

# Modeling and evaluation of structural reliability: current status and future directions

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## Abstract

An overview of developments in selected areas in the field of structural reliability analysis is presented. The topics covered include probabilistic modeling of uncertainty that includes discussions on non-Gaussian models, first and second order reliability methods, asymptotic analyses of the probability integral, simulation based techniques including variance reduction methods, system reliability analysis, time variant reliability analysis, probabilistic model reduction and importance measures, stochastic finite element method including discussions on simulation and discretization of Gaussian/non-Gaussian random fields, response surface methods, reliability analysis including structural nonlinearities, discussions on critical excitation models, convex models for uncertainty and robust reliability. The emphasis of the paper is on presenting a critical discussion on methods of reliability modeling and analysis and not so much to explore specific application areas. A discussion on avenues for future research in this field concludes the paper.

## 1.0 Introduction

The subject of structural reliability provides a logical framework within which the uncertainties, that invariably exist in dealing with problems of structural analysis and design, could be systematically addressed. Here, the uncertainties in structural and load characteristics are quantified using the mathematical theories of probability, random variables, random processes and statistics. The subject essentially aims to establish relationship between probability of structural failure to the uncertainty parameters connected with the structural and load characteristics. This, in turn, facilitates a rational basis for deciding upon optimal structural configurations for a given set of loading conditions consistent with desired levels of safety and affordable cost. In fact, many of the present day codes of practice for structural design employ concepts of limit state and partial load and resistance factors and have been calibrated based on probabilistic modeling of uncertainties. The application areas of this subject encompass analysis, design, and optimization of high risk structures such as nuclear power plants, dams and offshore structures. The range of structural behavior includes stress analysis, dynamics, deformation control, creep and relaxation, fracture and fatigue and structural stability.

Early studies in this field were carried out by Freudenthal (1945) who brought the methods of probability and statistics to bear on characterizing the nature of factor of safety. An illuminating account of the historical development of the subject is available in the book by Madsen, Krenk and Lind (1986). Besides, this book also provides a rigorous account of reliability index based approaches. Some of the books relevant to this field are those by Pugsley (1962), Bolotin(1969), Tichy and Vorlicek (1972), Benjamin and Cornell(1970), Ang and Tang (1975,1984), Leporati (1979), Ditlevsen (1981), Elishakoff(1983), Augusti, Baratta and Casciati (1984), Yao (1985), Thoft-Christensen and Murotsu(1986) and Melchers (1987). Books that deal with dynamical systems include those by Lin (1967), Nigam (1983) and Bolotin (1984). Works of more recent vintage include those by Wen (1990), Soong and Grigoriu (1993), Dai and Way (1992), Nigam and Narayanan (1994), Lin and Cai (1995), Sundararajan (1995), Ben Haim (1996), Ditlevsen and Madsen (1996), Haldar, Guran and Ayyub (1997), Maymon (1998), Melchers (1999), Ranganathan (1999) and Haldar and Mahadevan (2000a,b). Several specialized review articles and research monographs have also been published. Thus, Shinozuka (1983) presents a discussion on definitions and calculation of reliability indices. Methods for calculating probability of failure are critically reviewed by Schueller and Stix (1987). Bjerager (1990) presents a state-of-the-art review of structural reliability methods by classifying the problems into random variable problems, random process reliability models and random field reliability models. Schueller and Ang (1992) summarize results presented during the Structural Mechanics in Reactor Technology (SMiRT) conferences held up to 1992 on topics related to risk analysis of nuclear containment, primary piping and other similar structures. Schueller (1997a) outlines the developments in computational aspects of structural reliability analysis and treats the developments in probabilistic modeling of fatigue and fracture, stochastic stability analysis and structural control. Developments in the area of stochastic finite element method (SFEM) have been discussed by Nakagiri (1987), Benaroya and Rehak (1988), Shinozuka and Yamazaki (1988), Brenner (1991), Ghanem and Spanos (1991a), Shinozuka (1991), Der Kiureghian *et al.*, (1991), Kleiber and Hien (1992), Liu *et al.*, (1992), and Matthies *et al.*, (1997). A review of methods of structural reliability assessment in the context of seismic safety assessment has been presented by Der Kiureghian (1996). The special issue of Probabilistic Engineering Mechanics, edited by Schueller (1997b), documents the state-of-the-art in computational aspects of stochastic structural mechanics. The papers by Ibrahim (1987) and Manohar and Ibrahim (1998) provide comprehensive overviews on problems of structural dynamics with parameter uncertainties. The monograph by Scheidt and Purkert (1983) and the paper by Benaroya (1992) report on mathematical aspects of random eigenvalue problems. The collected works edited by Cheng and Yang (1993) cover both theoretical and computational aspects of stochastic structural mechanics. The research monograph by Grigoriu (1995) considers the specification and simulation of non-Gaussian random processes together with discussions on several applications. Casciati *et al.*, (1997) provide an overview of standard dynamic problems and algorithms within classical structural reliability schemes. Problems of time variant reliability with emphasis on techniques for load combination has been reviewed by Rackwitz (1998). A resume of industrial applications of structural reliability has been given by Thoft-Christensen (1998). Rackwitz (2000) has presented a discussion on counter examples that reveals possible weaknesses of methods of reliability analysis. Methods of digital simulation of random fields have been reviewed by Shinozuka and Deodatis (1991,1996) and Spanos and Zeldin (1998). In a series of articles, Elishakoff (1995a,b,1998,1999) has examined basic issues related to probabilistic modeling of structural uncertainties and explored alternative avenues for modeling uncertainties. Melchers (2001) has outlined the developments in the reliability analysis of existing structures and discussed the needs for further research.

