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An improved response surface method for the determination of failure probability and importance measures

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Abstract

The problem of response surface modeling of limit surface lying within two hyper spheres of prescribed radii is considered. The relevance of this problem in structural reliability analysis involving performance functions with multiple design points and/or multiple regions that make significant contributions to failure probability is discussed. The paper also proposes global measures of sensitivity of failure probability with respect to the basic random variables. The performance of the proposed improvements is examined by comparing simulation based results with results from the proposed procedure with reference to two specific structural reliability analysis problems.

Keywords: Response surface method; Multiple design points; Structural reliability; Global importance measures; Failure probability

1. Introduction

Predicting reliability of real engineering structures is a challenging task, especially for structures with high reliability. A detailed modeling of the structure is necessary which implies that the structural analysis can be carried out principally through computational tools, such as finite element codes. The uncertainties in the structural and load properties are represented through a vector of random variables having specified joint probability distributions. A straightforward method for estimating the structural reliability is to carry out Monte Carlo simulations. This involves a large number of deterministic structural analysis for different realizations of the random variables. For an accurate estimate of the failure probability P_f, the sample size, N, used in the simulation, must be at least of the order of $1/P_f$. For highly reliable complex structures, where the evaluation of each realization of performance function is computationally expensive, the direct Monte Carlo simulations is not a viable computational tool for estimating P_f. Since the

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performance function is usually available only in an implicit form, especially when finite element codes are used for the analysis, approximate analytical methods like first order reliability method (FORM) or second order reliability method (SORM) are difficult to apply. In such situations, the response surface method provides a powerful tool for estimating the structure failure probability.

2. Response surface method

The response surface method, was originally proposed by Box and Wilson [1] as a statistical tool, to find the operating conditions of a chemical process at which some response was optimized. Subsequent generalizations led to these methods being used to develop approximating functions that surrogate for long running computer codes. The books by Khuri and Cornell [2], Myers and Montgomery [3] and the paper by Romero et al. [4] provide modern perspectives in light of emerging computational power. In this paper, the focus is on studies that employ response surface method for structural reliability analyses. We denote by $\mathbf{X}$, the $n\times 1$ vector of basic random variables and by $g(\mathbf{X})$, the performance function. Typically, the response surface in an n-dimensional random variable space, is assumed to be a quadratic function of $\mathbf{X}$ where the constants are determined by evaluating $g(\mathbf{X})$ at certain specified sampling points. Various techniques have been explored to select the sampling points and determine the coefficients. Wong [5,6] and Faravelli [7,8] used factorial designs and regression methods to obtain least square estimates of the unknown coefficients. The sampling points were selected to be located at μ and $\mu \pm h\sigma$, where μ and σ are the vectors of the mean and standard deviations of the random variables $\mathbf{X}$, and h is an arbitrary factor. The number of $g(\mathbf{X})$ evaluations required for factorial designs is of the order of 2^n when a second order polynomial is to be fitted. The characteristics of the response surface obtained by selecting the central point to be situated at the mean, may not conform with the true failure surface. This is especially true in the case of highly reliable structures, where the failure point is located at a large distance from the mean vector.

To circumvent these difficulties, Bucher and Bourgund [9] proposed an adaptive interpolation scheme to arrive at quadratic response surfaces. They adopt a fully saturated experimental design. Initially, $g(\mathbf{X})$ is evaluated at a set of points given by $2n+1$ combinations of μ, $\mu+h\sigma$ and $\mu-h\sigma$ and an approximate surface $\tilde{g}(\mathbf{X})$ is constructed. The Hasofer-Lind reliability index and the associated design point $\mathbf{X_D}$, corresponding to $\tilde{g}(\mathbf{X})$, are determined. In doing so, it is assumed that $\mathbf{X}$ are in the standard normal space. An update on the location of the experimental design point is now obtained as $\mathbf{X}_M = \mu + (\mathbf{X}_D - \mu)g(\mu)/\{g(\mu) - g(\mathbf{X}_D)\}$. This helps to locate the center point closer to $g(\mathbf{X}) = 0$. A new surface using $\mathbf{X}_M$ as the center point is obtained and this is used as the final estimate of the response surface. Thus, this procedure requires $4n+3$ evaluations of $g(\mathbf{X})$.

Estimates of failure probability, obtained by this procedure, have been shown to be sensitive to the parameter h and may not always give acceptable approximation to the true failure probability [10]. While an understanding of the reasons behind the sensitivity of the parameter h is now possible, no clear guidelines are available on its correct choice. While Bucher and Bourgund obtained good results by choosing $h = 3$ in their numerical examples, Rajashekhar and Ellingwood [11] showed that using a constant value of $h = 3$ may not always yield good solutions, particularly when $g(\mathbf{X})$ is highly nonlinear. These authors also questioned if a single cycle of updating, is always adequate and demonstrated how a better fit of the response surface can be obtained by

subsequent updating of the center point and by lowering the value of the parameter h in subsequent cycles of updating. A modification to the Bucher and Bourgund approach has been proposed by Liu and Moses [12] where the updating of the response surface parameters are continued till a convergence criterion is satisfied.

Kim and Na [13] proposed a sequential approach to response surface method where they used the gradient projection method to ensure that the sampling points are located near the failure surface. These authors initially assume a linear response surface function and use the Rackwitz–Fiessler algorithm to obtain the design point. With this as the center point, a new set of sampling points are selected which are slightly perturbed from the failure surface. A response surface is now obtained using the procedure outlined by Bucher and Bourgund. The process is repeated till a convergence criterion on the reliability index is satisfied. In the study by Gayton et al. [14], the initial experimental point is chosen based on an understanding of the structural behavior. Next, the coordinates of the design points are assumed to be random variables whose properties are statistically determined from the database of computed experiments. Finally, the design point is located by making use of confidence intervals and defining a confidence area where the design point is located.

