

FREE VIBRATION ANALYSIS OF STOCHASTIC STRINGS

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The free vibration of strings with randomly varying mass and stiffness is considered. The joint probability density functions of the eigenvalues and eigenfunctions are characterized in terms of the solution of a pair of stochastic non-linear initial value problems. Analytical solutions of these equations based on the method of stochastic averaging are obtained. The effects of the mean and autocorrelation of the mass process are included in the analysis. Numerical results for the marginal probability density functions of eigenvalues and eigenfunctions are obtained and are found to compare well with Monte Carlo simulation results. The random eigenvalues, when normalized with respect to their corresponding deterministic values, are observed to tend to become first order stochastically stationary with respect to the mode count.

1. INTRODUCTION

The analysis of structural systems with stochastic stiffness and mass properties is currently an active area of research [1, 2]. These problems are also of fundamental importance in the study of linear systems under high frequency excitations by using statistical energy analysis formalisms [3]. The determination of natural frequencies and mode shapes constitutes a challenging class of problems in the study of stochastically defined systems. These problems are associated with the study of random matrices and stochastic boundary value problems. Some of the earliest works in this area of research have been by Boyce and his associates on random strings and bars [4, 5] and by Soong and Bogdanoff on discrete multi-degree-of-freedom systems [6, 7]. Reviews of the existing literature dealing with this class of problems have been presented by Boyce [8], Scheidt and Purkert [9] and Ibrahim [2]. A variety of methods based on perturbation analysis, the transfer matrix approach and variational formulations have been discussed in these references. By and large, the available studies on random eigenvalues aim at estimating the first two moments and often succeed only in establishing bounds on them.

The eigenvalues associated with a second order stochastic boundary value problem has been considered earlier by Iyengar and Athreya [10]. These authors have characterized the eigenvalues in terms of zeros of the solution of an associated initial value problem. This allows one to bring in the powerful concepts of diffusion processes to characterize the probability density functions of the eigenvalues. This approach has been studied further by Iyengar and Manohar [11] and Manohar and Iyengar [12], and it has been shown that, although an exact solution of the associated initial value problem is rarely possible, for

specific types of stochastic variations, however, the problem can be readily analyzed by using approximate techniques. The present study is a continuation of these earlier investigations and reports on the extension of the formulation to study the probability density functions of the eigenfunctions. It is demonstrated that the consideration of the joint density functions of eigenvalues and eigenfunctions requires the solution of a *pair* of coupled non-linear stochastic initial value problems. Exact solutions of these equations are again found not to be possible, but satisfactory approximate solutions are still obtainable.

2. ANALYSIS

The simplest of the eigenvalue problems which arise in the study of stochastic continuous systems is, perhaps, the second order boundary value problem associated with the stochastic wave equation. This equation is encountered in the vibration analysis of strings, bars, shafts and soil layers having randomly varying mass and stiffness properties. The eigenvalue problem in this case can be stated as finding the non-trivial solution of the equation

$$\frac{d}{dx} \left\{ [1 + \delta g(x)] \frac{dy}{dx} \right\} + \lambda^2 [1 + \epsilon f(x)] y = 0, \quad (1)$$

under the boundary conditions

$$y(0) = 0, \quad y(1) = 0. \quad (2, 3)$$

Here $g(x)$ and $f(x)$ are, respectively, the stochastic stiffness and mass processes which are taken to be jointly stationary and bounded in a mean square sense. ϵ and δ are parameters such that $1 + \epsilon f(x) > 0$ and $1 + \delta g(x) > 0$. The independent variable x has been normalized with respect to the length and, therefore, takes values in $(0, 1)$. λ is the eigenvalue parameter. As a consequence of modelling mass and stiffness as random processes, the eigensolutions, in turn, become stochastic in nature. Thus, the eigenvalues are now random variables and eigenfunctions are random processes. The aim of the present investigation is to formulate an analytical procedure to calculate the joint probability density function (pdf) of λ and $y(x)$.

2.1. RANDOM EIGENVALUES

The procedure for obtaining the marginal probability distribution of eigenvalues has already been outlined in references [10–12]. The central idea of this procedure is to seek the solution of the stochastic boundary value problem given by equation (1)–(3) in terms of solutions of an associated *initial* value problem. This, in turn, consists of forming the solution $y^*(x, \lambda)$ of equation (1) for an arbitrarily chosen value of λ and for the *initial* conditions at $x = 0$

$$y^* = 0, \quad (dy^*/dx) = 1. \quad (4)$$

It must be noted that the initial conditions here are being referred to the co-ordinate x , while in vibration literature the term “initial condition” normally refers to the time scale. For any realization of $g(x)$ and $f(x)$, a family of solutions $y^*(x, \lambda)$ can be obtained by solving equations (1) and (4) with λ as a parameter. The members of this family can further be grouped into two categories depending on whether or not they satisfy the condition $y^*(1, \lambda) = 0$. Those members which do satisfy this condition clearly correspond to the eigenfunctions of equations (1)–(3). In other words, the eigensolutions of equations (1)–(3) form a subset of a broader class solutions to equations (1) and (4) obtained for different

values of λ . The essence of the present approach is to characterize this subset of solutions by using the known properties of $y^*(x, \lambda)$.