As in other branches of structural mechanics, the emergence of computational power in recent years has strongly influenced the developments in this subject. Other factors that have contributed to the growth of the subject are the increased availability of data on natural hazards such as earthquakes and recent development of sensor technology in the field of structural health monitoring. The objective of the present paper is to provide a critical overview of methods of structural reliability assessment. To place the objective of this paper in a proper perspective, the class of structural reliability problems wherein the uncertainty in structural system is quantified in terms of a  $n$ -dimensional vector of random variables  $\{X_i\}_{i=1}^n$ , is first considered. These

random variables are taken to collectively and exhaustively represent the loading characteristics and also the geometric, elastic, inertial, and strength properties of the structure. Any single realization of the structure and loading system, can thus be interpreted as a point in the space spanned by the variables  $\{X_i\}_{i=1}^n$ . Let attention be focused on a local failure criterion explicitly characterized through a performance function  $g(\mathbf{X})$ , such that, the regions  $g(\mathbf{X}) > 0$  and  $g(\mathbf{X}) < 0$ , in the space of  $\{X_i\}_{i=1}^n$ , respectively, denote the safe and unsafe regions. The surface  $g(\mathbf{X}) = 0$  is the limit surface that separates the safe and unsafe regions. In this setting, the probability of failure  $P_f$  is known to be given by

$$P_f = \int_{g(\mathbf{x}) < 0} p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x}. \quad (1)$$

Here,  $p_{\mathbf{X}}(\mathbf{x})$  is the  $n$ -dimensional joint probability density function of the random variables  $\{X_i\}_{i=1}^n$ . Explicit evaluation of the above integral, in most cases of engineering interest, is not feasible. The primary sources of difficulty here are:

1. the random variables  $\{X_i\}_{i=1}^n$  could be mutually dependent and non-Gaussian in nature; complete knowledge of  $p_{\mathbf{X}}(\mathbf{x})$  is seldom available,
2. the failure surface  $g(\mathbf{X}) = 0$  could be a highly nonlinear function of the random variables  $\{X_i\}_{i=1}^n$ , and
3. the dimension of the integral ( $= n$ ) could be quite large.

Furthermore, there could be other difficulties. The material and geometric properties of the structure could be varying randomly in space; this calls for modeling these properties as random fields. Similarly, loads/structural resistance could be varying in time. Time varying loads such as earthquakes, wind and waves often cause dynamic effects. On the other hand, time variations in live loads are slow and thus warrant only static response analysis. There could be additional sources of uncertainties due to modeling and solution procedures adopted. Also, the failure surface  $g(\mathbf{X}) = 0$  may not be explicitly obtainable, such as, when the response of the structure is computed using a finite element code. The probability of failure, as given by equation (1), refers to a single performance criterion. For a large engineering structure, there would be several failure modes and many a times, there would be significant difficulty in enumerating these failure modes. Since structural failures are invariably associated with nonlinear structural behavior, accounting for this feature further complicates the problem.

The methods of structural reliability analysis provide a systematic framework that introduces simplifying assumptions in the evaluation of failure probability with an aim to treat the difficulties outlined above. The present review covers topics that include reliability index based techniques, simulation based methods that include variance reduction strategies, system reliability analysis, time variant reliability analysis, probabilistic model reduction and importance measures, random field modeling and stochastic finite element methods, critical load models, convex models of uncertainty and robust reliability, and nonlinear structural behavior. Parts of the paper that deal with random field discretization and dynamic response characterization are updates of the earlier review paper by Manohar and Ibrahim(1998). The paper neither aims to cover specific application areas, nor does it cover issues related to code formulations. The review encompasses static and dynamic behavior of structures but does not cover fracture, fatigue, stability and deformation control issues. It is aimed to focus mainly on literature that have been published over the last 10-15 years on methods of reliability analysis and to identify areas that require further development. Papers published prior to 1985 are also referenced with a view to build adequate background to discuss the recent developments.

At the outset, it may be noted that the uncertainties in loads and/or structural properties could be modeled either as random variables or in terms of spatially and/or temporally varying random processes. In the latter case, one can analyze the structural reliability within the framework of theory of random processes. The

class of problems that can be solved following this procedure would obviously be restricted. Alternatively, the spatially and/or temporarily varying random processes could be replaced by an equivalent set of random variables by using an appropriate discretization scheme. Thus, in recent years, the deterministic finite element methods have been generalized to deal with spatially inhomogeneous randomness and this class of methods have come to be known as stochastic finite element methods. Using these formulations, the problem of reliability assessment, in principle, could always be expressed in the form of equation (1), even when distributed randomness models are considered.

## 2.0 Uncertainty models

The sources of uncertainties in structural engineering problems can be schematized into the following four categories (Der Kiureghian 1989, Menezes and Schueller 1996)

- inherent or physical uncertainties,
- model uncertainties,
- estimation errors, and
- human errors.

Inherent uncertainties occur intrinsically and are beyond engineers' control. These can occur either in structural properties or in the loading characteristics. Mathematical idealizations employed in structural analysis lead to the model uncertainties. These uncertainties could be due to lack of understanding of structural behavior and due to simplifications knowingly introduced in arriving at models. Estimation errors represent the sampling fluctuations and their study belongs to realms of statistics. Finally, human errors can arise at the stage of design, construction or operation. The last three sources of errors are extrinsic in nature and in principle, their characteristics can be altered by human intervention. We restrict our attention here on the first two sources of uncertainties.

### 2.1 Models for inherent uncertainties

The inherent variability could be associated with structural properties such as elastic constants, density, strength characteristics, member sizes and geometry or in loads. The JCSS (Joint Committee on Structural Safety) model code proposes to discuss inherent variability in loads that include: self-weight, live load, industrial storage, cranes, traffic, car parks, silo load, liquid and gases, manmade temperature, earth pressure, water and ground water, snow, wind, temperature, wind waves, avalanches, earthquakes, impact, explosion, fire and chemical/physical agencies (Vrouwenvelder 1997). Similarly, the code proposes to cover properties of the following materials: concrete, reinforcement, prestressed steel, steel, timber, aluminum, soil and masonry. The present review does not aim to catalogue the details of models that have been proposed in the literature for these quantities, but instead, aims to address the mathematical questions that underlie the making of these models.

