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Multivariate probability distribution of ordered peaks of vector Gaussian random processes

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Abstract

The problem of determining the joint probability distribution of ordered peaks of jointly stationary Gaussian random processes is considered. The solution is obtained by modeling the number of times a specified threshold is crossed by the component processes as a multivariate Poisson process. Based on this, the joint probability distribution of the time required for the *n*th crossing of a specified level with positive slope is derived. This formulation is further extended to derive the joint distribution of ordered peaks in a given time interval. An illustrative example on a bivariate Gaussian random process is presented and the analytical predictions are shown to compare reasonably well with corresponding results from Monte Carlo simulations. Also presented is an analysis of response of a randomly driven multi-degree of freedom system with emphasis on the sensitivity of ordered peak characteristics with respect to changes in system parameters. It is demonstrated that higher order statistics are generally more sensitive to changes in system characteristics—a property that has potential for application in structural model updating and damage detection.

Keywords: Probability distribution; Gaussian random process; Ordered peaks

1. Introduction

Extreme values of structural capacities and loads play decisive role in the study of reliability of structural systems [14]. The development of response spectrum-based methods in earthquake engineering and gust factor method in wind engineering is strongly based on the theory of extremes of random processes [15]. The study of extreme values of a sequence of random variables forms a subset of a more broad-based study on order statistics. The properties of order statistics in general, and, extreme values in particular, of sequence of random variables have been widely studied in the existing literature by mathematicians and engineers. From a mathematical perspective, the books by Gumbel [7], Galambos [6], Resnick [17], Castillo [4] and Kotz and Nadarajah [13] provide comprehensive overviews on the subject. A few highlights of established results, as can be deduced from a study of these books, can be summarized briefly as follows. For a sequence of identical and independently distributed (IID) random variables, there exist fundamentally three forms of asymptotic distributions for the extremes. Methods to establish domains of attraction of these asymptotic distributions are available. The tail behavior of the underlying random variables forms the basis on which these domains are established. Similarly, it is possible to identify feasible limit distributions for *k*th order statistics in the IID case, with separate analysis for upper or lower, moderately upper and lower and central order statistics. Results on extremes of sequence of identical, but dependent, random variables include, conditions on dependence structure, which ensures the continued validity of limiting distributions that are strictly valid only for IID case and results on more general forms of dependence sequences. The treatment of asymptotic forms of extremes of nonstationary and dependent sequence of random

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variables is rather limited [11]. Studies on asymptotic forms of multivariate extreme value distributions of vector sequences of random variables have also been conducted. Here results are available mainly for distinct IID sequences such as U_i and V_i ($i=1,2,\ldots,n$), where the random variables U_i and V_j are mutually independent except when $i=j$. Results on asymptotic joint distributions of ordered statistics of U_i and $V_{i,}$ ($i=1,2,\ldots,n$) are also available.

In structural engineering applications, the study of ordered peaks of random processes has received limited attention. These studies are of interest, for example, in the study of progressive failure of ductile structures subjected to dynamic loads such as earthquake ground accelerations. Here, the exceedance of the structural response across a specified level only once over specified time duration may not signal the failure of the system. Consequently, properties of lower order peaks become important, especially, if attention is focused on response levels that make mild excursions into nonlinear regimes but nevertheless remain higher than permissible linear limits. The work by Amini and Trifunac [1], Gupta and Trifunac [8,9], Basu and Gupta and Basu et al. [2,3] contain discussions on related issues. The starting point in these studies is the model for the PDF of local maxima of Gaussian random processes [16]. Amini and Trifunac derive the probability distribution function (PDF) of the *n*th order peak by assuming that the peaks are independent, and, also, that the probability of exceeding a specified level by all the peaks is the same as that by the first order peak. Gupta and Trifunac [8] relax the second of these assumptions. Basu et al. [2,3] have employed simulation-based method and Markovian modeling procedures to investigate the properties of ordered peaks by relaxing the assumption of independence of peaks. The note by Gupta and Trifunac [8] reports on numerical experiments aimed at evaluation of relative performance of procedures due to Amini and Trifunac [1], Gupta and Trifunac [9] and Basu et al. [3]. This study concludes that the assumption of independence of peaks in deriving the PDF of ordered peaks is not restrictive.

In the present study, we develop a model for joint PDF of vector of ordered peaks associated with a vector stationary Gaussian random process. This distribution function is shown to be related to the joint distribution of time for crossing of a given threshold for the *n*th time. This, in turn, is deduced in terms of a multivariate counting process for modeling the number of times a specified threshold is crossed. This work is in continuation of a recent study by Gupta and Manohar [10] in which multivariate distributions for extremes of vector Gaussian random process have been developed and applied to problems of time variant structural system reliability analysis. We foresee that the properties of ordered peaks of random structural responses could serve as useful tools for structural model updating and damage detection. The present study offers preliminary evidence in support of this prognosis.

2. Analysis

Consider an $n\times 1$ vector random process $\boldsymbol{X}(t)$, $t\in T$, whose elements are made up of zero mean, jointly stationary, Gaussian random processes. Let $\boldsymbol{R}(\tau)$ and $\boldsymbol{S}(\omega)$ denote, respectively, the covariance matrix and the power spectral density (PSD) function matrix of $\boldsymbol{X}(t)$. The elements of $\boldsymbol{X}(t)$ are assumed to be differentiable at least once. In this study, we seek to obtain the following descriptors of $\boldsymbol{X}(t)$:

1. $N_i(\alpha_i,0,T)$ = the number of times the random process $X_i(t)$ crosses the level α_i with positive slope in the time interval 0–T. For a given T, $\boldsymbol{N}=\{N_i(\alpha_i,0,T\}$ constitutes a vector of discrete random variables. What is the multivariate PDF of the vector $\boldsymbol{N}$?
2. T_{ij} = the time taken by $X_i(t)$ to cross a threshold α_i with positive slope for the *j*th time. What is the joint PDF of T_{ij} for $i=1,2,\ldots,k\le n$ and $j=1,2,\ldots$?
3. X_{ij} = the *j*th highest peak of $X_i(t)$ over a duration 0–T. What is the joint PDF of X_{ij} for $i=1,2,\ldots,k\le n$ and $j=1,2,\ldots$?