Let $Z_n(\lambda)$ denote the n th zero of $y^*(x, \lambda)$. The eigenvalues of equations (1)–(3) can be defined in terms of $Z_n(\lambda)$ as being the roots of the equation

$$Z_n(\lambda_n) = 1. \tag{5}$$

This definition enables the study of λ_n to be carried out in terms of the properties of $Z_n(\lambda)$. It is known from the Sturm–Liouville theory of ordinary differential equations that $Z_n(\lambda)$ are non-increasing in λ [13]. The study of $Z_n(\lambda)$ is facilitated by the co-ordinate transformation

$$y^*(x, \lambda) = r(x, \lambda) \sin \phi(x, \lambda), \quad (1 + \delta g) dy^*/dx = r(x, \lambda)\lambda \cos \phi(x, \lambda). \tag{6, 7}$$

This leads to

$$\frac{d\phi}{dx} = \lambda \left\{ \frac{\cos^2 \phi}{1 + \delta g} + (1 + \epsilon f) \sin^2 \phi \right\}, \quad \frac{dr}{dx} = \frac{1}{2} \lambda r \sin 2\phi \left[\frac{1}{(1 + \delta g)} - (1 + \epsilon f) \right],$$

$$\phi(0) = 0, \quad r(0) = [1 + \delta g(0)]/\lambda. \tag{8-11}$$

It follows from equation (6) that $Z_n(\lambda)$ satisfy the equation

$$\phi[Z_n(\lambda), \lambda] = n\pi. \tag{12}$$

Thus the study of $Z_n(\lambda)$ reduces to the study of the process $\phi(x, \lambda)$. It may be observed from equation (8) that the right side is always positive, which implies that $\phi(x, \lambda)$ is a non-decreasing function in x and since $Z_n(\lambda)$ is non-increasing in λ it follows that

$$P[\lambda_n \leq \lambda] = P[Z_n(\lambda) \leq 1] = P[n\pi \leq \phi(1, \lambda)], \tag{13}$$

where $P[\]$ denotes the probability measure. Thus, in order to find the probability distribution function of λ_n , equations (8) and (10) must be solved to obtain the pdf of the process $\phi(x, \lambda)$ at $x = 1$.

2.2. RANDOM EIGENFUNCTIONS

The pdf of the eigenfunctions can also be characterized in terms of $r(x, \lambda)$ and $\phi(x, \lambda)$. For this purpose, consider the joint pdf between the n th eigenfunction and the n th eigenvalue. An expression for this pdf can be constructed using the properties of $y^*(x, \lambda)$ and $\phi(x, \lambda)$ and a few standard identities of probability theory. Thus,

$$\begin{aligned} p_{y_n, \lambda_n}(y, x; \lambda) &= P[y < y_n(x) < y + dy, \lambda < \lambda_n < \lambda + d\lambda] = p_{y_n | \lambda_n}[y, x | \lambda_n = \lambda] p_{\lambda_n}(\lambda) \\ &= p_{y^* | \lambda_n}[y, x | \lambda_n = \lambda \text{ and } x = 1 \text{ is the } n\text{th zero of } y^*(x, \lambda)] p_{\lambda_n}(\lambda) \\ &= p_{y^* | \lambda_n}[y, x | \lambda_n = \lambda \text{ and } \phi(1, \lambda) = n\pi] p_{\lambda_n}(\lambda) \\ &= p_{y^*, \phi | \lambda_n}[y, x; n\pi, 1 | \lambda_n = \lambda] p_{\lambda_n}(\lambda) / p_\phi(n\pi, 1, \lambda). \end{aligned} \tag{14}$$

The pdf of $y_n(x)$ can now be obtained as

$$p_{y_n}(y, x) = \int_0^\infty \frac{p_{y^*, \phi | \lambda_n}[y, x; n\pi, 1 | \lambda_n = \lambda] p_{\lambda_n}(\lambda)}{p_\phi(n\pi, 1, \lambda)} d\lambda. \tag{15}$$

Here $p_\phi(n\pi, 1, \lambda)$ denotes the pdf of $\phi(1, \lambda)$ evaluated at $\phi = n\pi$. To evaluate the conditional pdf appearing in the integrand of this expression, the joint pdf between $y^*(x, \lambda)$ and $\phi(1, \lambda)$ needs to be determined. This, in turn, necessitates knowledge of the joint pdf of $r(x, \lambda)$, $\phi(x, \lambda)$ and $\phi(1, \lambda)$. Notice that the equation for $r(x, \lambda)$ is coupled to $\phi(x, \lambda)$, while the equation for $\phi(x, \lambda)$ is independent of $r(x, \lambda)$. This means that the marginal pdf

of eigenvalues can be determined in terms of the solution of a first order equation, while the consideration of the pdf of the eigenfunctions requires the solution of two coupled first order equations.

3. STOCHASTIC AVERAGING

In order to apply the identities (13) and (14), it is necessary first to solve equations (8)–(11). These equations constitute a pair of stochastic initial value problems. The presence of non-linearity in these equations makes the solution procedures difficult. Only under very special circumstances can one obtain exact solutions to these equations. One such case arises when the condition

$$[1 + \delta g(x)][1 + \epsilon f(x)] = 1 \quad (16)$$

is satisfied [11, 14]. A discussion of the solutions for this special case is presented in Appendix A, where the exact solutions are used to illustrate the validity of equations (13) and (14). When dealing with more realistic problems, however, the analysis of equations (8)–(11) presents considerable difficulties. One way to approach this problem is to model $f(x)$ and $g(x)$ as filtered Gaussian white noise processes, and to employ the Markov process theory to derive the Fokker–Planck equation governing the transitional pdf of the vector (r, ϕ, f, g) [15]. Although the derivation of this equation is straightforward, finding exact solutions is, however, not possible. Thus, to proceed further, approximations become necessary. In this context, it may be recalled that one of the powerful approximation techniques in random vibration applications has been the method of stochastic averaging [16]. This technique is applicable to initial value problems involving weak, broadband stochastic coefficients and results in a diffusion process approximation to the solution. The