#### 2.1.1 Models for material and geometric properties

The strategy here consists of stochastic modeling of elastic constants, mass density and material damping of the structural material. This, in conjunction with possible uncertainties in specifying the geometry and boundary conditions of the structure, defines the structural mass, stiffness, damping matrices and the force vectors. While choosing an appropriate stochastic model for the material properties several questions arise:

1. Can the system uncertainties be adequately described by random variables or is it necessary to use random field models, which take into account spatial inhomogeneities?

2. When random field models are employed, what restrictions must be placed on differentiability of the fields?
3. What is the level at which the specification of the random quantities is possible? Is it in terms of moments, first order probability density functions or joint probability density functions?
4. Are Gaussian models acceptable for strictly positive quantities such as mass and elastic constants? What are the feasible non-Gaussian models for these quantities and how to describe and simulate them?
5. What is the correlation structure of random fields associated with different members of a given structure?
6. How to proceed with the analysis when uncertain quantities are incompletely specified?
7. What are the primary variables, which have to be modeled as random: thus, for example, while modeling the Young's modulus  $E(x)$  of a one dimensional structural member, should one model  $E(x)$ ,  $\frac{1}{E(x)}$  or  $\log E(x)$  in the first place?

Next, one has to consider the mechanism for arriving at the stochastic model for the associated structural matrices; this issue is related to the methods of discretizing a random field and selection of mesh sizes in a finite element/boundary element study and it will be considered later in the paper (section 8.0).

### 2.1.1.1 Gaussian Models

Gaussian, homogeneous, mean square bounded, random field models have been employed by several authors for elastic constants and mass density, see, for example, Bucher and Shinozuka (1988), Kardara *et al.*, (1989), Spanos and Ghanem (1989), Chang and Yang (1991) and Manohar and Iyengar (1994). In these models, the stochastic perturbations are imposed on the corresponding nominal values. For two or three dimensional fields, the assumption of isotropy of the field is also made. Several models for autocovariance/power spectral density functions of the stochastic perturbations have been used. Thus, for example, Shinozuka (1987) and Shinozuka and Deodatis (1988), in their studies on statical response variability of randomly parametered skeletal structures, have adopted the following types of models for the power spectral density functions for the deviations of Young's modulus around the mean value:

$$S(\kappa) = \alpha_n \kappa^{2n} \exp[-b|\kappa|] \quad n = 0, 1, \dots, 5 \quad (2)$$

$$S(\kappa) = \beta_n \kappa^{2n} \exp[-(\frac{b|\kappa|}{2})^2] \quad n = 0, 1, \dots, 5 \quad (3)$$

$$S(\kappa) = \gamma_n \frac{b}{(1 + b^2 \kappa^2)^{2n}} \quad n = 1, 2. \quad (4)$$

Here,  $S(\kappa)$  is the power spectral density function,  $\kappa$  is the wave number, the parameters  $\alpha_n$ ,  $\beta_n$  and  $\gamma_n$  control the variance of the process and the parameter  $b$  controls the shape of the power spectral density function. In a similar context, Spanos and Ghanem (1989) have utilized exponential and triangular autocovariance functions of the form

$$\begin{aligned} R(\xi) &= \sigma_s^2 \exp[-c|\xi|] \quad \text{and} \\ R(\xi) &= \sigma_s^2 (1 - c|\xi|) \end{aligned} \quad (5)$$

where,  $R(\xi)$  is the autocovariance function,  $\xi$  is the spatial lag,  $\sigma_s^2$  is the variance and the parameter  $c$  controls the correlation length. In their studies on flexural vibrations of random plates, Bucher and Brenner (1992)

have modeled Young's modulus and mass density of the plate as independent two dimensional homogeneous isotropic random fields each having covariance functions of the form

$$R(\xi, \zeta) = \sigma_{ff}^2 \exp\left(\frac{-\sqrt{\xi^2 + \zeta^2}}{l_f}\right). \quad (6)$$

Here,  $\xi$  and  $\zeta$  are space lags,  $R(\xi, \zeta)$  is the autocovariance function of the random field,  $l_f$  controls the correlation length and  $\sigma_{ff}^2$  is the variance. It may be pointed out that the models mentioned above have been largely selected for purpose of illustration of analytical results and little evidence by way of experimental/field data is available to ascertain the relative merits of alternative models in a given context and in a given scenario of quality control process.

### 2.1.1.2 Non-Gaussian Models

The Gaussian models are not admissible for strictly positive quantities such as mass and elastic constants. This can potentially be a serious drawback of the model for large values of coefficient of variations or when reliability issues are being examined. Furthermore, Gaussian distributions do not allow information on moments higher than the first two to enter the model. To circumvent the first difficulty, Yamazaki *et al.*, (1988) and Wall and Deodatis (1994) have restricted the variation of samples of Gaussian fields as follows:

$$-1 + \eta \leq f(x) \leq 1 - \eta; \quad 0 < \eta < 1. \quad (7)$$

The limitation on the upper value is imposed in order to achieve symmetry of the stochastic variations about the deterministic values. A similar approximation of a Gaussian distribution by a distribution with bounded range has also been made by Iwan and Jensen (1993). A more systematic way of constructing non-Gaussian field models is by making nonlinear memoryless transformations of a specified Gaussian field, that is, by considering  $w(x) = g[\nu(x)]$ , where,  $g$  is a 'memoryless' nonlinear function and  $\nu(x)$  is a Gaussian field (Grigoriu 1984, Yamazaki and Shinozuka 1988, Der Kiureghian and Liu 1986). This type of transformations enable characterization of  $w(x)$  in terms of mean and covariance of  $\nu(x)$ . Prominent among this type of models is the Nataf model which can produce any desired marginal distribution for  $w(x)$  (Grigoriu 1984, Der Kiureghian and Liu 1986). If  $w(x)$  is a non-Gaussian field with a specified mean  $\mu_w(x)$ , covariance  $\rho_{ww}(x, \tilde{x})$  and first order probability distribution function  $F_w(w; x)$ , then according to Nataf's model, the transformed process