3. Performance functions with multiple regions contributing to failure probability

In the methods discussed so far, it is assumed that the performance function has a single design point and only the region around this point contributes to the failure probability. This is, however, not true for all performance functions. Fig. 1 shows a few performance functions where there are multiple regions contributing to the failure probability. The limit state surfaces in Fig. 1(a) and (b), respectively, have two and four points which have minimal distance from the origin O. These are called multiple design points, each of which makes equally important contributions to the failure probability. Fig. 1(c) depicts a performance function where points a and b are respectively situated at distances β_1 and β_2 from the origin, where $\beta_2 - \beta_1 < \epsilon$ with ϵ being a very small number. In this case, even though there exists only a single design point a, it is obvious that point b also makes significant contribution to the failure probability. Fig. 1(d) shows two performance functions, one that is a perfect circle (full line) and the other an ellipse obtained by a minor modification to the circle (dotted line). For the performance function that is a circle, all points on the limit surface lie at same distance from the origin and hence, the number of design points are infinite in number. On the other hand, for the ellipse, there exists two design points, c and d, but infinitely many points that are not strictly the design points, nevertheless, make significant contributions to the failure probability.

In limit states which have such characteristics, application of FORM, SORM or asymptotic theory around a single design point yield erroneous failure probability estimates. In limit states with multiple design points, Ditlevsen and Madsen [15] estimate the failure probability by applying system reliability concepts on the FORM or SORM approximations computed for each design point. Breitung and Faravelli [16] proposed a method based on asymptotic theory to obtain estimates of failure probability for performance functions with multiple design points or surfaces at minimal distances from the origin. In all these studies, it is assumed that the performance function is available in an explicit form and the location of the regions which contribute to the failure probability are therefore, easy to identify.

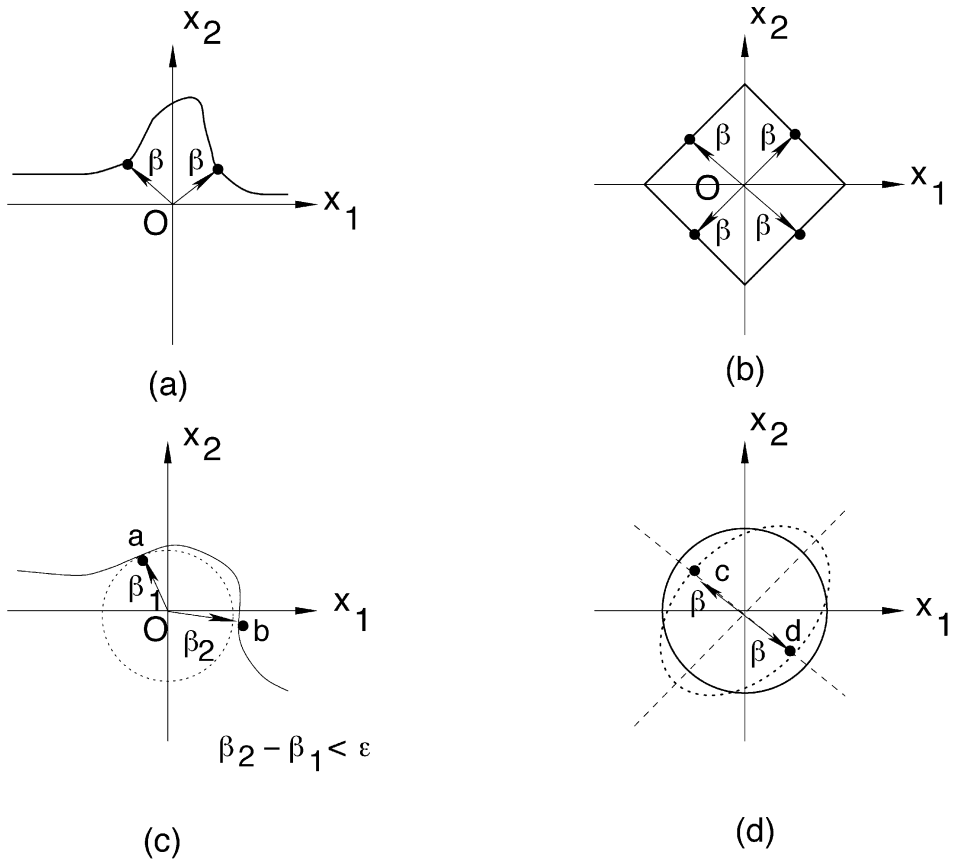

Fig. 1. Performance functions in standard normal space with multiple regions contributing to failure probability: (a) Function with two design points; (b) function with four design points; (c) function with one design point but more than one region which make significant contributions to failure probability; (d) full line: a function with infinite number of design points; dotted line: a function with two design points but with infinite number of points which make significant contributions to failure probability.

However, in situations where the use of the response surface method is essential, it is not possible to establish a priori the existence of a unique design point. The problem on hand, therefore, consists of identifying all the design points. Der Kiureghian and Dakessian [17] proposed a scheme for identifyiing multiple design points by constructing artificial barriers around the design points already identified using the Rackwitz–Fiessler [18] algorithm. At each iteration, a bulge is added at the design point so as to prevent the algorithm from converging to the same design point. Once all the design points, which are located at the same distance, are identified, the algorithm converges to the next most probable design point. At this juncture, the algorithm is terminated. Once all the points are identified, FORM or SORM approximations are constructed at these points and the failure probability is computed by series system reliability analysis. Recently, Mahadevan and Shi [19] proposed a multi-point linearization scheme for obtaining accurate estimates of failure probability by combining concepts of FORM and system reliability techniques for identifying points in the limit surface in regions which have significant contributions to the failure probability.

In this paper, a method has been proposed which identifies points and/or regions on the limit surface which have significant contributions to the failure probability. A global response surface is fitted through these points and Monte Carlo simulations are carried out on the fitted response surface to obtain failure probability estimates. The proposed method may not satisfy all the mathematical requirements of a response surface, inspired from statistical sampling theories [2,3,7,16]. However, there exist in the literature response surface methods which bypass some of these mathematical requirements, but obtain satisfactory results by incorporating reliability concepts. The studies by Bucher and Bourgund [9], Rajashekhar and Ellingwood [11] and Kim and Na [13], for instance belong to this category. It maybe noted that in the existing literature, the term response surface is being used for studies that are essentially rooted in statistical sampling theory [5–8,16] as well as in studies that combine theories of reliability indices with sampling procedures [9–13]. The method proposed in this paper is a study in a similar direction. The step by step procedure for the proposed method is outlined in the following section.