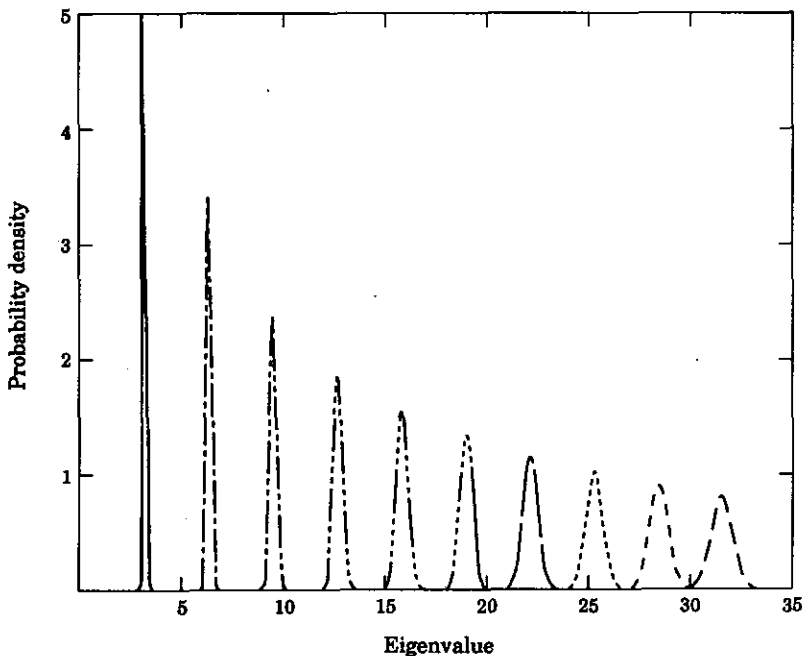


Figure 1. The probability density function of λ_n ; $n = 1, 10$; $\alpha = 20.8$, $\epsilon = 0.1$. —, $n = 1$; - - - - , $n = 2$; ·····, $n = 3$; - · - · - ·, $n = 4$; - - - - - , $n = 5$; - · - · - · - ·, $n = 6$; - · - · - · - · - ·, $n = 7$; - · - · - · - · - · - ·, $n = 8$; - · - · - · - · - · - · - ·, $n = 9$; - · - · - · - · - · - · - · - ·, $n = 10$.

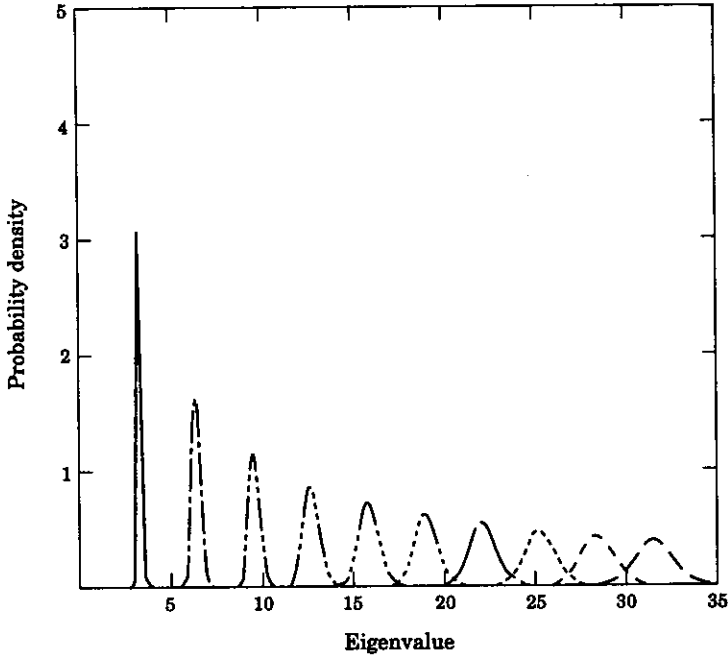


Figure 2. The probability density function of λ_n ; $n = 1, 10$; $\alpha = 20.8$, $\epsilon = 0.2$; legend as in Figure 1.

averaging procedure is a combination of “temporal” and ensemble averaging, and aims at eliminating rapid oscillations from the dominant slowly varying components and also at replacing randomly fluctuating components by equivalent delta correlated processes. The resulting simplified equations can often be solved within the framework of the Markov process theory. It is intended here to investigate the scope of this technique in analyzing equations (8)–(11) and also in the further analyses of the random eigenvalues and eigenfunctions.

In the further analysis it is assumed that $\delta = 0$. This means that the randomness in the resulting problem is taken to arise solely out of random variations in the mass process $f(x)$. Under the assumption that $\epsilon \ll 1$ and also that the correlation length of $f(x)$ is much less than the relaxation lengths of the processes $r(x)$ and $\phi(x)$, equations (8)–(11) can be analyzed by using the stochastic averaging approximations. For this purpose it is found advantageous to transform r and ϕ as

$$r(x, \lambda) = \exp[a(x, \lambda)], \quad \phi(x, \lambda) = \lambda x + \theta(x, \lambda). \tag{17, 18}$$

This leads to

$$\begin{aligned} \frac{da}{dx} &= -\frac{1}{2}\epsilon\lambda f(x) \sin 2(\theta + \lambda x), & \frac{d\theta}{dx} &= \frac{1}{2}\epsilon\lambda f(x)[1 - \cos 2(\theta + \lambda x)], \\ a(0) &= \ln r_0, & \theta(0) &= 0. \end{aligned} \tag{19–21}$$

This pair of equations can now be simplified by using the stochastic averaging theorem [16]. The details of this derivation are presented in Appendix B and, as has been shown there, the solution vector can be approximated by a two-dimensional Markov process which satisfies the Itô equations

$$da = c^2 m_1 dx + \epsilon \sigma_{11} dW_1(x), \quad d\theta = c^2 m_2 dx + \epsilon \sigma_{22} dW_2(x). \tag{22}$$

The solution of this pair of equations is straightforward and one obtains

$$a(x) = \epsilon^2 m_1 x + \epsilon \sigma_{11} \xi_1(x) + a_0, \quad \theta(x) = \epsilon^2 m_2 x + \epsilon \sigma_{22} \xi_2(x),$$

$$\xi_1(x) = \int_0^x dW_1(s), \quad \xi_2(x) = \int_0^x dW_2(s). \quad (23-26)$$

Notice that the processes $\xi_1(x)$ and $\xi_2(x)$ are independent and Gaussian distributed. Consequently, in the present approximation the processes $a(x)$ and $\theta(x)$ become stochastically independent Gaussian random processes. Equations (23) and (24) lead to

$$y^*(x, \lambda) = r_0 \exp[\epsilon^2 m_1 x + \epsilon \sigma_{11} \xi_1(x)] \sin [\lambda x + \epsilon^2 m_2 x + \epsilon \sigma_{22} \xi_2(x)]. \quad (27)$$

It may be noted that, $\lim_{\epsilon \rightarrow 0} y^*(x, \lambda) = \sin \lambda x$, which is as it should be.

