently, the process equations are taken to be free from noise.

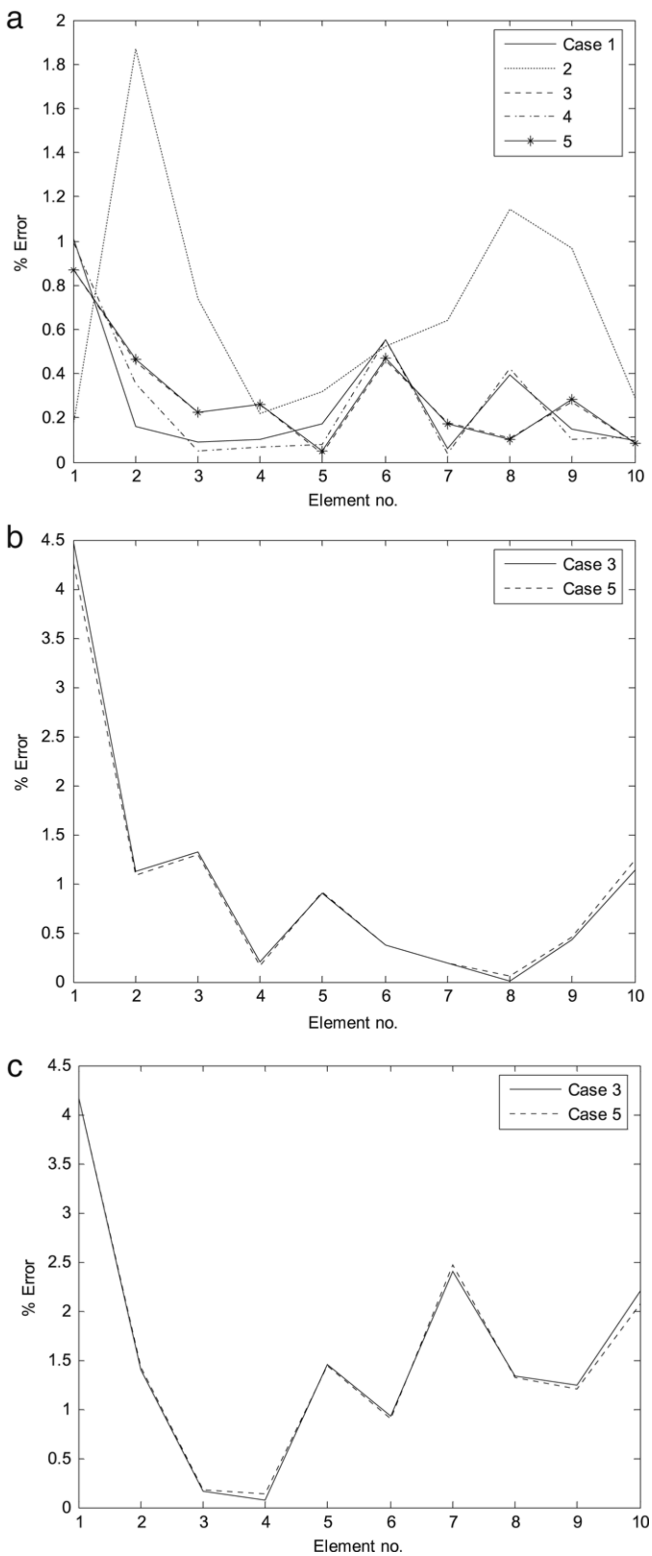

Fig. 3. Percentage errors in the system parameters estimated; (a) element flexural rigidity; (b) damping parameters; (c) mass parameter.