4. Algorithm for the proposed method

The first step in obtaining a global response surface fit is to identify multiple points which lie on the failure surface defined in the n-dimensional random variable space. This involves, first, defining a new set of coordinate system for each point to be identified. This is achieved by translating the origin along one of the axis by a prescribed distance. The procedure of shifting the origin is merely a geometric artifice and not a transformation of the reliability problem from one coordinate system to another. Thus, a point which lies in the safe region in the original coordinate system, continues to lie in the safe region even after the shift in the origin is made. Bucher and Bourgund's approach is subsequently used to identify the design point of the new performance function in the new coordinate system. The procedure is repeated by shifting all the axes a predetermined number of times, depending on the number of points one wishes to identify. Next, a polynomial response surface is obtained whose coefficients are determined by a least square regression analysis. Finally, Monte Carlo simulations are carried out on the response surface to obtain estimates of the failure probability. The algorithm is explained step by step below. Fig. 2 serves as the key for the nomenclature used in the following procedure: (1) The performance function $g(\mathbf{X})$, is first defined in the standard normal space. Thus, the random variables $\mathbf{X}$ are uncorrelated and the joint probability density function is rotationally symmetric about the origin O. (2) Bucher and Bourgund's algorithm is used to identify the design point on $g(\mathbf{X})$. It must be noted that if there are multiple design points, considering O to be the initial center point might lead the algorithm to jump from the vicinity of one design point to the other, leading to convergence problems. This can be avoided by selecting the central point to be at a slight perturbation from O. This also ensures that, multiple design points, if they exist, would be identified. (3) The Hasofer–Lind reliability index, β_0, corresponding to the identified point, is computed. (4) Assume that the origin O is now shifted along the ith axis, ($i = 1 \ldots n$) by a distance d in the standard normal space. This leads to a new definition of the performance function defined by the new coordinate system. The initial center point is once again chosen to be slightly perturbed from the new origin O_{i_j}. The subscript i denotes the shift along the ith ($i = 1,\ldots,n$) axis and the sub-subscript j indicates the jth ($j = 1,\ldots,k$) shift in the origin along the ith axis. To obtain a global fit of

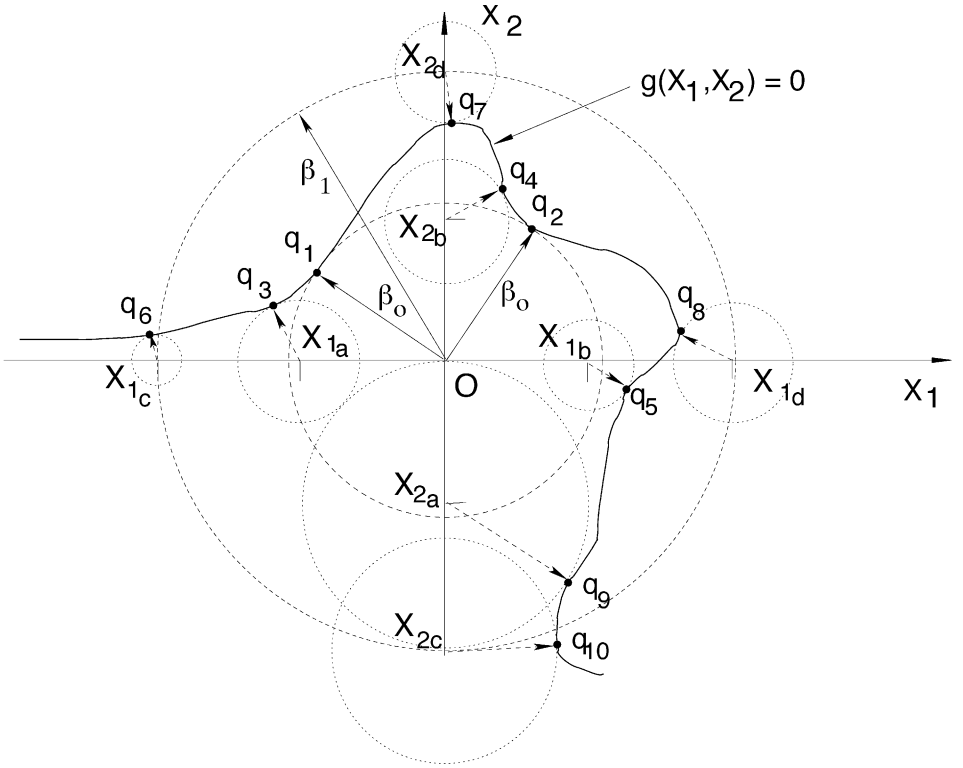

Fig. 2. Schematic description of the proposed method: q_i ($i=1,\ldots,10$) are the points identified on the limit surface, q_1 and q_2 are multiple design points, q_3, q_4, q_5, q_8, q_7, q_6, q_9, q_{10} are points arranged in decreasing order of importance in evaluating failure probability.

the performance function, it is necessary to identify points which span the entire standard normal space. This implies that the shift in the origins O_{i_j}, should span the entire standard normal space. However, points with a reliability index $\beta >> \beta_0$, do not have significant contributions to the failure probability. Thus, it is assumed that O_{i_j} lies within a hyper-sphere of radius β_1, ($\beta_1 > \beta_0$), given by $\beta_1 = -\Phi^{-1}\left[10^{-4}\Phi(-\beta_0)\right]$ in the original standard normal space. It has been observed that, as the origin is being shifted along the ith axis, if O_{i_j} and $O_{i_{j-1}}$, are situated at opposite sides of O, the design points obtained using Bucher and Bourgund's algorithm are spaced apart and span the entire hyper-sphere of radius β_1 in the original standard space. Otherwise, all the points obtained may form a cluster on one side of the standard normal space. Based on these observations, the distance d is given by $d = (\beta_1 - (-\beta_1))/k$ where k is the number of times the ith axis is shifted. A judicious choice of k is required such that the computational effort is minimized without compromising on the accuracy. The coordinates of the shift in the ith axis is given by $u_{i_k} = u_{i_0} + (-1)^j(j-1)d$ where u_{i_0} is the ith coordinate of the center point O chosen in step 2. (5) Bucher and Bourgund's algorithm is used to identify the design point in the new coordinate system. In the original standard normal space, this point may not be the design point, but is guaranteed to lie close to the failure surface, which implies $g(\mathbf{X})\approx 0$. The error, here, depends on the tolerance set in Bucher and Bourgund's algorithm. (6) Repeat the procedure by shifting the origin k times along each of the n axes in the n-dimensional random variable space. This would lead to a total of