The probability distribution function of the eigenvalues can now be determined by using equations (13) and (24). This leads to

$$P[\lambda_n \leq \lambda] = P[\epsilon^2 m_2 + \epsilon \sigma_{22} \xi_2(1) \geq n\pi - \lambda] = 1 - \int_{-\infty}^{\left[\frac{n\pi - \lambda - \epsilon^2 m_2}{\epsilon \sigma_{22}} \right]} p_{\xi_2}(\xi) d\xi. \quad (28)$$

Here $p_{\xi_2}(\xi)$ represents the Gaussian density function of $\xi_2(1)$. In turn, the probability density of λ_n is given by

$$p_{\lambda_n}(\lambda) = -\frac{d}{d\lambda} \int_{-\infty}^{\left[\frac{n\pi - \lambda - \epsilon^2 m_2}{\epsilon \sigma_{22}} \right]} p_{\xi_2}(\xi) d\xi. \quad (29)$$

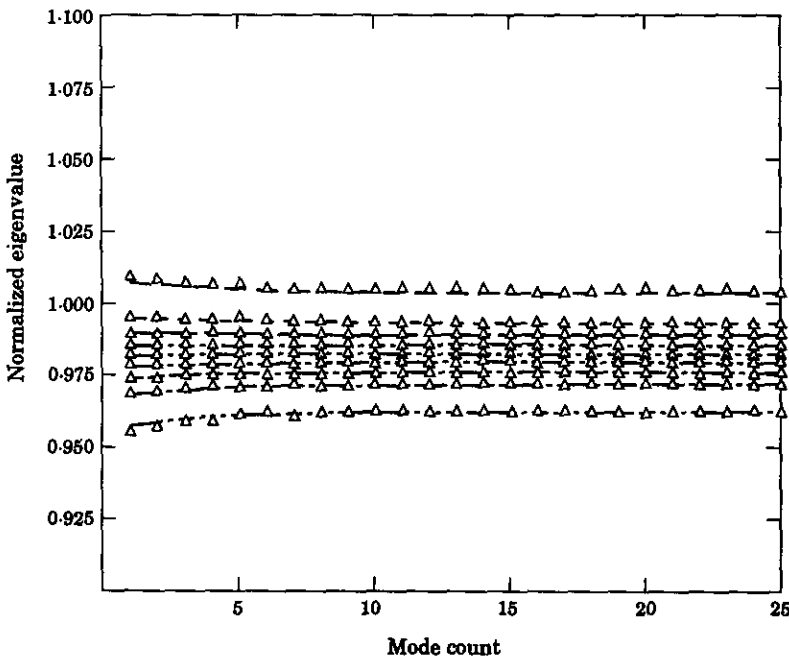


Figure 3. Contours of the probability distribution functions of the normalized eigenvalues $\lambda_n/n\pi$, $\epsilon = 0.1$, $\alpha = 20.8$. Δ , Digital simulations with 2000 samples; -----, $P = 0.05$; - - - - - , $P = 0.2$; - · - · - · , $P = 0.3$; - · - - - · , $P = 0.4$; - · - · - · - · , $P = 0.5$; - · - · - · - · - · , $P = 0.6$; - · - · - · - · - · - · , $P = 0.7$; - · - · - · - · - · - · - · , $P = 0.8$; - · - · - · - · - · - · - · - · - · , $P = 0.95$.

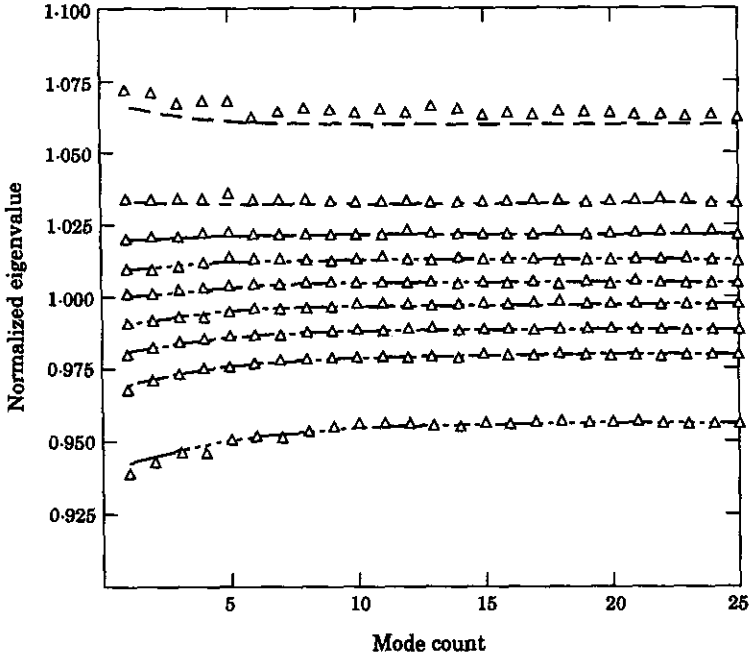


Figure 4. Contours of the probability distribution functions of the normalized eigenvalues $\lambda_n/n\pi$; $\epsilon = 0.2$, $\sigma = 20.8$; legend as in Figure 3.