It may be remarked here that for the special case of $n=1$, N can be modeled approximately as a Poisson random variable from which it can be shown that T_{11} is an exponential random variable and X_{11} is a Gumbel random variable [16]. In a recent study Gupta and Manohar [10] have considered the case of $n>1$ and has developed multivariate Gumbel models for X_{i1} ($i=1,2,\ldots,k\le n$).

2.1. Level crossings

To arrive at the joint PDF of $N_i(\alpha_i,0,T)$, $i=1,2,\ldots,k\le n$, for a given T, we assume that these counting processes constitute a vector of multi-variate Poisson random variables. To clarify this, we consider the case of $k=2$. We introduce the transformation

$$\begin{aligned} N_1(\alpha_1,0,T) &= U_1+U_3 \\ N_2(\alpha_2,0,T) &= U_2+U_3 \end{aligned} \tag{1}$$

where $U_i(i=1,2,3)$ are three mutually independent Poisson random variables with parameters λ_i ($i=1,2,3$), respectively. These parameters are as yet unknowns. It can be shown that $N_1(\alpha_1,0,T)$ and $N_2(\alpha_2,0,T)$ are Poisson random variables with parameters $(\lambda_1+\lambda_3)$ and $(\lambda_2+\lambda_3)$, respectively. It may be noted that this construct of multivariate Poisson random variables has been discussed in the existing literature [12]. It can also be shown that the covariance of $N_1(\alpha_1,0,T)$ and $N_2(\alpha_2,0,T)$ *is equal to* λ_3. Based on this we get the equation

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{Bmatrix} = \begin{Bmatrix} \langle N_1(\alpha_1,0,T)\rangle \\ \langle N_2(\alpha_2,0,T)\rangle \\ \mathrm{Cov}[N_1(\alpha_1,0,T),N_2(\alpha_2,0,T)] \end{Bmatrix} \tag{2}$$

Here $\langle . \rangle$ denotes the mathematical expectation operator and Cov[.] denotes the covariance. The counting processes $N_i(\alpha_I,0,T)$ $(I=1,2)$, are well known to be related to the parent processes $X_i(t)$ $(I=1,2)$, through the relations [16]

$$N_i(\alpha_i,0,T)=\int_0^T \delta[X_i(t)-\alpha_i]U[\dot{X}_i(t)]\dot{X}_i(t)\mathrm{d}t \quad (3)$$

Here $U(.)$ and $\delta(.)$ are, respectively, the Heaveside and Dirac's delta functions. Using the above relation, it is possible to calculate the quantities appearing on the right hand side of Eq. (2). The determination of covariance function appearing in Eq. (2) requires the evaluation of a six dimensional integral. Gupta and Manohar [10] have employed symbolic manipulation tools to reduce this integral to a two-dimensional integral that requires numerical evaluation. The unknowns $\lambda_i(i=1,2,3)$ can thus be evaluated once the right hand side in Eq. (2) is determined. To construct the joint PDF of $N_1(\alpha_1,0,T)$ and $N_2(\alpha_2,0,T)$ we first construct the joint characteristic function

$$\begin{aligned}\Phi_{12}(\omega_1,\omega_2)&=\langle\exp[i\omega_1N_1+i\omega_2N_2]\rangle\\&=\langle\exp[i\omega_1(U_1+U_3)+i\omega_2(U_2+U_3)]\rangle\\&=\langle\exp[i\omega_1U_1]\rangle\langle\exp[i\omega_2U_2]\rangle\langle\exp[iU_3(\omega_1+\omega_2)\rangle\\&=\exp[-(\lambda_1+\lambda_2+\lambda_3)+\lambda_1\exp[i\omega_1]+\lambda_2\exp[i\omega_2]\\&\quad+\lambda_3\exp[i(\omega_1+\omega_2)]\end{aligned} \quad (4)$$

We accept this as the definition of a bivariate Poisson random variable. It can also be shown that [12]

$$\begin{aligned}P[N_1=j\cap N_2=l]&=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\Phi_{12}(\omega_1,\omega_2)\exp[-i(\omega_1j+\omega_2l)]\mathrm{d}\omega_1\,\mathrm{d}\omega_2\\&=\exp[-(\lambda_1+\lambda_2+\lambda_3)]\sum_{i=0}^{\min(j,l)}\frac{\lambda_1^{j-i}\lambda_2^{l-i}\lambda_3^{i}}{(j-i)!(l-i)!i!}\end{aligned} \quad (5)$$

This result can be generalized for $k>2$ in a reasonably simple manner. The number of mutually independent Poisson random variables can be generalized to be given by $C_1^k+C_2^k$ where C_n^k denotes combination of k random variables taken n at a time. Thus, for $k=3$, we consider six mutually independent Poisson random variables U_i with parameters λ_i $(i=1,2,\ldots,6)$ and define the transformation

$$\begin{aligned}N_1(\alpha_1,0,T)&=U_1+U_4+U_5\\N_2(\alpha_2,0,T)&=U_2+U_4+U_6\\N_3(\alpha_3,0,T)&=U_3+U_5+U_6\end{aligned} \quad (6)$$

The equation relating λ_i $(i=1,2,\ldots,6)$ with moments of $N_i(\alpha_i,0,T)$, $i=1,2,3$ can be shown to be given by

$$\begin{bmatrix}1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{Bmatrix}\lambda_1\\\lambda_2\\\lambda_3\\\lambda_4\\\lambda_5\\\lambda_6\end{Bmatrix}=\begin{Bmatrix}\langle N_1(\alpha_1,0,T)\rangle\\\langle N_2(\alpha_2,0,T)\rangle\\\langle N_3(\alpha_3,0,T)\rangle\\\mathrm{Cov}[N_1(\alpha_1,0,T)N_2(\alpha_2,0,T)]\\\mathrm{Cov}[N_1(\alpha_1,0,T)N_3(\alpha_3,0,T)]\\\mathrm{Cov}[N_2(\alpha_2,0,T)N_3(\alpha_3,0,T)]\end{Bmatrix} \quad (7)$$