3. The measurement equations are derived based on the mechanics of the problem and the effects of measurement noise and modeling errors are subsumed into a set of noise processes which are taken to be made up of a sequence of zero mean, independent Gaussian random variables with

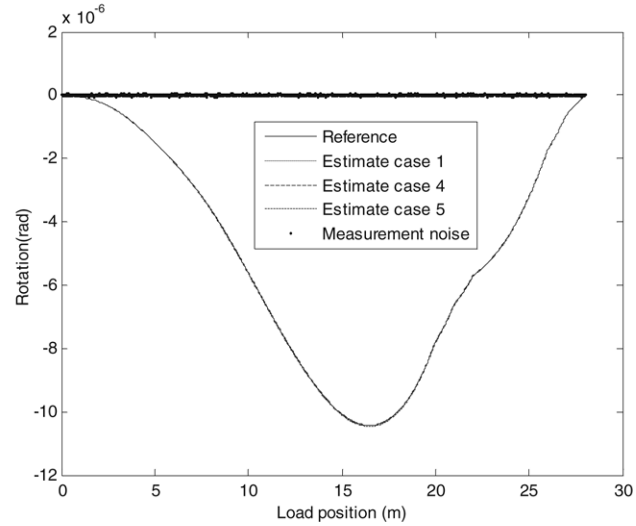

Fig. 4. Studies on the beam structure shown in Fig. 1a–c; rotation response at $x = 20$ m under the action of quasi-static moving loads; the measured data here has not been used in the identification process.

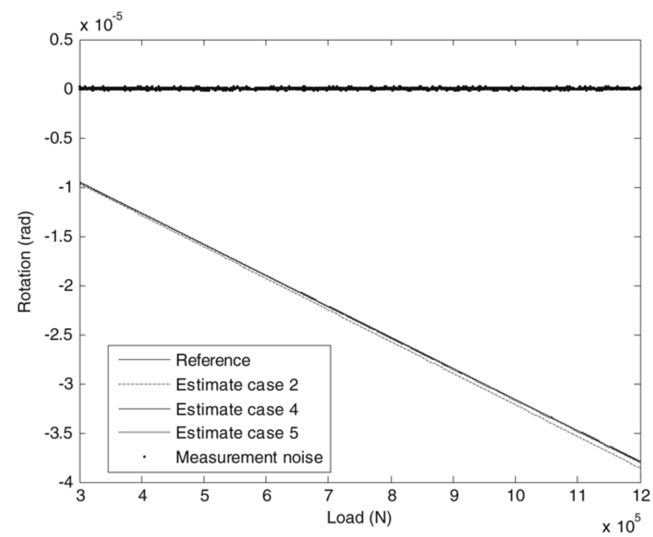

Fig. 5. Studies on the beam structure shown in Fig. 1a–c; load–displacement curve for rotation at $x = 20$ m; the measured data here has not been used in the identification process.

known covariance matrix. Furthermore, the noise processes are taken to be additive in nature.
4. The measurement equations are essentially algebraic in nature. This anyway is expected to be the case for statically loaded structures. However, by also treating the frequency as a parameterized independent variable, we obtain the measurement model under dynamic loads also as a set of algebraic equations.
5. In Kalman filter based approaches reported in the existing literature, especially for the problems of state and system identification for vibrating systems, the process equation is derived from the mechanics of the problem with the system parameters to be identified treated as auxiliary state variables. Thus the problem of estimating states (related to system response) and identification of system parameters become mutually coupled. In the present study, however, the