Furthermore, by using equations (24) and (18) it can be shown that

$$p_\phi(n\pi, \lambda, 1) = p_{\xi_{21}} \left[\frac{n\pi - \lambda - \epsilon^2 m_2}{\epsilon \sigma_{22}} \right] / \epsilon \sigma_{22}. \tag{30}$$

In order to determine $p_{y^*, \phi | \lambda_n}(y, n\pi | \lambda_n = \lambda)$, a dummy variable

$$z^*(x, \lambda) = \exp[\epsilon^2 m_1 x + \epsilon \sigma_{11} \xi_1(x)] \cos[\lambda x + \epsilon^2 m_2 x + \epsilon \sigma_{22} \xi_2(x)] \tag{31}$$

is introduced. By using this equation together with equation (27) and the relation

$$\phi(1, \lambda) = \lambda + \epsilon^2 m_2 + \epsilon \sigma_{22} \xi_2(1) \tag{32}$$

it can be shown that

$$p_{y^*, \phi | \lambda_n}(y, n\pi | \lambda_n = \lambda) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{p(\xi_{1x}, \xi_{2x}, \xi_{21})}{\epsilon^3 \sigma_{11} \sigma_{22}^2 (y^2 + z^2)} dz, \tag{33-36}$$

$$\xi_{1x} = (1/\epsilon \sigma_{11})[-\epsilon^2 m_1 x + \log \sqrt{y^2 + z^2}], \quad \xi_{21} = (1/\epsilon \sigma_{22})[n\pi - \lambda - \epsilon^2 m_2],$$

$$\xi_{2x} = (1/\epsilon \sigma_{22})[\tan^{-1}(y/z) + 2k\pi - \epsilon^2 m_2 x - \lambda x].$$

The function $p(\xi_{1x}, \xi_{2x}, \xi_{21})$ appearing in equation (33) represents the three-dimensional Gaussian density function of the random variables $\xi_1(x)$, $\xi_2(x)$ and $\xi_2(1)$. Finally, the joint pdf of $y_n(x)$ and λ_n can be determined by substituting equations (29), (30) and (33)-(36) into the identity (14).

As has already been pointed out, the equation of $\phi(x, \lambda)$ is uncoupled from $r(x, \lambda)$. It is of interest to note that the equation for $r(x, \lambda)$ can be integrated directly to obtain

$$r(x, \lambda) = r_0 \exp \left\{ \frac{-\epsilon \lambda}{2} \int_0^x f(s) \sin[\theta(s) + 2\lambda s] ds \right\}. \tag{37}$$

This solution in itself is, however, of limited use in the study of the random eigenfunctions, since the right side still contains $\theta(s)$ for which, no exact solution exists. Nevertheless, equation (37) provides a clue that the solution of $r(x, \lambda)$ can be sought in the form $r(x, \lambda) = \exp[a(x, \lambda)]$.

In summary, it is noted that the application of a stochastic averaging method to the initial value problem given by equations (8)–(11) converts the original stochastic eigenvalue problem into a problem involving memoryless transformation of Gaussian random variables. In the opinion of the authors, this is a significant simplification of the problem. Furthermore, a distinguishing feature of the solutions obtained here, as compared with the perturbational approaches discussed, for example, by Soong and Bogdanoff [6], Boyce [7] and Scheidt and Purkert [9], is that the present approach leads directly to the estimates of the probability distribution functions of the eigensolutions, while the expansion methods lead to the estimates of the moments of the eigensolutions and, consequently, the estimates of the probability density functions can only be obtained with additional statistical closure approximations.

4. NUMERICAL RESULTS AND DISCUSSION

For the purpose of illustration, a string the mass per unit length of which is modelled as a stationary random process with autocovariance given by

$$R_f(x_1, x_2) = \exp[-\alpha|x_2 - x_1|] \tag{38}$$

is considered. In Figures 1 and 2 are shown the pdf's of the first ten eigenvalues obtained for the case of $\alpha = 20.8$ and $\epsilon = 0.1$ and 0.2 , calculated in accordance with equation (29). As can be observed, the density functions are centered around the corresponding

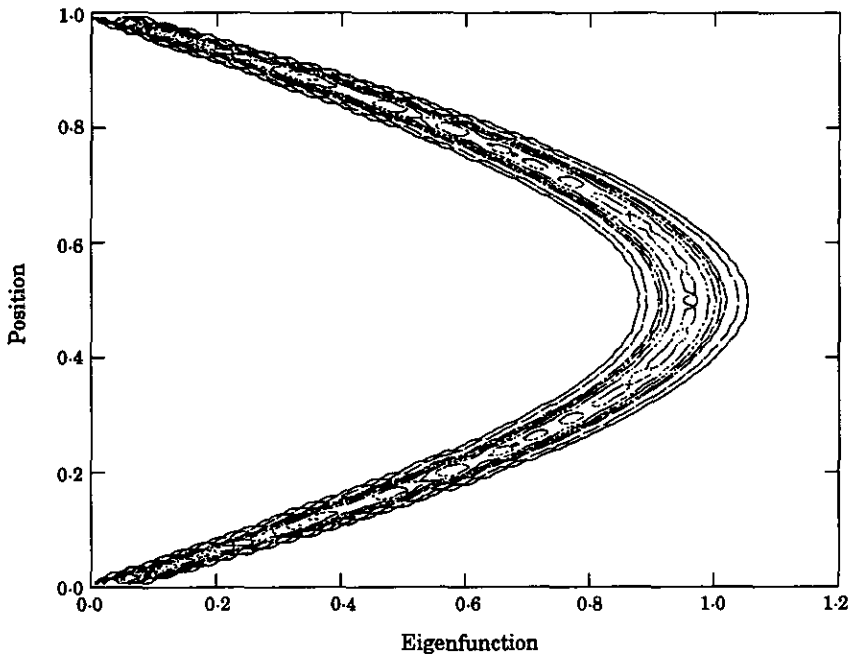


Figure 5. Contours of the probability density function of the first eigenfunction; $\alpha = 20.8$, $\epsilon = 0.1$. —, pdf = 0.01; ---, pdf = 0.1; - - - -, pdf = 1.0; - · - · - ·, pdf = 2.0; - · - · - ·, pdf = 4.0; - · - · - ·, pdf = 8.0; - · - · - ·, pdf = 16.0; - · - · - ·, pdf = 32.0; - · - · - ·, pdf = 64.0; - · - · - ·, pdf = 100.0.