$$\nu(x) = \Phi^{-1} [F_w(w; x)] \quad (8)$$

is taken to be Gaussian, in which  $\Phi$  = standard Gaussian probability distribution. It can be shown that  $\nu(x)$  has zero mean, unit standard deviation and autocovariance  $\rho(x, \tilde{x})$  satisfying the integral equation

$$\rho_{ww}(x, \tilde{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{F_w^{-1}[\Phi(u)] - \mu_w(x)}{\sigma_w(x)} \right\} \left\{ \frac{F_w^{-1}[\Phi(v)] - \mu_w(\tilde{x})}{\sigma_w(\tilde{x})} \right\} \phi_2\{u, v, \rho(x, \tilde{x})\} dudv \quad (9)$$

in which,  $\phi_2$  denotes the bivariate standard normal density. In general  $|\rho_{ww}(x, \tilde{x})| \leq |\rho(x, \tilde{x})|$  and for most processes  $\rho(x, \tilde{x}) \simeq \rho_{ww}(x, \tilde{x})$ . A set of empirical formulae relating  $\rho(x, \tilde{x})$  to  $\rho_{ww}(x, \tilde{x})$  for common distributions are available (Der Kiureghian *et al.*, 1991). Discussion on the use of this model in structural reliability analysis when random quantities involved are partially specified has been presented by Der Kiureghian and Liu (1986). This model has also been employed by Liu and Der Kiureghian (1991) and Li and Der Kiureghian(1993) in their studies on reliability of stochastic structures. Other examples of non-Gaussian field models for material properties can be found in the works of Elishakoff *et al.*, (1995) and Sobczyk *et al.*, (1996).

Manohar *et al.*, (1999) and Gupta and Manohar (2001) have considered problems of spatial variability of material properties when information available on the variability is limited to the mean, range and covariance

of the random fields. This limitation being considered as being realistic in practical applications. These authors construct a first order probability density function that is consistent with the given mean, range and standard deviation so that the entropy of the distribution is maximized. The resulting family of probability density functions are shown to have the form

$$p_X(x) = A \exp[-\lambda_1 x - \lambda_2 x^2]; \quad a \leq x \leq b. \quad (10)$$

Here,  $(a, b)$  is the given range of the random variable, and the constants  $A, \lambda_1, \lambda_2$  are to be determined so that the conditions on normalization, mean and standard deviation are satisfied. The authors have used the Nataf model in conjunction with the given covariance function in their investigation on reliability of vibrating skeletal structures.

Methods for obtaining non-Gaussian random processes as outputs of filters driven by white noise inputs have been considered within framework of Markov process theory. Thus, Liu and Munson (1982) and Sondhi (1983) employed a combination of linear filter and nonlinear memory-less transformations to model non-Gaussian random processes with specified first order probability density function (pdf) and power spectral density (psd) function. Kontrovich and Lyanders (1995) studied outputs of nonlinear filters when they are driven by Gaussian white noise processes. These authors adjusted drift and diffusion coefficients in the governing Fokker Planck equations, to realize target pdf and psd functions. This approach has been studied further by Cai and Lin (1996, 2001). In their study the matching of the spectral density is achieved by adjusting the drift coefficient alone, followed by adjustment of diffusion coefficient, to match the target pdf function.

At the level of random variables, the treatment of non-Gaussianity is well known: thus, the classical Rosenblatt transformations to handle multi-variate non-Gaussian random variables have been discussed in detail by various authors (see, for example, the books by Ang and Tang 1984 and Ditlevsen and Madsen 1996). Non-Gaussian random variable models for system parameters in the context of single degree of freedom (sdof) dynamical systems or individual structural elements of built-up vibrating structure have been considered by several authors. Thus, Udawadia (1987a,b) considered maximum entropy probability distributions for incompletely specified system parameters of a sdof system. Other examples include lognormal models (Shinozuka and Yamazaki 1988, Cruse *et al.*, 1988), beta distributions (Shinozuka and Yamazaki 1988) and ultraspherical random variables (Jensen and Iwan 1991).

Manohar and Bhattacharyya (1999) quantitatively examined the influence of different non-Gaussian pdf models and autocovariance function parameters of system property random fields on the system response. The study reveals that the differences produced in the response quantities for different first order non-Gaussian system property pdf are, at times, of the same order as those produced by differences in their correlation lengths. Figure (1) shows the spectrum of mean and standard deviation of the octahedral shear stress in a random beam on elastic foundation that is driven by a harmonic concentrated force. The beam flexural rigidity, mass density and foundation modulus are modeled as a vector of non-Gaussian random fields. Three alternative models are adopted for these properties: the three models share the same mean and covariance but differ from each other in their first order pdf. The results shown in figure (1) are expressed as fractions of result from one of the models that is normalized to unity. The significant differences between results of the three models are clearly discernible from this figure. This underlines the need for accurate modeling of not only the auto-correlation functions, but also the first order probability distributions of the loads and system properties. This is particularly true for studies of rare events, as is encountered in structural reliability, where the tail characteristics of the probability distribution functions play a significant role.

## 2.2 Models for loads

In modeling external actions on the structure one needs to consider uncertainties associated with frequency of occurrence, duration, spatio-temporal/frequency variations, and severity of the loads. Additional considerations are also due on uncertainties associated with multi-presence of more than one source of loading. The present review does not address the details of probabilistic models adopted for various sources of loads. Instead, specific issues arising in treating partially specified loads and also questions on combined effects of more than one source of loading are covered later in the review (see section 5.3). The details of available models for loads such as earthquakes, wind guideway unevenness, and waves are available in the book by Nigam and Narayanan (1994). The book by Madsen and Krenk (1986) and Ranganathan (1999) provide further details on other sources of loads. As has been already noted, the paper by Vrouwenvelder (1997) reports on the progress made in documenting models for various sources of loading. The book by Wen (1990) provides a comprehensive treatise on probabilistic modeling of loads.