$R = kn + 1$ points being identified which lie on the failure surface and the total number of $g(\mathrm{X})$ evaluations required is $(4n + 3)(1 + kn)$. (7) Next, fix the order l of the polynomial of the response surface that is to be fitted passing through the R points. If the effect of the cross terms are neglected, the polynomial is taken to be of the form

$$G = a_0 + \sum_{i=1}^{n} a_i X_i + \sum_{i=1}^{n} b_i X_i^2 + \sum_{i=1}^{n} c_i X_i^3 + \ldots \text{up to } l\text{th order} \tag{1}$$

The number of unknown coefficients that need to be determined is $ln + 1$. The choice of l should be such that $ln + 1 \leqslant R$. In matrix form, the relationship between the values of $g(\mathbf{X})$ at the R points and the response surface parameters, is written as $\mathbf{G} = \mathbf{WA}$ where $\mathbf{G}$ is a $R \times 1$ vector consisting of the values $g(\mathbf{X}_i)$, $(i = 1,\ldots,R)$, A is a $(ln + 1) \times 1$ vector of the unknown coefficients and $\mathbf{W}$ is a $R \times (ln + 1)$ dimensional matrix, given by

$$W = \begin{bmatrix} 1 X_{1a} X_{2a} \ldots X_{1a}^2 X_{2a}^2 \ldots X_{1a}^l X_{2a}^l \ldots \\ \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\ 1 X_{1r} X_{2r} \ldots X_{1r}^2 X_{2r}^2 \ldots X_{1r}^l X_{2r}^l \ldots \end{bmatrix} \tag{2}$$

A least square estimate of the unknown coefficients $\mathbf{A}$ is given by, $\hat{\mathbf{A}} = E[\mathbf{A}] = (\mathbf{W}'\mathbf{W})^{-1}\mathbf{WG}$. $(\mathbf{W}'\mathbf{W})$ should be of full rank for it to be invertible. This implies that the R points identified in step 6, should be distinct. It is therefore necessary to check the coordinates of all the R points. For each identical row in $\mathbf{W}$, one row is eliminated and the dimension of $\mathbf{W}$ is reduced by 1. This process ensures that $(\mathbf{W}'\mathbf{W})$ is of full rank and invertible. (8) Full scale Monte Carlo simulations are carried out on the response surface and an estimate of the failure probability is obtained from the relative frequency of failures.

The procedure of using a least square regression analysis to obtain the parameters of a response surface around a design point has earlier been used by Faravelli [7]. Select points from the vicinity of the mean vector of the random variables were used in the regression analysis. This, however, does not guarantee that the response surface would match the true failure surface even near the design point, especially in highly reliable structures, where the failure surface is located at a large distance from the mean vector. Moreover, the location of the sampling points were fixed and were independent of the geometry of the limit surface. On the other hand, in this study, the sampling points for regression analysis, are chosen to lie close to the true failure surface. Also, the selection of these points is dictated by the geometry of the global failure surface. Hence, the fitted response surface has a better correspondence to the true failure surface.

5. Importance measures of the random variables

Studies on sensitivities of the random variables have been carried out in the literature in order to establish a systematic framework for identifying relative importance of parameters that merit descriptions through random variables. Studies on sensitivity of reliability index at the design point with respect to distribution parameter and/or a deterministic design parameter are available [20–22]. Brenner and Bucher [23], in their study on SFEM based reliability assessment of non-

linear structures under random dynamic loads, assessed the relative importance of the random variables by a sensitivity analysis around the mean values of the random variables. This practice of ranking the random variables on the basis of their sensitivities at the mean is, however, questionable as there is no way of knowing whether the ranking remains valid if the sensitivities were computed at the design point. Madsen [24] introduced the concept of omission sensitivity factors to characterize the relative error in the reliability index when a basic random variable is replaced by a deterministic number. It was noted that the direction cosines of the limit surface at the design point, for a random variable in the standard normal space, denote the gradient of the reliability index with respect to the design variable. It is thus, a measure of the sensitivity of the reliability index to inaccuracies in the value of the design variable at the design point. Madsen defined the omission sensitivity factor for a basic random variable X_i as the inverse ratio between the value of the first-order reliability index and the first order reliability index with X_i replaced by a deterministic value. It was shown that if the absolute value of the direction cosines was less than 0.14, then the relative error of the reliability index is less than 1% when X_i is replaced by its median. Madsen suggested using only a few random variables having large values of direction cosines in the FORM computations as means of reducing the computational effort in subsequent iterations. However, at the first iteration, a FORM analysis involving all the random variables is necessary to obtain an estimate of the direction cosines. Further discussions on omission sensitivity factors with details of important measures related to input parameters in series/parallel system reliability is presented in the book by Ditlevsen and Madsen [25]. The converse of ommission sensitivity is the ignorance sensitivity [26,27], and is obtained when a deterministic number is replaced by a random variable.