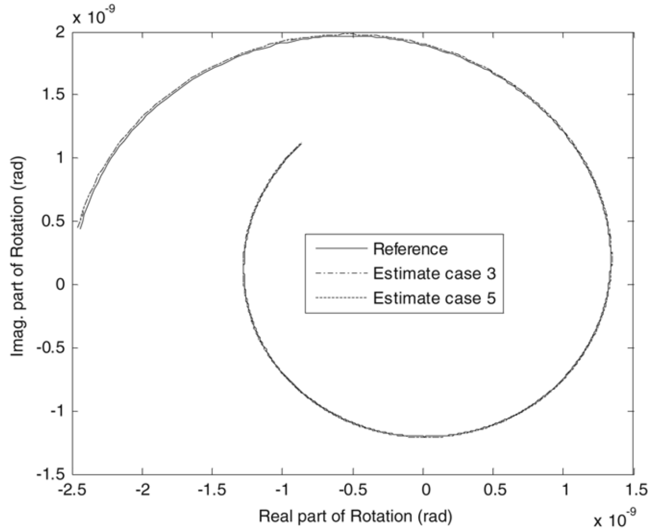

Fig. 6. Studies on the beam structure shown in Fig. 1a–c; Nyquist's plot for the FRF of rotation at $x = 20$ m; the measured data here has not been used in the identification process.

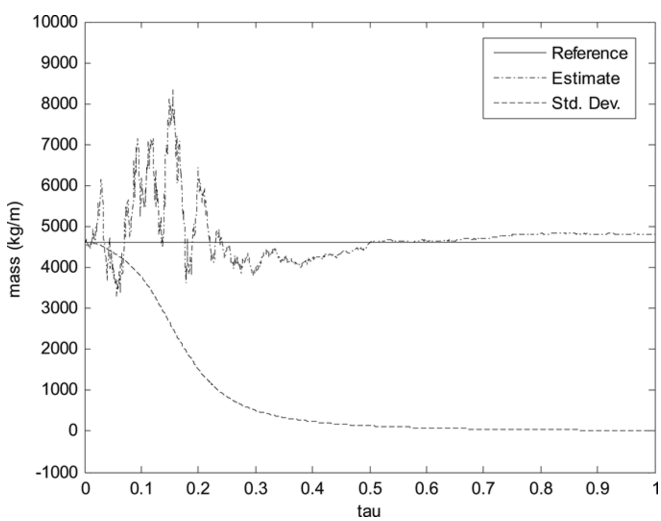

Fig. 7. Studies on the beam structure shown in Fig. 1a; Case 5; evolution of expected value and standard deviation of mass per unit length in element 2.

problem of the identification of system parameters is treated independent of the problem of estimation of system states.

6. The identification procedure here directly leads to the estimation of spatial model parameters. This is in contrast to identification methods based on the experimental modal analysis procedures [8] in which the eigenparameters of the system are identified first by curve-fitting strategies and the spatial models are deduced subsequently.
7. One of the problems in system identification is associated with spatial incompleteness of measurements that lead to a mismatch of sizes of experimental models and computational models [9]. This problem is tackled in the existing literature by using model reduction or expansion techniques. Alternatively, the unmeasured states are estimated within the framework of dynamic state estimation methods. The present approach employs Neumann's expansion technique to approximately obtain

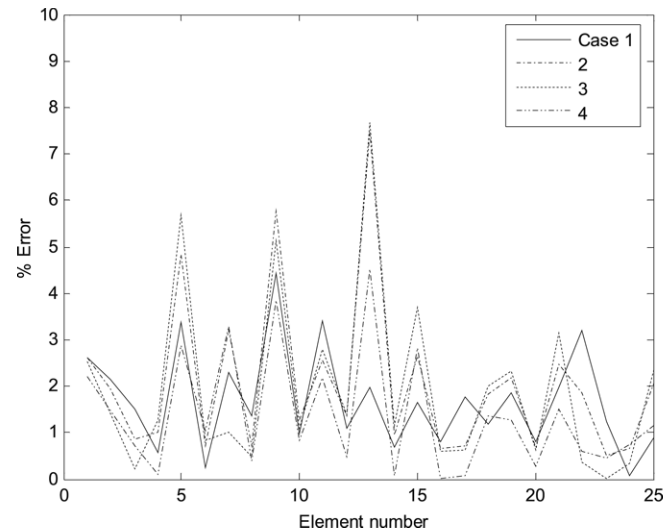

Fig. 8. Percentage errors in the system parameters estimated.