## 2.3 Mechanical modeling uncertainties

As has been noted already, mechanical model uncertainties arise due to imperfections in arriving at the mathematical model for the structural system. Two factors contribute to these imperfections: the first is due to lack of satisfactory understanding of the underlying physical phenomena and the second is due to assumptions that the engineer consciously introduces to make the modeling and solution procedures tractable. Specific examples as sources of mechanical model uncertainties include: assumptions made in constitutive laws (linearity, isotropy, homogeneity), geometric simplification, nature of strain-displacement relations used (linear/nonlinear), use of specific engineering theories (Euler-Bernoulli *versus* Timoshenko beam theory, Kirchoff-Love *versus* Mindlin plate theories, alternative shell theories), use of specific damping models (viscous / structural, proportional / nonproportional), modeling of interaction effects (*e.g.*, structure-foundation-fluid interactions), choice of finite elements and mesh details, occurrence of series truncation (as in modal expansions), numerical errors, and modeling of joints (flexible/rigid) and boundary conditions. Discussions on probabilistic modeling of mechanical modeling uncertainties are available, for example, in the works of Ditlevsen (1981), Der Kiureghian (1989), Maes (1996), Matthies *et al.*, (1997), Menezes and Schueller (1996), and Saffarini (2000). Ditlevsen and Der Kiureghian have indicated the usefulness of Bayesian decision theory for treatment of mechanical model uncertainties. Menezes and Schueller write the performance function in the form  $g(\mathbf{X}, \mathbf{A})$ , where,  $\mathbf{X}$  is the vector of basic random variables as in equation (1) and  $\mathbf{A}$  is the vector of random variables that models mechanical model uncertainties. Any specific evaluation of  $p_f$  using a given mechanical model can be interpreted as the failure probability conditioned on  $\mathbf{A}$ . The unconditional probability of failure can, in principle, be evaluated by using the joint pdf of  $\mathbf{A}$ . Saffarini (2000) investigates the significance of uncertainties arising due to various assumptions made in analysis of reinforced concrete structures on the assessment of structural reliability. According to his study, wide variability in results of analyses are possible even when the analyses are performed by competent engineers using acceptable assumptions.

Examples of round robin surveys on a few structural analysis problems have been reported by Menezes and Schueller (1996). These include studies on fatigue modeling of defects and analysis of transmission line towers. The latter study revealed a coefficient of variation in the range of 10% to 37 % on predictions of axial forces and member strengths made by an ensemble of twenty established companies. More recently, Schueller *et al.*, (1997c) report on a benchmark study on stochastic dynamic response analysis of single and multi-degree of freedom nonlinear systems. The objective of the study has been to compare various methods for response analysis with respect to their accuracy, efficiency and limitations. A set of eight example structures encompassing several types of nonlinearities and loading conditions were analyzed by a set of independent researchers using methods of their choice. The paper reports on differences in responses obtained by various participants with respect to exact solutions, when available, and Monte Carlo simulation results for problems which defy exact



solution.

It is of interest to note in this context that in many dynamic situations, especially in automobile and aerospace applications, a part of the structure could be modeled using experimental tools and these models are integrated with computational models for remaining components (Silva and Maia 1999). In such situations, there could be newer sources of uncertainties originating from experimental procedures. A round robin survey on measurement of frequency response functions and subsequent modal analysis reported by Ewins and Griffin (1981) reveals significant variability across results from leading vibration laboratories. The recent study by Ashroy (1999) divides the sources of lack of precision in modal testing procedures into three categories: (a) experimental and data acquisition errors, (b) signal processing errors and (c) modal analysis errors. The first category of errors include qualitative mechanical errors due to mass loading effect of transducer, shaker-structure interaction, supporting of the structure, measurement noise and presence of nonlinearity. Quantitatively, experimental errors arise due to inability to measure adequate points on the structure and adequate degrees of freedom. Signal processing errors are well studied and these include errors due to leakage, aliasing, effect of window functions, effect of discrete fourier transforms and effects of averaging. Modal analysis errors arise due to approximations made in various methods such as circle-fit analysis, line-fit analysis and global modal analysis procedures. While most current technologies aim towards avoidance of these errors, given the complexity of issues involved, further research involving probabilistic modeling clearly is needed.

## 2.4 Additional remarks on uncertainty modeling

1. The estimates of  $p_f$  obtained using equation (1) are known to be sensitive to the nature of tails of  $p_{\mathbf{X}}(\mathbf{x})$ , especially when values of  $p_f$  are low, as in the case of important structures. Consequently, the computed failure probabilities could only be used in comparing reliability of alternative designs. This comparison itself is meaningful only if the computed reliabilities are based on the same set of distribution types for the vector random variable  $\mathbf{X}$ . Ditlevsen (1994,1997) has discussed this and other related problems. Based on studies on snow load statistics, Ditlevsen points out that a best fit criterion is not sufficient as the basis for choosing distribution models for reliability analysis. Instead, it is argued that standardizations of distribution types need to be imposed on alternative designs if estimated reliabilities need to be compared.
2. Just as there is an element of arbitrariness in selection of parametric models for probability density functions, there exists a similar problem in choosing variables for random field modeling. The study by Ditlevsen and Johansen (1999) addresses the issues related to modeling and discretization of stiffness and flexibility fields of beam elements. With reference to a study of buckling of a spring supported Euler-Beam, the authors demonstrate the effect of discretizing either compliance field or the stiffness field on distribution of buckling load.
3. When random fields are used to model structural property variability, considerations need to be given to smoothness and regularity properties of the random fields (Matthies *et al.*,1997). In the studies on dynamics of randomly parametered beams, by Adhikari and Manohar (2000) and Gupta and Manohar (2001b), the following restrictions are imposed on random fields  $f_1(x)$ =foundation modulus,  $f_2(x)$ =mass density per unit length,  $f_3(x)$ =flexural rigidity and  $f_4(x)$ =axial rigidity:
  - (a)  $f_i(x)$  ( $i=1,2,3,4$ ) are meansquare bounded, that is,  $\langle f_i(x)^2 \rangle < \infty$ ; here  $\langle \cdot \rangle$  is the mathematical expectation operator;
  - (b)  $f_3(x)$  is twice differentiable in a meansquare sense; this requires that  $\frac{\partial^4 R_{33}(x_1, x_2)}{\partial x_1^2 \partial x_2^2}$  must exist for all  $x_1$  and  $x_2$  in the interval  $(0, L)$ ;