For the case of $k=3$, the joint characteristic function of N_1, N_2 and N_3 can be shown to be given by

$$\begin{aligned}&\Phi_{123}(\omega_1,\omega_2,\omega_3)\\&=\exp\left[-\sum_{j=1}^{6}\lambda_j+\sum_{j=1}^{3}\lambda_j\exp\{i\omega_j\}+\lambda_4\exp\{i(\omega_1+\omega_2+\omega_3)\}\right.\\&\quad\left.+\lambda_5\exp\{i(\omega_1+\omega_3)\}+\lambda_6\exp\{i(\omega_2+\omega_3)\}\right]\end{aligned} \quad (8)$$

It should be noted here that in constructing this model we need information on only the mean and covariance of $N_i(\alpha_i,0,T)$, $i=1,2,\ldots,k\leq n$ even when $k>2$. It is also to be emphasized in this context that the assumption that the number of crossings of a given level in a given duration is Poisson distributed is not always valid. Cramer [5] has shown that this assumption is asymptotically exact when the threshold levels increase to infinity. However, as pointed out by Vanmarcke [18, 19], for barrier levels of practical interest, this assumption results in errors whose size and effect depend upon the bandwidth of the underlying parent process. For wide band processes, the Poisson approximation makes no allowance for the time the process spends above the threshold levels, while, for narrow band processes, the level crossings tend to occur in clumps thereby introducing statistical dependence between occurrence of level crossings. Thus the models for extremes and other order statistics developed in the present study are not free of limitations resulting from Poisson approximations for the number of level crossings.

2.2. Passage times

For the purpose of illustration we begin by considering the case of $k=1$. It is well known that the first passage time $T_{11}(\alpha)$ is related to the counting process $N_1(\alpha,0,T)$ through the relation

$$P[T_{11}(\alpha)>T]=P[N_1(\alpha,0,T)=0]=\exp[-\lambda_1T] \quad (9)$$

Here it is assumed that $P[T_{11}(\alpha)=0]=0$. The above result can be generalized to characterize the time for the nth crossing of level α with positive slope by $X_1(t)$. Thus one gets

$$P[T_{1n}(\alpha) > T] = P[N_1(\alpha, 0, T) \leq n-1]$$

$$= \sum_{r=0}^{n-1} \exp(-\lambda_1 T) \frac{(\lambda_1 T)^r}{r!} \quad (10)$$

Based on this, we can also derive the joint distribution

$$P[T_{1n}(\alpha) > T \cap T_{1m}(\alpha) > T]$$

$$= P[N_1(\alpha, 0, T) \leq (n-1) \cap N_1(\alpha, 0, T) \leq (m-1)]$$

$$= P[N_1(\alpha, 0, T) \leq \min(n-1, m-1)] \quad (11)$$

The extension of the above results for case of $k>1$ can be achieved in a reasonably straightforward manner. Thus for $k=2$, one gets

$$P[T_{ij}(\alpha) > T \cap T_{rs}(\beta) > T]$$

$$= P[N_i(\alpha, 0, T) < (j-1) \cap N_r(\beta, 0, T) \leq (s-1)] \quad (12)$$

This probability can be evaluated knowing the joint distribution of the two counting processes (Eq. (5)).

2.3. Ordered peaks

Fig. 1 shows a sample of a Gaussian stationary random process in which the positive peaks are marked with a star. It is well known that Gumbel random variables serve as acceptable models for the highest peak in a given time interval. The parameters of this model can be derived in terms of number of times a given level is crossed with positive slope. This counting process can approximately be modeled as a Poisson process [16]. This result can be generalized on two fronts:

(a) Determination of the joint PDF of highest peaks of a vector of Gaussian random processes.

(b) Determination of the joint PDF of ordered peaks, other than just the highest, for a vector of Gaussian random processes.

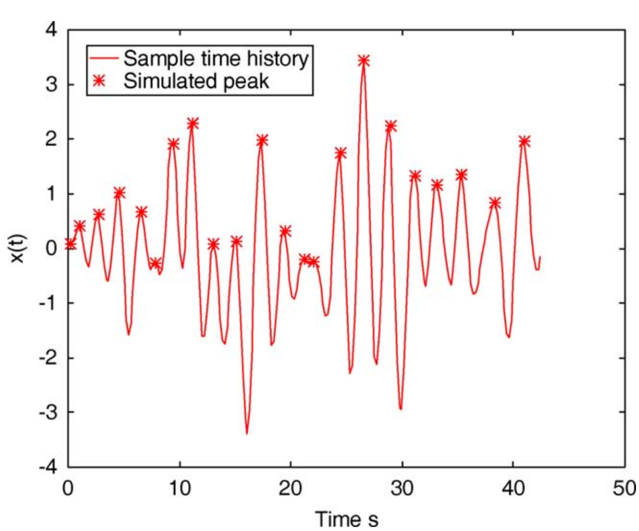

Fig. 1. Sample of $X(t)$ with positive peaks shown.

The first of these generalizations has been discussed in the recent study by Gupta and Manohar [10] in the context of time variant system reliability analysis. In the present study we focus attention on the second generalization. We again begin by considering the scalar case of $k=1$. The jth highest peak is related to the jth crossing statistic by

$$P[X_{1j} \leq \alpha] = P[T_{1j}(\alpha) > T] \quad (13)$$

Thus, for the first three highest peaks, one gets

$$P[X_{11} \leq \alpha] = P[T_{11}(\alpha) > T] = P[N_1(\alpha, 0, T) = 0] = \exp(-\lambda_1 T)$$

$$P[X_{12} \leq \alpha] = P[T_{12}(\alpha) > T] = P[N_1(\alpha, 0, T) \leq 1] = \exp(-\lambda_1 T)[1 + \lambda_1 T]$$

$$P[X_{13} \leq \alpha] = P[T_{13}(\alpha) > T] = P[N_1(\alpha, 0, T) \leq 2] = \exp(-\lambda_1 T)[1 + \lambda_1 T + 0.5(\lambda_1 T)^2] \quad (14)$$