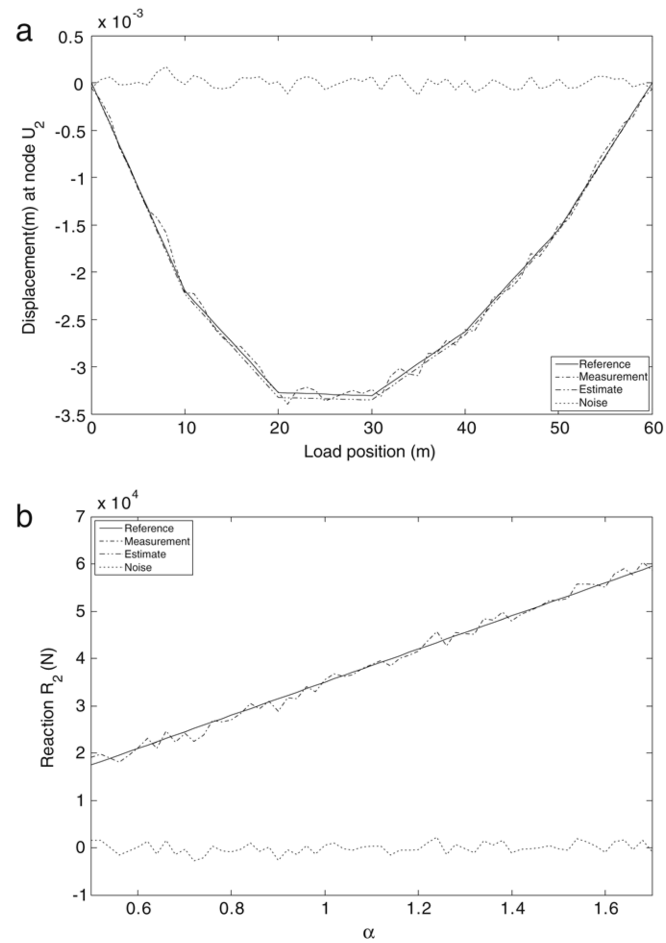

Fig. 9. Studies on the truss structure (Fig. 1d; Case 4); (a) displacement at node U_2 on the upper chord under the action of quasi-static moving load; (b) Reaction at the right hand support under the action of incrementally varied static load. Note that the measured data here have not been used in the identification process.

the inverse of the structural static/dynamic stiffness matrix, circumventing the need for these intermediate steps. This

Table 5
Studies on the truss structure shown in Fig. 1d and e; results on the standard deviation of the system parameters after the measurements have been assimilated

Element no.	Case (1)	Case (2)	Case (3)	Case (4)
	10e8 N/m^2	10e8 N/m^2	10e8 N/m^2	10e8 N/m^2
1	1.96660	2.88460	1.29000	1.53140
2	1.97260	2.84850	1.27390	1.51940
3	2.33550	3.13000	1.39980	1.61070
4	1.51290	2.18230	0.97600	1.13540
5	3.01950	4.09910	1.83320	2.07200
6	1.79720	2.57070	1.14960	1.33990
7	4.11630	8.70950	3.89510	3.93890
8	1.29630	1.75420	0.78440	0.92300
9	3.79450	5.62880	2.51730	2.83760
10	1.43610	1.87280	0.83750	0.98920
11	4.00620	5.65930	2.53090	2.73910
12	1.25900	1.72880	0.77320	0.90800
13	3.65090	5.33110	2.38420	2.71080
14	1.40290	1.84560	0.82540	0.97360
15	4.03580	8.48300	3.79370	3.84140
16	1.45620	2.11520	0.94600	1.10240
17	2.85230	3.88680	1.73820	1.97560
18	1.78560	2.54450	1.13800	1.33240
19	2.25410	3.05430	1.36590	1.58280
20	1.55650	2.26950	1.01490	1.20690
21	2.08510	3.01240	1.34710	1.60670
22	3.28670	5.29260	2.36700	2.53220
23	3.41620	7.01500	3.13720	3.17300
24	3.08760	6.28840	2.81230	2.85040
25	2.93040	4.79060	2.14240	2.28370

would mean that any refinement in the size of the underlying FE model for the structure does not increase the size of the state vector, thereby resulting in significant reduction in computational effort. Here, the unmeasured states, if desired, could be approximated in a subsequent stage following the system identification step.