(c)  $f_4(x)$  is differentiable in a meansquare sense; this requires that  $\frac{\partial^2 R_{44}(x_1, x_2)}{\partial x_1 \partial x_2}$  must exist for all  $x_1$  and  $x_2$  in the interval  $(0, L)$ ;

(d) for a specified deterministic function  $F(x)$ , which is bounded and continuous in  $(0, L)$ , integrals of the type  $\int_0^L F(x) f_i(x) dx$  exist in a meansquare sense; this requires that  $\int_0^L \int_0^L |F(x_1) F(x_2) R_{ii}(x_1, x_2)| dx_1 dx_2 < \infty$  ( $i = 1, 2, 3, 4$ ).

Here,  $R_{ij}(x_1, x_2)$  is the covariance function of the random fields  $f_i(x_1)$  and  $f_j(x_2)$ . The first three conditions ensure that the sample realizations of the beam have sufficiently smooth behavior so that the various stress resultants and boundary conditions (such as those at a free edge) are satisfactorily described. The last of the above requirements was necessitated due to the nature of random field discretization used in subsequent analysis.

4. In models for randomly parametered structural dynamical systems, uncertainty models could be introduced at the level of system normal modes and natural frequencies themselves. Thus, in the statistical energy analysis formalisms (Lyon and De Jong 1995) the natural frequencies are taken to constitute a set of Poisson points on the frequency axis. The study by Brown and Ferri (1996) on combining substructuring techniques with probabilistic analysis reveals difficulties in correctly identifying the statistical properties of primitive variables such as geometry, stiffness and mass. These authors propose an alternative method where the measured dynamic properties of the substructures are considered as random quantities. Apparently no work has been done in combining these types of uncertainty modeling with methods of structural reliability.

### 3.0 First Order Reliability Method (FORM), Second Order Reliability Method (SORM), and Asymptotic analyses

The exact evaluation of  $p_f$  from equation (1) is possible when  $g(\mathbf{X})$  is a linear function of  $\mathbf{X}$  of the form

$$g(\mathbf{X}) = a_0 + \sum_{i=1}^N a_i X_i \quad (11)$$

and the random variables  $\{X_i\}_{i=1}^n$  are jointly normal. Let the mean vector be  $\mu$  and covariance matrix be  $\mathbf{C}$ . In this case, it can be shown that

$$p_f = \Phi(-\beta) \quad (12)$$

where

$$\beta = \frac{a_0 + \sum_{i=1}^n a_i \mu_i}{\sqrt{\sum_{i=1}^n \sum_{j=1}^n a_i a_j C_{ij}}} \quad (13)$$

It thus follows that  $\beta$  is a single parameter that encapsulates the properties of the random variables  $\{X_i\}_{i=1}^n$  that are needed for determination of the probability of failure. Therefore,  $\beta$  can be termed as an index of reliability of the system. It may be re-emphasized that this result is true only under the assumption that  $\{X_i\}_{i=1}^n$  are jointly Gaussian and  $g(\mathbf{X})$  is a linear function of  $\{X_i\}_{i=1}^n$ . Nevertheless, this raises the question of existence of such similar single parameters, that are related to  $p_f$  in some sense, when the assumptions, linearity of  $g(\mathbf{X})$  and Gaussianity of  $\{X_i\}_{i=1}^n$  are not satisfied. Such indices would clearly side step the need for evaluation of multi-dimensional integrals and, in themselves, could form the basis for reliability characterization. The development and application of such reliability indices, in fact, has remained a central theme of research in structural reliability. These studies are based on replacing the nonlinear performance function  $g(\mathbf{X})$  by its Taylor's expansion around a specified point. Depending upon whether this series is truncated after the linear term or after the quadratic term, the associated methods are termed, respectively, as first

order and second order reliability methods.

### 3.1 First order reliability methods

Cornell (1969) considered the problem of  $g(\mathbf{X})$  being nonlinear in  $\mathbf{X}$  and  $\{X_i\}_{i=1}^n$  being mutually independent normal random variables. Assuming that  $g(\mathbf{X})$  is differentiable in the neighborhood of the mean,  $g(\mathbf{X})$  was replaced by a first order Taylor's expansion around  $\mu$ . Based on his study, a reliability index can be defined as

$$\beta_c = \frac{g(\mu_1, \mu_2, \dots, \mu_n) + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j}(\mu_i, \mu_j) < (X_i - \mu_i)(X_j - \mu_j) >}{\sum_{i=1}^n \sum_{j=1}^n \frac{\partial g}{\partial x_i}(\mu_i) \frac{\partial g}{\partial x_j}(\mu_j) < (X_i - \mu_i)(X_j - \mu_j) >}. \quad (14)$$