The associated probability density functions (PDFs) can be easily by derived by differentiation with respect to α. It is of interest to note that in this formulation, ordered peaks are mutually dependent. Thus, for instance

$$P[X_{1r} \leq \alpha \cap X_{1(r+s)} \leq \alpha]$$

$$= P[N_1(\alpha, 0, T) \leq (r-1) \cap N_1(\alpha, 0, T) \leq (r+s-1)]$$

$$= P[N_1(\alpha, 0, T) \leq r-1] \neq P[X_{1r} \leq \alpha] P[X_{1(r+s)} \leq \alpha] \quad (15)$$

It is important to note that the proposed model imposes a specific type of dependence structure on the ordered peaks. To illustrate the formulation for the case of $k>1$, we consider $k=2$. Here one gets

$$P[X_{1j} \leq \alpha \cap X_{2r} \leq \beta] = P[T_{1j}(\alpha) > T \cap T_{2r}(\beta) > T]$$

$$= P[N_1(\alpha, 0, T) \leq (j-1) \cap N_2(\beta, 0, T) \leq (r-1)] \quad (16)$$

This probability can be deduced knowing the joint PDF of the counting processes N_1 and N_2 (Eq. (5)). Similar extensions for $k>2$ are straightforward albeit requiring involved formulary.

3. Numerical illustrations

3.1. Case 1

Here we consider a zero mean, scalar, stationary, Gaussian random process $X(t)$ with autocovariance and

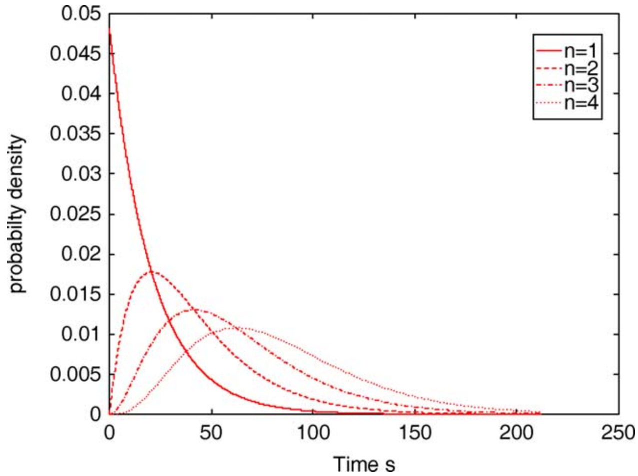

Fig. 2. Probability density function of nth passage times of $X(t)$; $\alpha=2\sigma_1$.

PSD functions given, respectively, by

$$R_{xx}(\tau)=2\exp(-\alpha_0\tau^2)\cos(\beta_0\tau);\quad -\infty<\tau<\infty\quad S_{xx}(\omega)=\sqrt{\frac{\pi}{\alpha_0}}\left\{\exp\left[-\frac{(\omega-\beta_0)^2}{4\alpha_0}\right]+\exp\left[-\frac{(\omega+\beta_0)^2}{4\alpha_0}\right]\right\};\quad -\infty<\omega<\infty \tag{17}$$

It can be shown that [16]

$$\langle N_X(\alpha,0,T)\rangle=\frac{\sigma_2}{2\pi\sigma_1}\exp\left[-\frac{\alpha^2}{2\sigma_1^2}\right]\quad \sigma_1^2=2;\ \sigma_2^2=2(\beta^2+2\alpha) \tag{18}$$

Fig. 2 shows the PDF of nth passage time for crossing a threshold of $2\sigma_1$ for $n=1,2,3$, and 4. The PDF of the first four ordered peaks over a time period of 0–$T=42.4$ s is shown in Fig. 3 along with the first order PDF of the parent process. The time duration of 42.5 s here correspond to about 25 times the correlation length of the process $X(t)$. The results on PDF of the first four ordered peaks are compared with corresponding results from Monte Carlo simulations with 700 samples in Fig. 4. It can be observed from this figure that there exists reasonable mutual agreement between theoretical and simulation results with the agreement being closer at the upper tails of the distribution functions. This feature is consistent with the fact that the assumption of Poisson model for $N(\alpha,0,T)$ is increasingly valid for higher values of α. In these results it is assumed that $\alpha_0=0.5$, and $\beta_0=2.0$.

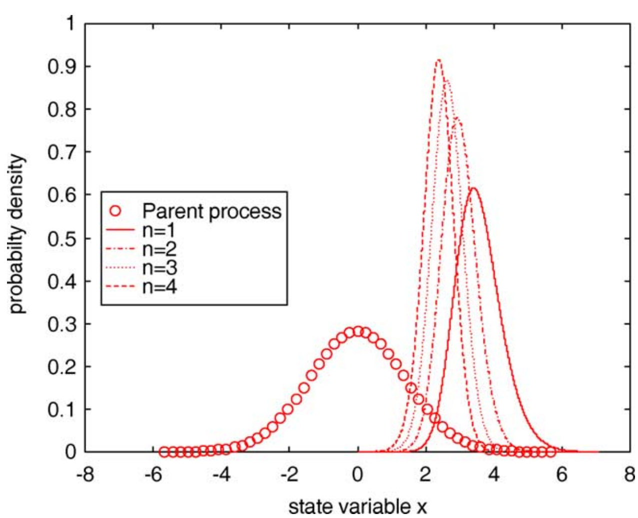

Fig. 3. Probability density function of parent process and nth order statistics of $X(t)$; $T=42.4$ s.

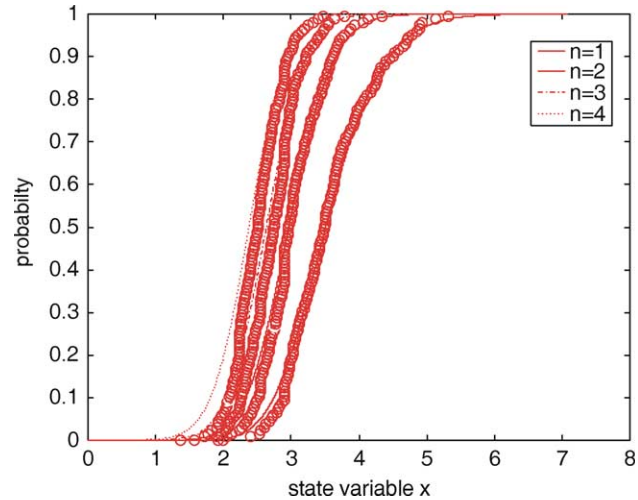

Fig. 4. Probability distribution of the first four highest peaks; 'o' denotes results from a 700 samples Monte Carlo simulations; $T=42.4$ s.