The generalization of the above studies [20–27] to performance functions with multiple regions of comparable importance is not straightforward, since, random variables that are deemed to be less important in one region could become important in other regions. Consequently, in this type of problems, there exists a need to develop methods to rank random variables in order of their relative importance, and, also, to investigate if such ranking could be used to reduce the number of significant random variables before or during the reliability evaluation process. The present study explores these issues within the framework of algorithm presented in Section 4. Thus, we propose that, if R distinct points are identified in the n-dimensional standard normal space, the global importance measure of Z_i, is expressed as

$$\gamma_i = \sum_{l=1}^{R} w_l \lambda_l^{i^2} \Big/ \sum_{i=1}^{n} \sum_{l=1}^{R} w_l \lambda_l^{i^2} \tag{3}$$

where, the parameters w_l and λ_l^i are defined later. The vector of correlated non-Gaussian random variables $\mathbf{Z}$ are related to the vector of standard normal variates $\mathbf{Y}$, through Nataf's memorvless transformation, given by $\mathbf{Y} = \Phi^{-1}[P_{\mathbf{z}}(\mathbf{Z})]$. Here, P_z and $\Phi(\cdot)$ are, respectively, the marginal probability distribution functions of $\mathbf{Z}$ and $\mathbf{Y}$. The joint probability density function of the normal variates, $p_{y_1 \ldots y_n}(.)$ is characterized by the unknown correlation coefficient. matrix $\mathbf{C_Y}$**.** The correlation coefficients $\mathbf{C}_{\mathbf{Y}_{ij}}$ are expressed in terms of the known correlation coefficients $\mathbf{C}_{\mathbf{z}_{ij}}$ through an integral equation which is solved iteratively to obtain $\mathbf{C_Y}$ [28]. The vector of correlated normal random variables $\mathbf{Y}$ are related to the standard normal variates $\mathbf{X}$ through the expression $\mathbf{Y} = \mathbf{LX}$ where $\mathbf{L}$ is the lower triangular matrix, such that $\mathbf{C_Y} = \mathbf{LL}^t$. $\mathbf{L}$ is determined by Cholesky decomposition of

$\mathbf{C_Y}$. Since, the proposed algorithm is carried out in the standard normal space, the direction cosines for the identified points, $\left\{\frac{\partial g}{\partial \mathbf{X}}\right\}$ are obtained in the **X**-space. The corresponding direction cosines in the **Z**-space are expressed as

$$\left\{\frac{\partial g}{\partial \mathbf{Z}}\right\} = \mathbf{J}\left\{\frac{\partial g}{\partial \mathbf{X}}\right\}, \tag{4}$$

where, **J** is the Jacobian. The parameter λ_l^i in Eq. (3). is now defined as

$$\lambda_l^i = \sum_{j=1}^{R} w_l \left\{\left(\frac{\partial g}{\partial \mathbf{Z}}\right)_j^i\right\}^2. \tag{5}$$

where w_l are weighting functions. To give greater weightage to the points closer to the origin in the **X**-space, the weights, w_l, are defined in terms of probability content associated with each point in the **X**-space, and are expressed as

$$w_l = \Phi(-\beta_l)/\sum_{j=1}^{R}\Phi(-\beta_j) \tag{6}$$

where, β_l is the Hasofer–Lind reliability index of the lth point in the **X**-space. Thus, points in the failure domain more likely to fail are given greater weightage.

An alternative procedure for establishing importance measures could be by performing Monte Carlo simulations on the global response surface. This involves computing the change in failure probability when one of the random variables is assumed to be deterministic at the mean value. The importance measure of a random variable is then expressed as

$$\Gamma_i = (P_f - P_{f_i})^2/\sum_{j=1}^{n}(P_f - P_{f_j})^2$$

where, P_f is the true failure probability when all the random variables are considered and P_{f_i} is the failure probability obtained from Monte Carlo simulations on the response surface when the ith random variable is held deterministic at its mean. This procedure of establishing importance measures however, requires additional n runs of Monte Carlo simulations.

The global importance measures can be used to rank the random variables according to their importance with respect to the reliability of the structure in question. This is particularly useful in structural design and control as the knowledge of the global importance measures of the random variables helps to identify those structural parameters which have greater effect on the reliability of the structure. In problems of structural reanalysis in a design iteration process, these measures have the potential to enable probabilistic model reduction. This is possible because in conducting iterations on the preliminary design, the random variables that are lowly ranked in order of importance can be treated as being deterministic.

In this context, it is also of interest to investigate if the importance measures are helpful in reducing the number of random variables before or during the reliability calculations. In the

proposed algorithm, it is important to note that the global importance measures, $\gamma_i (i = 1,2,\ldots,n)$, are computed *before* the probability of failure is computed using the global response surface (step 8 in the algorithm). Therefore, the computational effort in evaluating the failure probability can be reduced if we eliminate the lowly ranked random variables from the analysis at this stage. This feature has been illustrated in the numerical examples to follow. It may also be noted that the importance measure $\Gamma_i (i = 1,2\ldots,n)$ can only be defined *after* the failure probability has been evaluated, and therefore, these measures would not be helpful in probabilistic model reduction. Besides, at the stage of identifying the individual points on the limit surface, we can employ sensitivity measures, as developed, for example, by Madsen [24], to reduce the computational effort at each stage. This avenue for model reduction, has not however been explored in the present study.

6. Numerical examples

The scope of the proposed method is illustrated through two numerical examples. The failure estimates have been compared through the following three procedures. (1) Method 1: Full scale Monte Carlo simulations using the exact performance function, which may be implicitly defined. An estimate of the failure probability is obtained from the relative frequency of failures. This is assumed to be the ''true'' failure probability and is used to benchmark other methods. (2) Method 2: The failure probability estimate is obtained from the Hasofer–Lind reliability index, computed by fitting a response surface using Bucher and Bourgund's algorithm. (3) Method 3: A global response surface is fitted using the proposed algorithm, followed by full scale Monte Carlo simulations. The failure probability is computed from the relative frequency of failures. (4) Method 4: Similar to method 3 except that the importance measures γ_i, are used to eliminate lowly ranked random variables in failure probability calculations. The global importance measures for the random variables, γ_i and Γ_i, are compared with the importance measures S_i, defined as

$$S_i = \left(P_f - P_{f_i}\right)^2 / \sum_{j=1}^{n} \left(P_f - P_{f_j}\right)^2 \tag{8}$$

where, P_{f_i} is the estimate of the failure probability obtained by full scale Monte Carlo simulations on the actual limit surface, when the ith random variable is held deterministic at the mean value and P_f is the corresponding true failure probability.