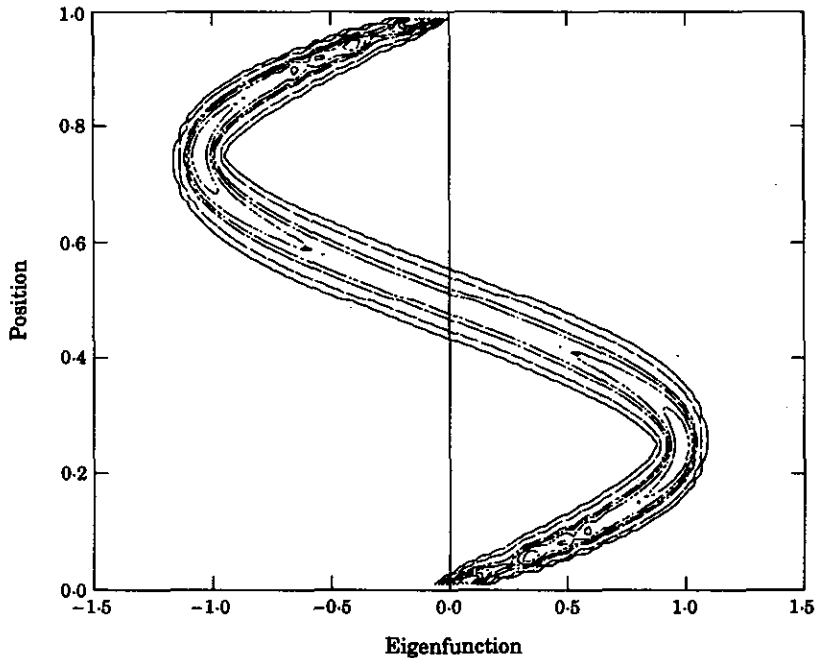


Figure 6. Contours of the probability density function of the second eigenfunction; $\alpha = 20.8$, $\epsilon = 0.1$; legend as in Figure 5.

deterministic solution of $\lambda_n = n\pi$. For larger ϵ and for higher mode counts the “overlap” between the probability density functions of neighbouring eigenvalues is observed to increase. With a view to studying the distribution of the higher eigenvalues, the pdf’s of the first 25 eigenvalues have been obtained for the case of $\alpha = 20.8$ and $\epsilon = 0.1$ and 0.2 . The contours of the probability distributions as a function of the mode count are shown in Figures 3 and 4 for different values of the probability measure P . Notice that in these plots the eigenvalues have been normalized with respect to the corresponding deterministic value: that is, the state variable in these plots is $\lambda_n/n\pi$. These figures also show the results of digital simulations with 2000 samples. In the simulation work the mass process is taken to be Gaussian distributed. The realizations of this process are computed by passing samples of Gaussian white noise process through a first order linear filter: that is, the samples of $f(x)$ are obtained by numerically solving the equation

$$df/dx + \alpha f = \zeta(x). \quad (39)$$

Here $\zeta(x)$ is a Gaussian white noise process with strength 2α . The calculation of the sample eigenvalues is based on numerically searching the poles of the tip receptance function [17, 18]. For this purpose, each realization of the string is divided into 30 piece-wise uniform sections of equal length, and the receptance function is evaluated by using the transmission matrices of the individual sections.

It may be observed from Figures 3 and 4 that, for the parameter values considered, the theoretical results compared very well with the simulation results. As might be expected, the accuracy of the theoretical results is found to be better for smaller values of ϵ . An interesting feature to be observed in these figures is that the probability contours tend to constant values beyond about the second or third mode count. This, in turn, implies that the variable $\lambda_n/n\pi$ tends to become stochastically stationary with respect to the mode index n and also that all the first order statistics, such as the mean and standard deviation, of

the eigenvalues λ_n are linearly proportional to n . In this context it may be recalled that the modal spacing for strings with uniform properties remains constant throughout the entire frequency axis. However, this property does not hold good for non-uniform strings. In the present case, the modal spacing S_n is a random variable and its average value is given by

$$\langle S_n \rangle = \langle [\lambda_{n+1} - \lambda_n] \rangle = (n + 1)\langle [\lambda_{n+1}/(n + 1)] \rangle - n\langle [\lambda_n/n] \rangle. \tag{40}$$

In view of the observation that, for large n , $\langle \lambda_n/n\pi \rangle$ remains constant with respect to n , it follows that

$$\langle S_n \rangle = \langle [\lambda_n/n] \rangle = \text{constant}. \tag{41}$$

This means that the average modal density for the random strings considered in this study remains constant along the frequency axis. Further work on the joint pdf of distinct eigenvalues is necessary before the question of the higher order moments and the possible stationarity of the pdf of S_n with respect to n can be analyzed.

The contours of the theoretical probability density functions of the first two eigenfunctions as a function of position x and for $\epsilon = 0.1$ and $\alpha = 20.8$ are shown in Figures 5 and 6. The contours are constructed for ten different values of the probability density function (pdf). While reading these plots it may be noted that the pdf's of the eigenfunctions are unimodal in nature. Clearly, the density functions are centered around the corresponding deterministic solutions. Notice that at $x = 0$ and $x = 1$ the functions degenerate into Dirac delta functions centered at zero. The spiky nature observed in these figures, especially near the boundaries, arises due to the aliasing problems occurring in plotting the contours and is not present in the actual solution. Given the approximate nature of the theoretical solution it is desirable to compare the solution with digital simulation results. For this purpose, four sections along the length of the system which roughly span the antinode and

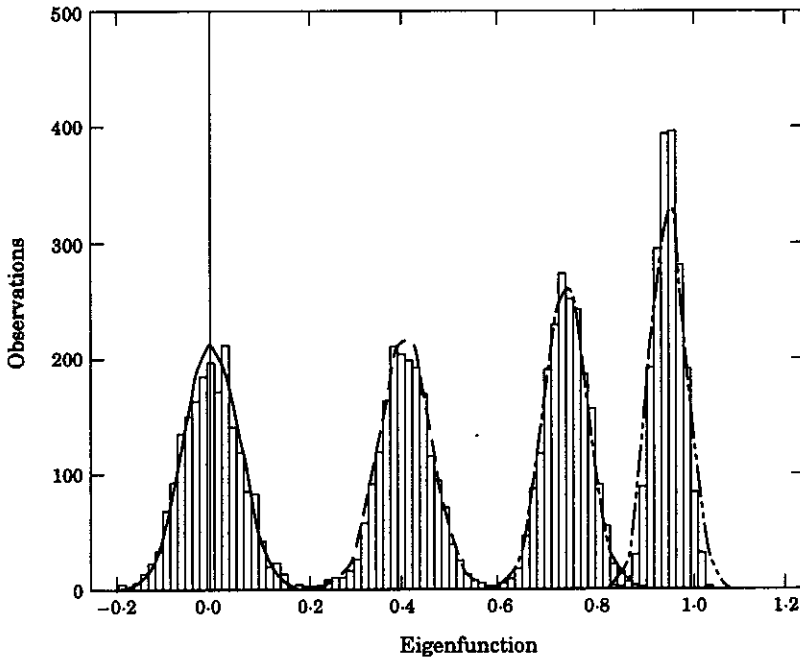


Figure 7. Histogram of the second eigenfunction; bars are simulated results and extended lines are theoretical solutions. —, $x = 0.5$; ----, $x = 0.44$; - - - - , $x = 0.38$; - · - · - , $x = 0.32$.