3.2. Case 2

Here we examine the case of two mutually dependent and jointly stationary Gaussian random processes $X(t)$ and $Y(t)$. We take $X(t)$ to be identical with the scalar random process considered in the previous case and define $Y(t)=A_1X(t)+A_2X(t+\varepsilon)$. Fig. 5 shows the autocovariance,

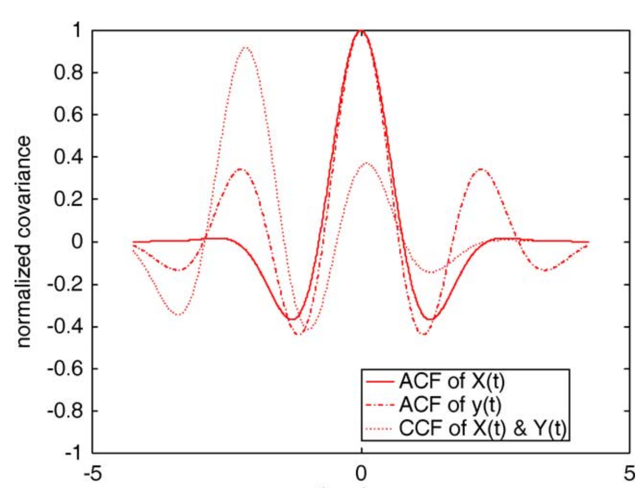

Fig. 5. Normalized autocovariance and cross covariance of $X(t)$ and $Y(t)$.

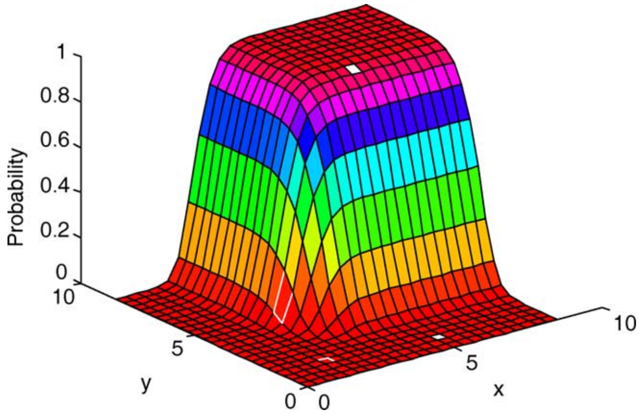

Fig. 6. Joint probability distribution of highest peaks of $X(t)$ and $Y(t)$; $T=42.4$ s.

normalized with respect to variance, and cross correlation coefficient of $X(t)$ and $Y(t)$ for $A_1=0.3$, $A_2=0.7$ and $\varepsilon=2.12$ s. It can be observed that $X(t)$ and $Y(t)$ are away from the extreme cases of being fully correlated or completely uncorrelated. Figs. 6 and 7, respectively, show the joint PDF of the first highest peaks and the second highest peaks of $X(t)$ and $Y(t)$ computed using Eq. (16). Here time duration over which ordered peaks are studied is $T=42.4$ s which is about 25 times the correlation length of the process $X(t)$ and about 20 times the correlation length of the process $Y(t)$. For the four random variables consisting of the first two highest peaks of $X(t)$ and $Y(t)$, respectively, the results on moments up to second order were obtained using theory and Monte Carlo simulations with 700 samples. The results on mean are found to be 3.56 (3.57), 2.98 (3.03), 2.68 (2.67), and 2.27 (2.30). Here the numbers in the parenthesis correspond to the simulation results. Similarly, the standard deviations of the four random variables were obtained as 0.67 (0.64), 0.50 (0.46), 0.46 (0.49), and 0.35 (0.37). The correlation coefficient matrix for these four random variables was determined to be

$$\rho = \begin{bmatrix} 1 & 0.63(0.61) & 0.62(0.64) & 0.45(0.47) \\ & 1 & 0.51(0.48) & 0.59(0.56) \\ & & 1 & 0.73(0.68) \\ \text{Sym} & & & 1 \end{bmatrix}$$

The reasonable agreement that is found to exist between the results of theory and simulations lends credence to the formulations developed in this study.

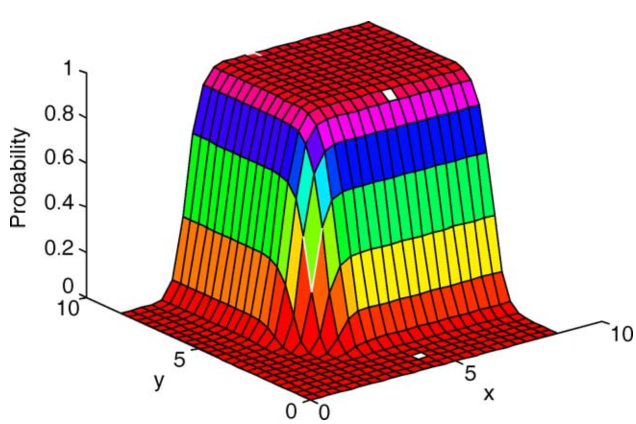

Fig. 7. Joint probability distribution of the second highest peaks of $X(t)$ and $Y(t)$; $T=42.4$ s.