6.1. Example 1

The structural reliability of a 1.0 m cantilever steel beam with cross sectional dimensions (0.1×0.01) m is studied. The beam is subjected to an in-plane moment at the free end and an in-plane concentrated load at 0.4 m from the free end (see Fig. 3). The structural analysis is carried out using the finite element method. The beam is discretized using 10 number of four noded plane stress elements with each node having two translational degrees of freedom along the global coordinates. The partitioned global stiffness matrix that needs to be inverted is of dimensions 40×40. The structure is assumed to have failed if the square of the Von Mises stress at the

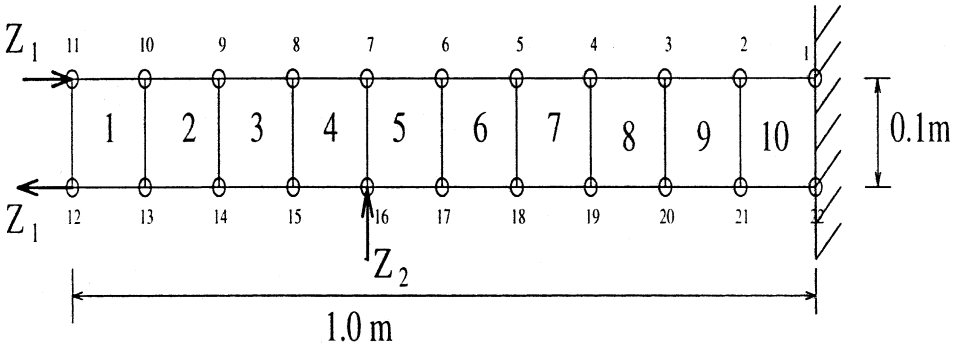

Fig. 3. Finite element model for cantilever beam using plane stress elements (example 1).

support (node 1) exceeds a specified threshold. Mathematically, the limit surface is written as $g(\mathbf{Z}) = \alpha^2 - V$. Here, $\mathbf{Z}$ represents the vector of basic random variables and V is the square of the Von Mises stress, expressed as a quadratic operator on the stress vector $\sigma = \left[\sigma_{xx}\sigma_{yy}\gamma_{xy}\right]^t$ and is given by $V = \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} + 3\gamma_{xy}^2$. In this example, the loads Z_1 and Z_2, the threshold α and the Young's modulus of elasticity E are considered to be random variables that are mutually independent. The fluctuations in α and E are expressed as $E = E_0[1 + \epsilon_2 Z_3]$ and $\alpha = \alpha_0[1 + \epsilon_1 Z_4]$. Here, ϵ_1 and ϵ_2 are small deterministic constants denoting the coefficient of variation of the random variables and are taken to be equal to 0.05, $E_0 = 2.07 \times 10^5$ N/m^2 denotes the deterministic component of Young's modulus and $\alpha_0 = 500 \times 10^6$ N/m^2 denotes the deterministic component of the threshold. The probability distributional properties of $\mathbf{Z}$ are tabulated in Table 1. For each realization of $\mathbf{Z}$, $g(\mathbf{Z})$ is evaluated through the finite element code and is thus, an implicit function of Z. The stress components are linear functions of Z_1 and Z_2 and nonlinear functions of Z_3. Since V is obtained by a quadratic operator on the stress components, the implicit performance function $g(\mathbf{Z})$ is a highly nonlinear function of the random variables. Transforming the performance function into the standard normal space $\mathbf{X}$ introduces further nonlinearities. As such, the failure estimate is expected to have significant sensitivity to the parameter h. The reliability analysis is carried out using methods 1–4 and the results are tabulated in Table 2. It is observed that the failure probability estimates by method 2 are at wide variance with the true failure probability (method 1) and show a strong dependency on the choice of h. In contrast, the failure probability estimates by method 3, with $k = 5$, are in much better agreement with simulation results. Here, initially the origin is considered as the central point but in subsequent iterations, the central point is considered to be shifted from the origin by a distance of 10^{-2} on all axes. When $h = 3$, β_0 and β_1

Table 1
Distributional properties and importance measures of the randomn variables in example 1

Random variable	Probability distribution	Mean	Standard deviation	γ_i	Γ_i	S_i
X_1	Gaussian	1.0	1.0	0.9528	0.8214	0.8182
X_2	Gaussian	0.0	1.0	0.0100	0.0010	0.0000
X_3	Lognormal	2.0	1.0	0.0174	0.0613	0.0316
X_4	Lognormal	0.1	0.05	0.0198	0.1163	0.1502

Table 2
Estimates of failure probability for example 1

Method	P_f ($h=1$)	P_f ($h=3$)	$g(x)$ evaluations	CPU time (s)
Method 1	5.90×10^{-3}	–	10^4	495.78
Method 2	4.94×10^{-1}	2.09×10^{-2}	19	0.81
Method 3	6.30×10^{-3}	4.60×10^{-3}	399	15.88 + 0.12
Method 4	5.60×10^{-3}	4.60×10^{-3}	399	15.88 + 0.08

are respectively, 2.0362 and 3.5289. Even though method 3 is built on the same principles as method 2, the effect of changing the parameter h is observed to be not significant. This example also illustrates the efficiency of the proposed method with regard to computational time. The computing time involved in implementing methods 1–4 is also summarized in Table 2, from which it is observed that method 2 requires significantly less computing time but has difficulties associated with it as regards to selection of h and accuracy realized. The computing times for methods 3 and 4 are reported in two parts. The first part indicates the time needed for fitting the global response surface and the second, the time for simulation runs on the response surface. The total time required here is still far less as compared to simulation results from method 1. A point that can be mentioned in favor of method 3 is that the computing time is not significantly dependent upon the magnitude of P_f, whereas, in method 1, as has been already noted, number of samples required is crucially dependent on P_f.