As may be observed, this index is a function of mean and covariance of  $\{X_i\}_{i=1}^n$  and consequently, the method of reliability characterization *via* this index is called mean value first order second moment (MVFOSM) method. Based on this concept, Cornell(1969) proposed a framework for structural codes that systematically take into account the uncertainties. A significant draw back of the index, as proposed above, was that it was not invariant with respect to alternative, but mechanically equivalent, definitions of  $g(\mathbf{X})$ . Thus, for instance, the reliability index for a set of mechanically equivalent performance functions, namely,  $g_1(\mathbf{X}) = X_1 - X_2$ ,  $g_2(\mathbf{X}) = (X_1 - X_2)^2$ ,  $g_3(\mathbf{X}) = \log \frac{X_1}{X_2}$  and  $g_4(\mathbf{X}) = \frac{X_1}{X_2} - 1$  would not be identical. Furthermore, this index uses information on only the first two moments of  $\mathbf{X}$  and makes no allowance for utilizing additional knowledge on  $\mathbf{X}$  should such a knowledge be available. Some of these difficulties were overcome by Hasofer and Lind (1974), who noted that  $\beta_c$ , as given by equation (14) also represents the shortest distance from origin to the limit surface when the surface is plotted in the space of standard normal random variables. Accordingly, these authors considered the vector  $\mathbf{X}$  to be made up of Gaussian random variables and introduced a new vector  $\mathbf{Y}$  of standard normal random variables given by

$$\mathbf{Y} = \mathbf{A}[\mathbf{X} - \mu] \quad (15)$$

where the vector  $\mathbf{Y}$  and matrix  $\mathbf{A}$  are such that

$$< \mathbf{Y} > = \mathbf{0}; \quad < \mathbf{Y}\mathbf{Y}^t > = \mathbf{A} < [\mathbf{X} - \mu]^t [\mathbf{X} - \mu] > \mathbf{A}^t = \mathbf{I} \quad (16)$$

The transformation on  $\mathbf{X}$  also implies a transformation on the performance function  $g(\mathbf{X})$  and the transformed performance function is denoted by  $G(\mathbf{Y})$ . The definition of reliability index, as proposed by Hasofer and Lind, reads

$$\beta_{HL} = \min_{G(\mathbf{y})=0} \sqrt{\{\mathbf{y}\mathbf{y}^t\}} \quad (17)$$

This reliability index has the following notable properties (Hasofer and Lind, 1974):

1. Given the geometrical interpretation of  $\beta_{HL}$  as the shortest distance from the origin to the limit surface in the  $Y$ -space, it follows that it is invariant with respect to mechanically equivalent transformation of the performance function. Also, at no stage of the calculation does the designer have to specify which variables are load and which are resistances. Any change of the resistance definition consistent with the rules of mechanics will lead to the same failure region, and, therefore, the same value of reliability index.
2. The set of basic random variables need not have any direct physical significance as long as their joint distribution determines the distribution of physical variables which enter into the problem. This feature permits non-Gaussian models to be adopted for the basic random variables.
3. The index can take into account correlated basic variables. Also, issues such as stress reversal and multiple failure modes, imply an increase in the failure region.

The overcoming of the problem of invariance by the use of  $\beta_{HL}$  represented a significant development in structural reliability theory.

Veneziano (1979) noted that despite the simplicity and generality of  $\beta_{HL}$ , the index was still not completely satisfactory. For the purpose of illustration, he considered a vector of independent basic random variables with zero mean and unit standard deviation. The safe region was assumed to be of the form  $|X_i| < d$  ( $i = 1, 2, \dots, n$ ). Clearly, for this case,  $\beta_{HL} = d$  and hence is independent of  $n$ . The reliability, on the other hand, can be easily shown to be a function of  $n$ . Similarly, if the safe region were to be now replaced by the region  $X_i < d$  ( $i = 1, 2, \dots, n$ ), the reliability index would not change its value although the reliability indeed changes. Thus, it can be inferred that  $\beta_{HL}$  is not consistent and it may not account adequately the geometry of the safe region. To overcome these difficulties, Veneziano noted that, given complete/partial description of  $p_{\mathbf{X}}(\mathbf{x})$  and  $g(\mathbf{X})$ , the probability of failure can always be bounded as

$$P_f^L \leq P_f \leq P_f^U. \quad (18)$$

Based on this observation, the author defined

$$\gamma = \sqrt{P_f^U} \quad (19)$$

as the alternative definition of reliability index. Methods based on generalized Chebychev bounds were proposed to evaluate  $\gamma$ . A list of closed form solutions to certain set of safe region and for certain level of specification of basic random variables were also provided. Ditlevsen (1979) pointed out that  $\beta_{HL}$  does not distinguish between limit surfaces that are tangential to each other at the common point closest to the origin in the normalized space. These distinct limit surfaces get assigned the same reliability index although the associated reliabilities for these surfaces differ from each other. This leads to the question of whether  $\beta_{HL}$  has a sufficient degree of resolution to satisfy engineering goals. To remedy this situation Ditlevsen proposed a generalized reliability index defined by

$$\beta_g = \Phi^{-1} \left[ \int_{g(\mathbf{x}) \leq 0} \phi(x_1)\phi(x_1)\phi(x_2)\dots\phi(x_n)dx_1dx_2\dots dx_n \right] \quad (20)$$

Here,  $\phi(x)$  denotes the first order Gaussian probability density function and  $\Phi$  denotes the  $n$ -th order standard normal probability distribution function. This reliability index takes into account the entire probability space and hence possess the property of orderability. The choice of  $\phi(x)$  as being Gaussian is formal and this does not imply that  $\mathbf{X}$  is Gaussian: this choice is guided by the rotational symmetry of the  $n$ -dimensional Gaussian probability density function. Evaluation of  $\beta_g$  involves evaluation of a multi-dimensional integral and Ditlevsen proposed approximating the nonlinear limit surface by a polyhedral surface consisting of tangent hyperplanes at selected points on the surface. It may be noted that when  $\mathbf{X}$  is a Gaussian vector and  $g(\mathbf{X})$  is linear function of  $\mathbf{X}$ ,  $\beta_c = \beta_{HL} = \beta_g = \gamma$ .