3.3. Case 3

Here we consider the ordered peaks of response of linear multi degree of freedom (MDOF) systems subject to random excitations. Specifically we focus on examining the sensitivity of properties of ordered peaks of the response processes with respect to changes in values of structural parameters. This objective, in turn, is motivated by the question if properties of ordered peaks could serve as effective tools for detecting damage in the structure using vibration signatures. We believe that such a possibility has not been explored in the existing literature. The term damage here is taken to denote any changes in structural stiffness, mass or damping characteristics. We consider an MDOF system in its undamaged state governed by the equation

$$M\ddot{Z} + C\dot{Z} + KZ = F(t) \tag{19}$$

The system is assumed to start from rest. The excitation vector $F(t)$ is taken to consist of zero mean, jointly stationary Gaussian random processes. Upon the occurrence of damage, the structure equation of motion is modified to read

$$[M + \Delta M]\{\ddot{Z} + \Delta\ddot{Z}\} + [C + \Delta C]\{\dot{Z} + \Delta\dot{Z}\} + [K + \Delta K]\{Z + \Delta Z\} = F(t) \tag{20}$$

Here ΔM, ΔC and ΔK denote the changes in mass, damping and stiffness matrices due to occurrence of damage and ΔZ is the concomitant change in the response. The matrix of receptance functions for the structure, in its damaged state, can be written as

$$H(\omega) + \Delta H(\omega) = [-\omega^2(M + \Delta M) + i\omega(C + \Delta C) + (K + \Delta K)]^{-1} \tag{21}$$

Using the notation $D(\omega) = [-\omega^2 M + i\omega C + K]$ and $\Delta D(\omega) = [-\omega^2 \Delta M + i\omega \Delta C + \Delta K]$, we can write

$$H(\omega) + \Delta H(\omega) = [D(\omega) + \Delta D(\omega)]^{-1} \tag{22}$$

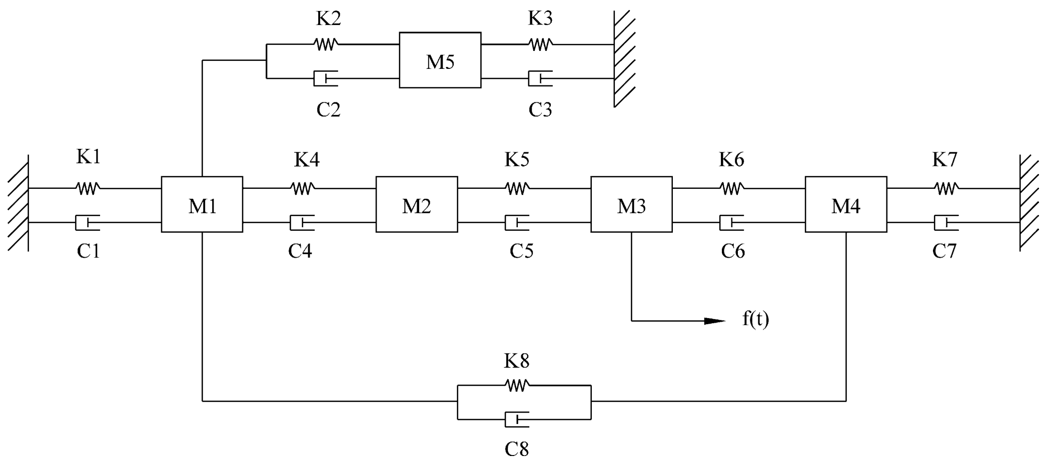

Fig. 8. Randomly excited MDOF system.

Using Neumann's expansion, we get

$$H(\omega) + \Delta H(\omega) = [I - P(\omega) + 0.5P^2(\omega) + \ldots]D^{-1}(\omega) \tag{23}$$

Here $P(\omega) = D^{-1}(\omega)\Delta D(\omega)$. Based on this, a second order expansion for the PSD matrix of response of the damaged structure can be obtained as

$$\begin{aligned} S(\omega) + \Delta S(\omega) = S(\omega) &- [S(\omega)P^{t*}(\omega) + P(\omega)S(\omega)] \\ &+ 0.5[S(\omega)(P^{t*}(\omega))^2 + P^2(\omega)S(\omega)] \\ &+ P(\omega)S(\omega)P^{t*}(\omega) \end{aligned} \tag{24}$$

Here the superscripts t and $*$ denote, respectively, the matrix transposition and conjugation. Furthermore, $S(\omega)$ denotes the PSD matrix of the response of the structure in its undamaged state and is given by

$$S(\omega) = D(\omega)S_{FF}(\omega)D^{t*}(\omega) \tag{25}$$

Based on Eqs. (24) and (25) one can compute the properties of ordered peaks of any desired response quantity for the damaged and undamaged structures using formulary developed in Section 2. We limit our attention in this study to investigate the relative magnitudes of changes in response quantities due to a given set of damage scenarios. To illustrate the formulation we consider the MDOF system shown in Fig. 8. We take, for the undamaged structure, $M_1 = 1.5M$, $M_2 = M$, $M_3 = 3M$, $M_4 = 2.5M$, $M_5 = 2M$, $K_1 = K$, $K_2 = 2K$, $K_3 = 1.5K$, $K_4 = 2K$, $K_5 = 3K$, $K_6 = 1.5K$, $K_7 = 2K$ and $K_8 = 4K$ with $M = 10$ kg and $K = 1000$ N/m. The natural frequencies of this structure are determined to be 5.75, 11.77, 15.75, 22.19, and 28.50 rad/s. The damping in the structure is taken to be viscous and proportional with damping ratios η_1 and η_2 in the first two modes being 0.03. Furthermore, mass 3 is taken to be driven by a band-limited white noise with a strength of unity and frequency range of 0–50 rad/s. To denote the changes in structural parameters we introduce the notation

$$\chi = \{M_1, M_2, \ldots, M_5, K_1, K_2, \ldots, K_8, \eta_1, \eta_2\}^t$$

The structural parameters, after the occurrence of damage, are denoted by $\chi_i + \Delta\chi_i = \chi_i \xi_i$ $(i = 1,2,\ldots,15)$. We first consider the scenario in which

$$\xi = \{0.99\ 0.98\ 0.97\ 0.97\ 0.98\ 0.97\ 0.98\ 0.99\ 0.99\ 0.97\ 0.99\ 0.98\ 0.98\ 0.97\ 0.99\}$$