The global importance measures γ_i, Γ_i, and S_i are listed in Table 1. Since $\mathbf{Z}$ is a vector of mutually independent, non-Gaussian random variables, $\mathbf{J}$ in Eq. (4) is a diagonal matrix with the elements corresponding to Z_1 and Z_2 being unity. For $k=5$, the total number of points $R=21$ and the weights w_1 are observed to vary from 1.9691×10^{-4} to 0.4404. The sample size for computing Γ_i and S_i is taken to be 2×10^4. It is observed that all the three measures lead to identical conclusion that the ranking of the random variables, in the order of relative importance, reads as Z_1, Z_4, Z_3 and Z_2. A parametric study was carried out by changing h and k to study their effect on the importance measures. It has been observed that, even though there is a change in the relative importance of the random variables, the ranking of random variables, which have significant sensitivities, remains unaltered. The evaluation of failure probability by treating Z_1, with $\gamma_i=0.95$, as the only significant random variable was also carried out by treating Z_2. Z_3 and Z_4 as being deterministic at their mean values. The corresponding results are shown in the fourth row of Table 2. These results are observed to be in reasonably good agreement, with results from methods 1 and 3. It is important to note that the actual saving in computations in Method 4, in this case, is marginal. This is to be expected since the use of importance measures does not reduce the computational efforts in implementing steps (1–7) of the algorithm and, also, in the present problem, the number of random variables considered is rather small.

6.2. Example 2

In this example, the problem of a 10-story building with a tuned mass damper (TMD) excited by random harmonic base accelerations is considered. A similar example has been earlier considered by Der Kiureghian and Dakessian [17] in the context of illustrating the treatment of

multiple design points in reliability analysis. The TMD in their study was tuned in such a way that the TMD natural frequency was designed to coincide with the fundamental building frequency. However, due to structural uncertainties, a perfect tuning would not occur all au the sample realizations of the problem. Thus, depending on the relative values of the TMD frequency and building first mode frequency, there would be either overtuning or undertuning that consequently leads to two regions of importance in reliability calculations. The building (Fig. 4) has floor masses m_i and story stiffnesses k_i, ($i=1,\ldots,10$), and the TMD has mass m_0 and stiffness k_0. The combined system is assumed to have modal damping ratios ζ_i, ($i=0,\ldots,10$). The structural parameters $m_0,\ldots,m_{10}$, $k_0\ldots k_{10}$, $\zeta_0,\ldots,\zeta_{10}$ have been considered to be random along with the frequency Ω of the harmonic ground acceleration $F\sin(\Omega t)$. Thus, a total of 34 random variables enter the formulation. For the purpose of illustration. all these random variables are assumed to be mutually independent. Table 3 shows the details of probability distributions assumed for these variables. For the mean values listed in this table, the fundamental frequency of the structure and the frequency of the TMD are identical and equal to 7.34 rad/s. The first two natural frequencies of the mean building-TMD system, ω_1 and ω_2 are 5.99 and 8.84 rad/s respectively. The excitation frequency Ω is assumed to be uniformly distributed within the range defined in Table 3.

For reliability analysis, the performance function is expressed in terms of base shear and is given by $g(\mathbf{Z})=V_0-V_{\text{base}}(\mathbf{Z})$ where $\mathbf{Z}$ represents the 34-dimensional vector of mutually independent, non-Gaussian random variables. For each set of realization of $\mathbf{Z}$, an eigenvalue analysis is first preformed and the dynamic analysis of the 11-degree-of-freedom, combined building–TMD system, is carried out to obtain the base shear V_{base}. This implies that the performance function is implicit and highly nonlinear. It is assumed that the threshold base shear $V_0=2\times10^5$ kN. The reliability analysis is carried out using methods 1–4. A comparison of the estimated failure probabilities is shown in Fig. (5) for F varying from 0.2 to 0.5 m/s^2. In methods 2 and 3, $h=3$ and the origin is taken to be the initial central point. The failure probability estimates reported in Fig. (5) for method 3 are when $k=3$. For $h=3$, β_0 and β_1 are, respectively, 0.6490 and 4.0481. A parametric study is carried out for $F=0.40$ m/s^2 to study the effects of changing k on P_f estimates. For $k=3$, 4, 5, 6 and 10, the P_f estimates obtained are respectively, 0.1568, 0.1360, 0.1462, 0.1502 and 0.1506. The sample size in Monte Carlo simulations is taken to be 5000.

The global importance measures γ_i, for $\mathbf{Z}$ have been computed for the case $h=3$ and $k=3$. The number of points identified on the failure surface is $R=103$. The weights w_l are observed to vary from 5.54×10^{-6} to 0.0990. Since $\mathbf{Z}$ is a vector of mutually independent, random variables, $\mathbf{J}$ is a

Table 3
Distributional properties of random variables in example 2

Variable	Distribution	Mean	c.o.v.
$m_1,\ldots,m_{10}$	Lognormal	8.75×10^4 kg	0.2
$k_1,\ldots,k_{10}$	Lognormal	2.1×10^8 N/m	0.2
m_0	Lognormal	7.16×10^4 kg	0.2
k_0	Lognormal	3.85×10^6 N/m	0.2
$\zeta_0,\zeta_1,\ldots,\zeta_{10}$	Lognormal	0.05	0.3
Ω	Uniform in [5.39, 7.79] rad/s	6.59 rad/s	0.1