node of the second eigenfunction were selected. The simulated histograms of the second eigenfunction at these sections with 2000 samples are compared with the corresponding theoretical solutions in Figure 7. The procedure for simulating the realisations of the eigenfunctions is as described in reference [18]. The agreement between the simulation and theoretical results is again found to be satisfactory in this case also.

Finally, it must be noted that no assumption has been made about the probability distribution of $f(x)$ in obtaining the analytical results based on the stochastic averaging approximation. In fact, knowledge of the pdf of $f(x)$ is not needed for the application of the averaging theorem. Thus, irrespective of the actual distribution of $f(x)$, one always obtains Gaussian approximations for the processes $a(x, \lambda)$ and $\phi(x, \lambda)$ as given in equations (23) and (24). This, in turn, automatically fixes the *form* of the probability distribution functions of the eigensolutions, although the parameters of these distributions can still vary with changes in the mean and autocorrelation of $f(x)$. In this context it must be emphasized that the identities derived in equations (13) and (14) crucially depend on the fact that $\phi(x, \lambda)$ is non-decreasing with respect to x . Thus, the fact that $\phi(x, \lambda)$ is Gaussian in the averaging approximation is indicative of a limitation of the analysis. It may also be noted that in the simulation work, and also in the discussion presented in section 3, the process $f(x)$ is taken to be a Gaussian random process. Consequently, $P[1 + \epsilon f(x) < 0] \neq 0$, which, in view of the fact that $f(x)$ represents the mass process, is, strictly speaking, inadmissible. Also, the stipulation that $\phi(x, \lambda)$ be non-decreasing with respect to x again becomes violated on account of this assumption. The resulting error in the results can, however, be expected to be small if $\epsilon \leq 0.25$.

5. CONCLUSIONS

A new procedure for the determination of the joint pdf of the eigenvalues and eigenfunctions of a second order stochastic boundary value problem is developed. It is shown that the joint pdf can be characterized in terms of the solutions of a pair of stochastic initial value problems. The validity of the formulation is illustrated with the help of a special example for which the exact joint pdf of eigenvalues and eigenfunctions is obtainable by using alternative methods. An approximate procedure based on the theory of stochastic averaging is outlined for the determination of the pdf of eigensolutions for strings with random mass variations. The performance of this procedure for the calculation of the pdf of eigenvalues and eigenfunctions is examined with the help of digital simulation results. For the parameter values considered, the theoretical solutions are shown to compare well with the corresponding simulation results. It is observed from the numerical results that the first order probability distribution functions of the eigenvalues normalized with respect to their corresponding deterministic values tend to become stationary with respect to the mode count.

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APPENDIX A

Consider the special case in which the processes $g(x)$ and $f(x)$ satisfy the condition

$$[1 + \delta g(x)][1 + \epsilon f(x)] = 1. \quad (\text{A1})$$

In this case equations (8) and (9) yield

$$\phi(x) = \lambda x + \epsilon \lambda \int_0^x f(s) ds, \quad r(x) = r_0. \quad (\text{A2})$$

This further leads to

$$y^*(x, \lambda) = r_0 \sin \left[\lambda x + \epsilon \lambda \int_0^x f(s) ds \right]. \quad (\text{A3})$$

By using the condition $y^*(1, \lambda) = 0$, the expressions for the eigenvalues and eigenfunction can now be obtained, respectively, as

$$\lambda_n = n\pi / (1 + \epsilon F_1), \quad y_n(x) = \sin [\lambda_n(x + F_x)], \quad (\text{A4, A5})$$

where

$$F_1 = \int_0^1 f(s) ds, \quad F_x = \int_0^x f(s) ds. \quad (\text{A6})$$

Given the knowledge of the joint density of the random variables F_x and F_1 , the joint density of the n th eigenfunction and n th eigenvalue can easily be obtained by applying the rules of transformation of random variables to equations (A4) and (A5). Thus, if it is assumed that F_x and F_1 have zero mean and are jointly Gaussian, it is straightforward to show that

$$p_{y_n, \lambda_n}(y, \lambda) = \sum_{k=-\infty}^{\infty} \frac{n}{\epsilon^2 \lambda^3 \sigma_1 \sigma_x \sqrt{(1 - \sigma_{1x}^2)} \sqrt{(1 - y^2)}} \exp\left[\frac{-1}{2(1 - \sigma_{1x}^2)} \times \left\{ \frac{f_x^2}{\sigma_x^2} + \frac{f_1^2}{\sigma_1^2} - \frac{2\sigma_{1x} f_1 f_x}{\sigma_1 \sigma_x} \right\}\right], \quad (A7)$$

where

$$f_x = \frac{-x}{\epsilon} + \frac{1}{\epsilon \lambda} [\sin^{-1} y + 2k\pi], \quad f_1 = \frac{-1}{\epsilon} + \frac{n\pi}{\epsilon \lambda},$$

$$\sigma_1^2 = \left\langle \int_0^1 \int_0^1 f(u)f(v) du dv \right\rangle, \quad \sigma_x^2 = \left\langle \int_0^x \int_0^x f(u)f(v) du dv \right\rangle,$$

$$\sigma_{1x} = \frac{1}{\sigma_1 \sigma_x} \left\langle \int_0^1 \int_0^x f(u)f(v) du dv \right\rangle. \quad (A8-A12)$$