This means that the structural damage is spatially distributed and affects mass, stiffness and damping properties. In the discussion to follow we designate this damage scenario as case 3.1. On account of these changes, the structure natural frequencies change to 5.77, 11.84, 15.83, 22.13 and 28.48 rad/s. Fig. 9 shows the PSD of response at the third mass for the undamaged and the damaged structure. As can be observed the difference is hardly perceptible. Also shown in this graph is the difference between the two PSD functions. In these plots, the results on

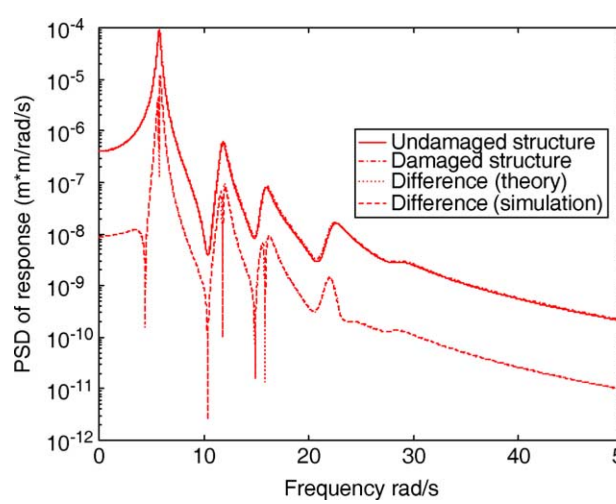

Fig. 9. PSD of response for undamaged and damaged structures. The difference of the two PSD functions is also shown.

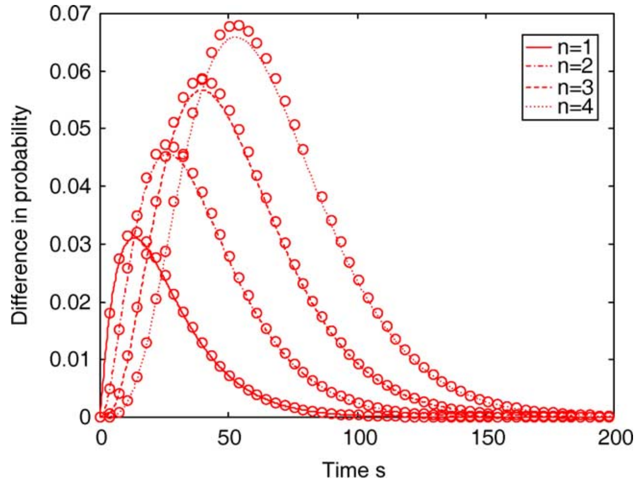

Fig. 10. Difference in PDF of the nth passage time for response at mass 3; case 3.1; extended lines denote theoretical results and 'o' denotes simulation results; $\sigma = 2\sigma$.

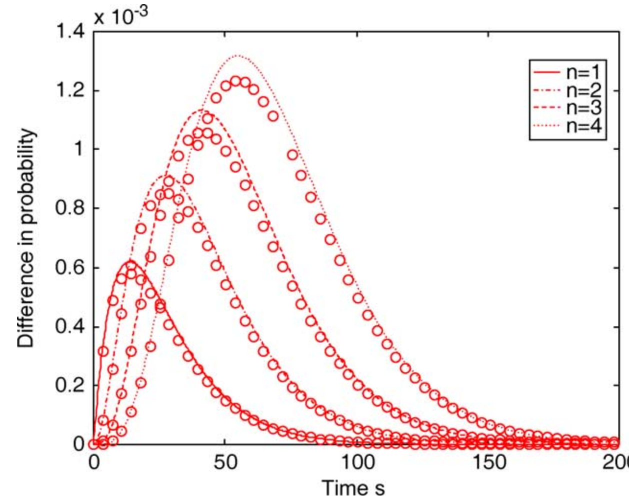

Fig. 12. Difference in PDF of nth passage time for response at mass 3; case 3.2; extended lines denote theoretical results and 'o' denotes simulation results; $\alpha = 2\sigma$.

modified structure, obtained using the formulation in Eq. (24), is compared with the simulation results. It can be observed that the sensitivity analysis predicts the changes in response fairly accurately. Figs. 10 and 11 show the changes that are observed in the probability distribution of passage times and the PDF of the ordered peaks. In computing the passage times, a threshold of two times the response standard deviation is set. Similarly, in computing the order statistics a time duration of 20 s is used. Fig. 11 also shows the changes observed in the PDF of the parent response process due to occurrence of damage. It is interesting to note that the changes in the distribution of higher order passage times are more pronounced than similar results for lower order passage times. Conversely, the changes produced in the density functions of the lower order peaks are higher than that observed for the higher order peaks. In any case, the changes in PDF of ordered peaks are far higher than that observed in the PDF of the parent process. This is, perhaps, to be expected, since, PDF of higher order passage times and ordered peaks depends upon many details of time evolution of response process which a first order PDF of the parent response process does not explicitly take into account. It may also be noted that Figs. 11 and 12 contain results from sensitivity analysis as well as simulated results and these results are in reasonably good mutual agreement. To illustrate the effect of isolated changes to structural properties we consider the case when one of the springs (K_1, see Fig. 8) is changed by an amount of 1% (case 3.2). Figs. 12 and 13, respectively, show the results on differences on probability distribution of passage times and the ordered peaks. These figures show trends that

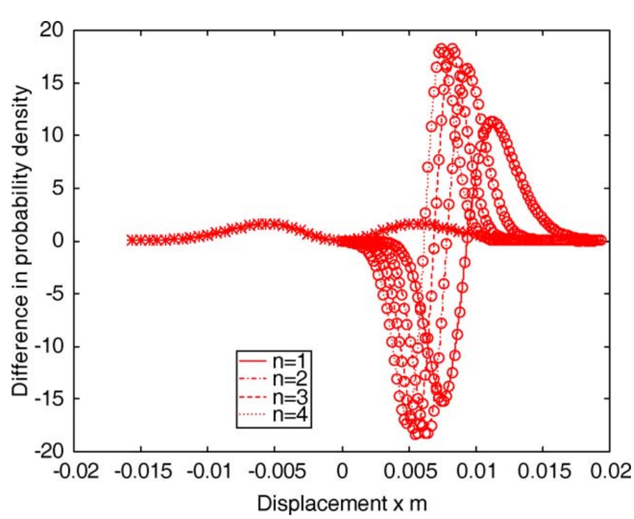

Fig. 11. Difference in probability density function of nth order statistics of response of undamaged and damaged structures; case 3.1; extended lines denote theoretical results and 'o' denotes simulation results; '*' denotes the theoretical results for the parent process; the full line close to '*' denotes simulation results; $T = 20$ s.