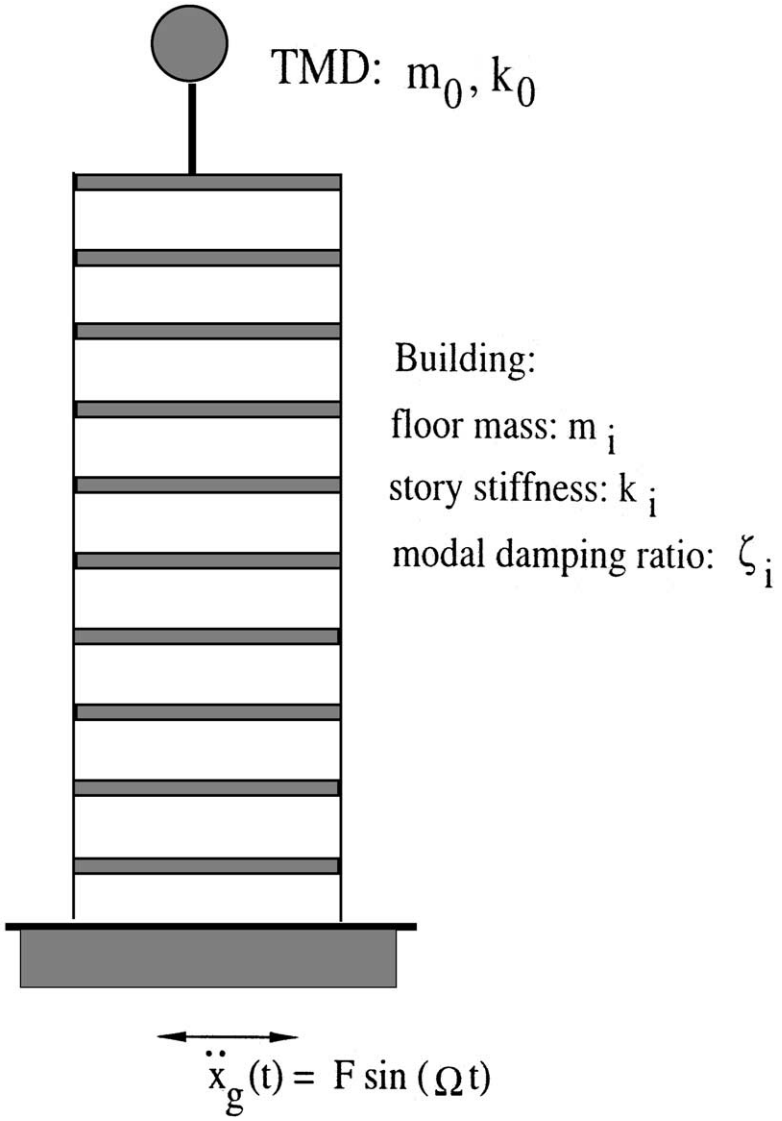

Fig. 4. Ten-story building with TMD (example 2).

diagonal matrix. The sample size for computing Γ_i and S_i is taken to be 5000. The first three ranked random variables were found to be Ω, m_1 and k_1 with the associated γ_i being 0.9930, 0.0069 and 0.0001, Γ_i being 0.9448, 0.0304 and 0.0081 and S_i, being 0.9498, 0.0231 and 0.0159 respectively. The computations for method 4 were carried out by considering the set of 33 random variables, other than Ω, to be deterministic at their mean values and Monte Carlo simulations were carried out on the fitted response surface by assuming only Ω to be random. The estimated failure probabilities are plotted in Fig. 5 and are seen to compare well with those obtained from methods 3 and 1.

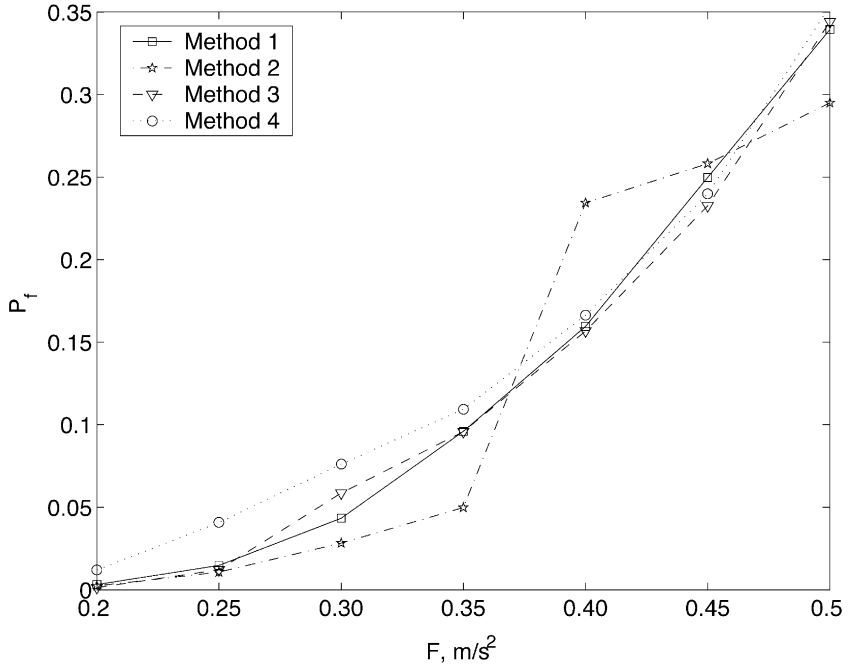

Fig. 5. Failure probability estimates by methods 1–4 (example 2).

7. Concluding remarks

In realistic problems involving structural reliability assessment, the performance functions are most often specified implicitly through a finite element code. Among the currently available procedures for reliability assessment, the response surface method offers the best alternative to deal with this class of problems. One version of this method, as developed by Faravelli, is inspired from the earlier work done in statistical sampling theory. The other version, by Bucher and Bourgund, incorporates the first order reliability method into response surface methodology. The computational effort in Faravelli's approach significantly increases with number of random variables and the method does not explicitly take into account the geometry of the true failure surface while fitting the response surface. The Bucher and Bourgund approach, on the other hand, has the potential to lead to acceptable results in problems where there exist a single design point. The results from this method are found to be sensitive to selection of the parameters inherent in the algorithm. In the present study, an attempt has been made to combine the above two approaches and this has led to a new procedure which incorporates the following improvements: (a) The proposed method is capable of dealing with multiple design points and/or multiple regions on the failure surface that could make significant contributions to the failure probability. (b) The study proposes a set of importance measures which could be used in ranking basic random variables in terms of their relative importance vis à vis the failure probability. These measures incorporate the geometry of the failure surface over regions that contribute significantly to the failure probability. These measures also serve, to a limited extent, to reduce the number of significant random variables during reliability calculations. (c) The accuracy of the proposed method, as applied to

nonlinear performance functions, is found to be better than the accuracy realized in Bucher and Bourgund's method; this is particularly true when there exists multiple design points and/or regions which have significant contributions to the failure probability, in which case, Bucher and Bourgund's method fares poorly.

The proposed method proves to be computationally more efficient than full scale Monte Carlo simulations, especially when the performance function evaluations consume significant computer time. Further studies are needed to employ sensitivity factors and global importance measures in achieving more effective probabilistic model reduction for reliability assessment. Also, extension of the proposed method to problems of system reliability and time variant reliability are currently being pursued by the present authors.

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