8. As has been noted already, the identification procedure here is implemented in an off-line manner. This also permits the implementation of an additional global iteration step in which the initial assumptions on the characteristics of the system parameter can be successively be refined. In the numerical work it has been found that this global iteration step leads to useful improvements to the values of the parameter identified. In most of the calculations it was noted that about 3–5 global iterations steps lead to convergent results on the moments of the system parameters.
9. The present authors are currently working on addressing the following research issues:
   (a) If the linearization step employed in the proposed study is not used, the resulting measurement model becomes nonlinear in nature. The associated state estimation problem can then be solved by using Monte Carlo filtering methods [7]. The application of these methods also would enable the treatment of structural nonlinearities. It is to be noted that the question of characterizing the extent of errors resulting from the linearization step has not been addressed in the present study: this requires further research.
   (b) The identification procedure in this study has been illustrated using synthetically generated structural response data. The robustness of the proposed method needs to be examined when the data originate from laboratory/field measurements.
   (c) The assumption of Gaussian white noise models for measurement and modeling errors is one of the basic assumptions in the theory of Kalman filtering. Non-white noise models could be accommodated in the formulation by treating such processes as outputs of white noise driven dynamical systems. Also, the mathematical developments in the area of dynamic state estimation methods (encompassing the various particle filtering methods) indeed permit non-Gaussian models for these errors. However, arriving at parameters of these noise models remains as one of the difficult questions in the area of structural system identification. This is particularly true for the case of characterization of un-modeled dynamics where the epistemic uncertainties need to be quantified.

Acknowledgement

A part of the work reported in this paper has been supported by funds from the Aeronautical Research and Development Board, Government of India. The authors thank the agency for the support.

References

[1] Banan MR, Banan MR, Hjelmstad KD. Parameter estimation of structures from static response. I: Computational aspects. ASCE Journal of Structural Engineering 1994;120(11):3243–59.
[2] Banan MR, Banan MR, Hjelmstad KD. Parameter estimation of structures from static response. II: Numerical simulations. ASCE Journal of Structural Engineering 1994;120(11):3260–83.
[3] Brown RG, Hwang PYC. Introduction to random signals and applied Kalman filtering. 2nd ed. New York: John Wiley and Sons, Inc.; 1992.
[4] Ching J, Beck JL, Porter KA. Bayesian state and parameter estimation of uncertain dynamical systems. Probabilistic Engineering Mechanics 2006; 21(1):81–96.
[5] Doebling SW, Farrar CR, Prime MB. A summary review of vibration-based damage identification methods. The Shock and Vibration Digest 1998;30:91–105.
[6] Doucet A, Godsill S, Andrieu C. On sequential Monte Carlo sampling methods for Bayesian filtering. Statistics and Computing 2000;10: 197–208.
[7] Doucet A, de Freitas N, Gordon N. Sequential Monte Carlo methods in practice. New York: Springer; 2001.
[8] Ewins DJ. Modal testing: Theory, procedures and applications. Badlock: Research Studies Press; 2000.
[9] Friswell MI, Mottershead JE. Finite element model updating in structural dynamics. Dordrecht: Kluwer Academic Publishers; 1996.
[10] Ghanem R, Shinozuka M. Structural system identification I: Theory. Journal of Engineering Mechanics, ASCE 1995;121(2):264.
[11] Ghosh S, Manohar CS, Roy D. A sequential sampling filter with a new proposal distribution for state and parameter estimation of nonlinear dynamical systems. Proceedings of Royal Society of London, A 2008; 464:25–47.
[12] Gordon NJ, Salmond DJ, Smith AFM. Novel approach to nonlinear/non-Gaussian Bayesian state estimation. IEE Proceedings-F 1993;140(2): 107–13.