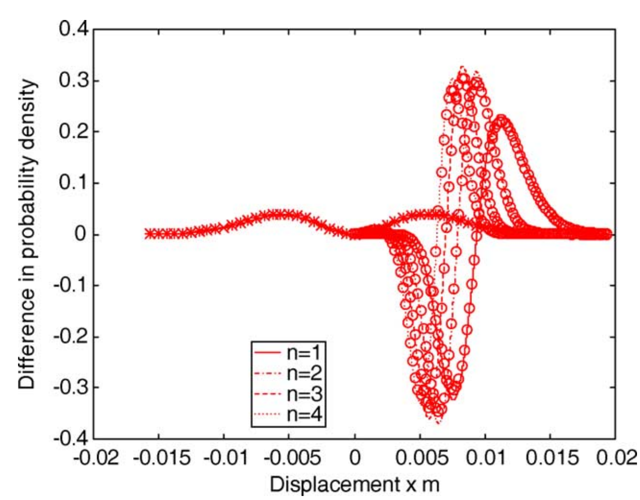

Fig. 13. Difference in probability density function of nth order statistics of response of undamaged and damaged structures; case 3.2; extended lines denote theoretical results and 'o' denotes simulation results; '*' denotes the theoretical results for the parent process; the full line close to '*' denotes simulation results; $T = 20$ s.

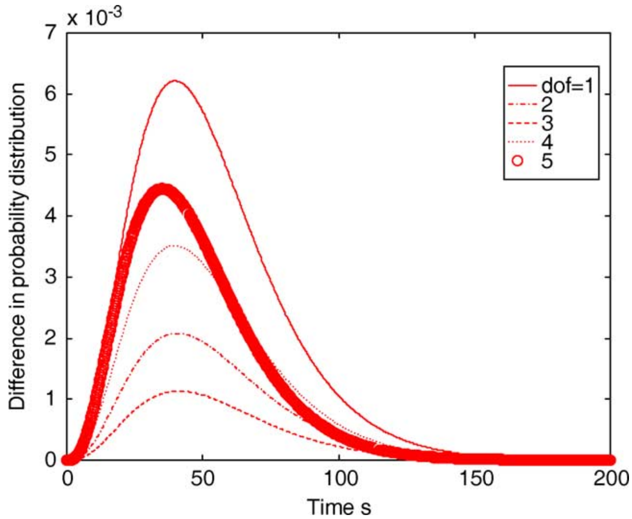

Fig. 14. Difference in PDF of third passage time for response at masses 1–5; case 3.2; $\alpha=2\sigma$.

are similar to those observed in Figs. 10 and 11. To examine the relative positioning of measurement point and the location of damage, the choice of response DOF has been varied and the resulting differences in distribution of passage times and the density functions of the ordered peaks are shown in Figs. 14 and 15. As one might expect, responses at locations closer to the location of the damage show relatively higher sensitivity with respect to the damage.

4. Conclusions

The focus of the present study has been on characterizing the probability distribution of ordered peaks of stationary Gaussian random processes. Specifically, the study has explored the relationship between processes that count the number of times a specified threshold level is crossed, the times for crossing of a specified threshold level for the nth time ($n=1,2,\ldots$) and the probability distribution of ordered peaks. Using the results from multi-variate Poisson processes, models for multi-variate probability distributions of ordered peaks of vector Gaussian random processes have been derived. The analytical predictions are shown to compare reasonably well with limited results from Monte Carlo simulations. One of the key features of the present study has been the assumption of independent crossings that is implicit in the Poissonian models for counting the level crossings. Improvements to the model can be achieved by using refined Markov models for the crossings. Similarly, the extension of the procedures developed to handle nonstationary and/or nonGaussian random processes require further work. The present study also explores the sensitivity of ordered peak characteristics of response processes of randomly driven linear MDOF systems to small changes in structural properties. Based on the illustrative example considered, it is found that higher order passage times and ordered peaks are more sensitive to structural changes than the response process itself. This one might expect, since, the examination of higher order passage times and ordered peaks permits a scrutiny of response processes in greater detail than the study of the probability structure of the response process itself. We believe that this feature has important applications in problems of structural model updating and vibration signature based structural damage detection. Such applications can be developed either using inverse sensitivity procedures or by employing Bayesian updation procedures. The present authors are currently pursuing studies on these lines.

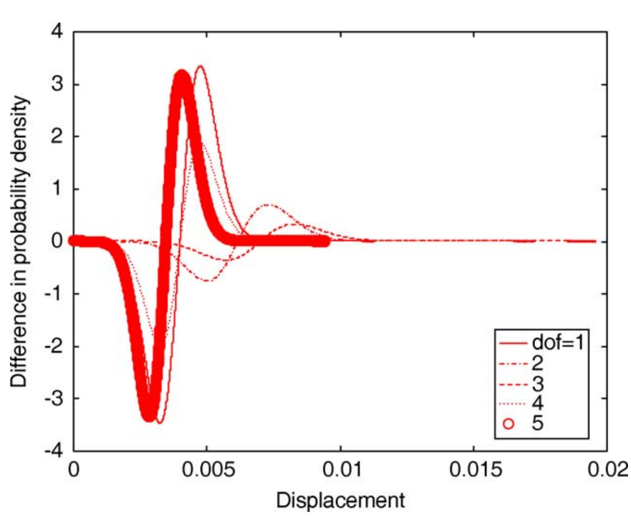

Fig. 15. Difference in probability density function of third order statistics of response at masses 1–5 of undamaged and damaged structures; case 3.2; $T=20$ s.

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