Here $\langle \rangle$ denotes the mathematical expectation operator. The marginal pdf of λ_n can be obtained directly from equation (A4), and this pdf can be shown to be given by

$$p_{\lambda_n}(\lambda) = \frac{n\pi}{\sqrt{2\pi\sigma_1\epsilon\lambda^2}} \exp\left[\frac{-1}{2\sigma_1^2} \left(\frac{n\pi}{\epsilon\lambda} - \frac{1}{\epsilon}\right)^2\right]. \quad (A13)$$

It is important to note that the probability density functions given in equations (A7)–(A13) have been obtained *without* taking recourse to the identities derived in equations (13) and (14). It is of interest to ascertain that the application of these identities also lead to the same solutions. Thus equations (13) yields

$$P[\lambda_n \leq \lambda] = P[n\pi \leq \lambda(1 + \epsilon F_1)] = 1 - \int_{-\infty}^{\left[\frac{n\pi - \lambda}{\epsilon \lambda}\right]} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{u^2}{2\sigma_1^2}\right] du. \quad (A14)$$

Upon differentiating this function with respect to λ , the pdf of λ_n can be obtained, which, as might be expected, agrees with the pdf derived in equation (A13). In order to apply the identity (14), consider the joint pdf of $y^*(x, \lambda)$ given by equation (A5) and $\phi(1, \lambda)$ given by

$$\phi(1, \lambda) = \lambda[1 + \epsilon F_1]. \quad (A15)$$

It follows that

$$p_{\phi, \lambda_n}(n\pi, 1, \lambda) = \frac{1}{\sqrt{2\pi\sigma_1\epsilon\lambda}} \exp\left[\frac{-1}{2\sigma_1^2} \left(\frac{n\pi - \lambda}{\epsilon\lambda}\right)^2\right],$$

$$p_{y^*, \phi, \lambda_n}(y, n\pi | \lambda_n = \lambda) = \sum_{k=-\infty}^{\infty} \frac{1}{\pi \epsilon^2 \lambda^2 \sigma_1 \sigma_x \sqrt{(1 - \sigma_{1x}^2)} \sqrt{(1 - y^2)}} \exp\left[\frac{-1}{2(1 - \sigma_{1x}^2)} \times \left\{ \frac{f_x^2}{\sigma_x^2} + \frac{f_1^2}{\sigma_1^2} - \frac{2\sigma_{1x} f_1 f_x}{\sigma_1 \sigma_x} \right\}\right]. \quad (A16)$$

Here f_x and f_1 are as defined in equations (A8) and (A9) respectively. Substitution of equation (A16) and $p_{\lambda_n}(\lambda)$ obtained by using equation (A14) into the identity (14) leads

to the joint pdf of y_n and λ_n and the resulting expression, again as might be expected, is identical to the pdf obtained in equation (A7) by alternative means.

APPENDIX B

By using the notations

$$\begin{aligned} X_1(x) &= a(x), & X_2(x) &= \theta(x), & X_{10} &= a(0), & X_{20} &= \theta(0), \\ g_1[x, X, f(x)] &= -0.5\lambda f(x) \sin 2(\theta + \lambda x), & g_2[x, X, f(x)] &= 0.5\lambda f(x)[1 - \cos 2(\theta + \lambda x)] \end{aligned} \quad (\text{B1})$$

equations (19)–(21) can be recast as

$$X' = \epsilon g[X, x, f(x)], \quad X(0) = X_0. \quad (\text{B2})$$

The prime here denotes the derivative with respect to x . This equation is in a form suitable for the application of stochastic averaging theorem (see the paper by Roberts and Spanos [16]). The solution process $X(x)$ is now approximated by a two-dimensional Markov process which satisfies the Itô equation

$$dX = \epsilon^2 m(X) dx + \epsilon \sigma(X) dW(x). \quad (\text{B3})$$

The symbol $W(x)$ denotes a two-dimensional vector of independent Brownian motion processes with unit variances. The drift vector m and diffusion matrix σ are given by

$$m = T^{av} \int_{-\infty}^0 E\{(\partial g / \partial X)_x (g^*)_{x+\tau}\} d\tau, \quad \sigma \sigma^* = T^{av} \int_{-\infty}^0 E\{g_x g_{x+\tau}^*\} d\tau, \quad (\text{B4})$$

where (*) denotes matrix transposition and $E\{\}$ the mathematical expectation. Furthermore, T^{av} is an averaging operator in x given by

$$T^{av}(\cdot) = \lim_{T \rightarrow \infty} \left(\frac{1}{T} \right) \int_{x_0}^{x_0+T} (\cdot) dx, \quad (\text{B5})$$

with the integration being performed on explicit x . Upon substituting equation (B1) into equation (B4), the drift and diffusion coefficients can be shown to be given by

$$\begin{aligned} m_1 &= \frac{\lambda^2}{4} \int_0^\infty R_f(\tau) \cos 2\lambda\tau d\tau, & \sigma_{11} &= \left[\frac{\lambda^2}{8} \int_{-\infty}^\infty R_f(\tau) \cos 2\lambda\tau d\tau \right]^{1/2}, \\ m_2 &= -\frac{\lambda^2}{4} \int_0^\infty R_f(\tau) \sin 2\lambda\tau d\tau, & \sigma_{22} &= \left[\frac{\lambda^2}{4} \int_{-\infty}^\infty R_f(\tau) d\tau + \frac{\lambda^2}{8} \int_{-\infty}^\infty R_f(\tau) \cos 2\lambda\tau d\tau \right]^{1/2}, \\ \sigma_{12} &= \sigma_{21} = 0. \end{aligned} \quad (\text{B6})$$

Here $R_f(\tau)$ denotes the autocorrelation function of the process $f(x)$. Thus, the pair of simplified equations for further analysis is given by

$$dX_1 = \epsilon^2 m_1 + \epsilon \sigma_{11} dW_1(x), \quad dX_2 = \epsilon^2 m_2 + \epsilon \sigma_{22} dW_2(x). \quad (\text{B7})$$