[13] Grewal MS, Andrews AP. Kalman filtering: Theory and practice using matlab. New York: John Wiley and Sons, Inc.; 2001.

[14] Hjelstad KD, Shin S. Damage detection and assessment of structures from static response. ASCE Journal of Engineering Mechanics 1997;123(6): 568–76.

[15] Hoshiya M, Sato E. Structural identification by extended Kalman filter. ASCE Journal of Engineering Mechanics 1984;112(12).

[16] Hoshiya M, Sutoh A. Kalman filter-finite element method in identification. ASCE Journal of Engineering Mechanics 1993;119(2): 197–210.

[17] Imai H, Yun CB, Maruyama O, Shinozuka M. Fundamentals of system identification in structural dynamics. Probabilistic Engineering Mechanics 1989;4(4):162–73.

[18] Kailath T. Lectures on Wiener and Kalman filtering. New York: Springer Verlag; 1981.

[19] Kalman RE. A new approach to linear filtering and prediction problems, Transactions of ASME. Journal of Basic Engineering 1960;82(Series D): 35–45.

[20] Ljung L. System identification: Theory for the user. New Jersey: Prentice-Hall, Inc.; 1997.

[21] Manohar CS, Roy D. Monte Carlo filters for identification of nonlinear systems. Sadhana 2006;31(4):399–427.

[22] Namdeo V, Manohar CS. Nonlinear structural dynamical system identification using adaptive particle filters. Journal of Sound and Vibration 2007;306:524–63.

[23] Natke HG. Problems of model updating procedures: A perspective resumption. Mechanical Systems and Signal Processing 1998;12(1): 65–74.

[24] Nejad FB, Rahai A, Esfandiari A. A structural damage detection method using static noisy data. Engineering Structures 2005;27:1784–93.

[25] Paola MD, Bilello C. An integral equation for damage identification of Euler–Bernoulli beams under static loads. ASCE Journal of Engineering Mechanics 2004;130(2):225–34.

[26] Pintelon R, Schoukens J. System identification: A frequency domain approach. New York: IEEE Press; 2001.

[27] Ristic B, Arulampallam S, Gordon N. Beyond the Kalman filter: Particle filters for tracking applications. Boston: Artech House; 2004.

[28] Sajeeb R, Manohar CS, Roy D. Rao–Blackwellization with substructuring for state and parameter estimations of a class of nonlinear dynamical systems, Transactions of ASME. Journal of Applied Mechanics. 2007 [under review].

[29] Sanayei M, Saletnik MJ. Parameter estimation of structures from static strain measurements. I: Formulation. ASCE Journal of Structural Engineering 1996;122(5):555–62.

[30] Sanayei M, Saletnik MJ. Parameter estimation of structures from static strain measurements. II: Error sensitivity analysis. ASCE Journal of Structural Engineering 1996;122(5):563–72.

[31] Sanayei M, Imbaro GR, McClain JAS, Brown LC. Structural model updating using experimental static measurements. ASCE Journal of Structural Engineering 1997;123(6):792–7.

[32] Shinozuka M, Ghanem R. Structural system identification II: Experimental verification. Journal of Engineering Mechanics, ASCE 1995;121(2): 265–73.

[33] Tanizaki H. Nonlinear filters: Estimation and applications. 2nd ed. Berlin: Springer Verlag; 1996.

[34] Yeo I, Shin S, Lee HS, Chang SP. Statistical assessment of framed structure from static responses. ASCE Journal of Engineering Mechanics 2000;126(4):414–21.

[35] Yun CB, Shinozuka M. Identification of nonlinear structural dynamic systems. Journal of Structural Mechanics 1980;8(2):187–203.