Der Kiureghian (1989), in an insightful paper, considered the problem of defining reliability index that includes the uncertainties due to estimation error and model imperfection. He laid out the following fundamental requirements that any reliability index need to satisfy:

1. Consistency: The reliability index shall asymptotically approach the strict reliability index as the state of knowledge approaches the perfect state. It is to be noted that under imperfect state of knowledge, the reliability index is to be interpreted as a point estimator of the strict reliability index.
2. Completeness: The reliability index shall incorporate all available information and account for all uncertainties.
3. Invariance: For a given state of knowledge, the reliability index shall be invariant for mutually consistent formulations of the reliability problem.

4. Remunerability: The reliability index shall remunerate improvements in the state of knowledge.
5. Orderability: Any ordering of reliability indices shall be consistent with the corresponding ordering of strict safeties at a prescribed probability level.
6. Simplicity: The required effort for computing the reliability index shall be commensurate with its approximate nature.

Judging by these criteria, it may be inferred that  $\beta_c$  lacks invariance;  $\beta_{HL}$  lacks orderability;  $\beta_c, \beta_{HL}$  and  $\beta_g$  lack completeness;  $\gamma$  perhaps lacks simplicity. In an effort to define reliability index that accounts for estimation error and model imperfection, Der Kiureghian denoted the performance function as  $g(\mathbf{X}, \Theta)$ . Here  $\mathbf{X}$  describes irreducible uncertainties that arise due to inherent uncertainties and  $\Theta$  denotes the reducible set of uncertainties arising due to estimation errors and model imperfection. Conditioned on  $\Theta$ , the probability of failure and the reliability index are given by

$$P_{f|\Theta}(\theta) = \int_{g(\mathbf{x}, \theta) \leq 0} p_{\mathbf{X}|\Theta}(\mathbf{x}, \theta) d\mathbf{x} \quad (21)$$

$$\beta_{\Theta}(\theta) = \Phi^{-1}[1 - P_{f|\Theta}(\theta)]. \quad (22)$$

Introducing the random variable  $B = \beta_{\Theta}(\theta)$  with probability density function  $p_B(b)$ , Der Kiureghian proposed a new reliability index given by

$$\beta_{mp} = \min_{\hat{\beta}} \langle \mathcal{P}(B - \hat{\beta}) \rangle \quad (23)$$

where  $\mathcal{P}$  is a penalty function,  $\hat{\beta}$  is the point estimator of  $\beta$ , and  $\beta_{mp}$  is called the minimum penalty reliability index. Specific forms of penalty functions considered include

$$\begin{aligned} \mathcal{P}(B - \hat{\beta}) &= a(B - \hat{\beta}) \quad \hat{\beta} \leq B \\ &= Ka(B - \hat{\beta}) \quad \hat{\beta} > B \end{aligned} \quad (24)$$

$$\begin{aligned} \mathcal{P}(B - \hat{\beta}) &= a(B - \hat{\beta})^2 \quad \hat{\beta} \leq B \\ &= Ka(B - \hat{\beta})^2 \quad \hat{\beta} > B. \end{aligned} \quad (25)$$

The first of these functions is a linear penalty function and the second is a quadratic penalty function. In the above equations,  $a$  = scale factor and  $K$  = a measure of asymmetry of penalty function. If  $B$  is taken to be Gaussian distributed, upon minimizing the expected penalty, one gets,

$$\beta_{mp} = \mu_B(1 - \delta_{\beta}u) \quad (26)$$

where  $\delta_{\beta} = \frac{\sigma_B}{\mu_B}$  is the coefficient of variation of  $B$  and  $u = u(K)$  is a function which depends upon the penalty function used. For the linear penalty function, one gets

$$u = \Phi^{-1}\left\{\frac{K}{K+1}\right\} \quad (27)$$

and for the quadratic penalty function  $u$  is the solution of the equation

$$Ku - (K-1)[u\Phi(u) + \phi(u)] = 0. \quad (28)$$

The reliability index  $\beta_{mp}$  has three components:  $\mu_B, \sigma_B$ , and  $u(K)$ . The first term provides a central measure of safety, the second term accounts for uncertainty in the estimate of safety, and the third term accounts for the asymmetry in the penalty function. When the state of knowledge is perfect,  $\delta_{\beta}$  is zero and  $\beta_{mp}$  coincides

with the strict reliability index. Thus,  $\beta_{mp}$  satisfies the consistency requirement. When the state of knowledge is imperfect,  $\delta_\beta$  is non-zero and  $\beta_{mp}$  deviates from the central measure of safety by  $u$  units of the standard deviation  $\sigma_B$ . As the state of knowledge improves,  $\delta_\beta$  decreases and  $\frac{\beta_{mp}}{\mu_\beta}$  increases for  $K > 1$ . Thus, for  $K > 1$ ,  $\beta_{mp}$  satisfies the remunerability requirement. The requirement of orderability is also satisfied for a fixed value of  $u$ , since,  $\beta_{mp}$  is then a fixed number of standard deviations from the mean, which corresponds to a fixed probability level for the assumed normal distribution.

Methods to compute reliability indices have been studied by several authors (Hasofer and Lind 1974, Rackwitz and Fiessler 1978, Shinozuka 1983, Chen and Lind 1983, Wu and Wirsching 1987, Liu and Der Kiureghian 1991, Haldar and Mahadevan 2000). The problem of computing  $\beta_{HL}$  constitutes a constrained nonlinear optimization problem. An iterative procedure based on the use of Lagrangian multiplier was proposed by Rackwitz and Fiessler (1978). The paper by Shinozuka (1983) provides many useful insights. It is observed that the design point, defined as the point in standard normal space that is closest to the origin, represents the point of maximum likelihood. Furthermore, under the assumption that  $g(\mathbf{X})$  is well behaved and concave towards the origin, and, also, that  $\mathbf{X}$  is a Gaussian vector, it is shown that

$$1 - \chi_n^2(\beta_{HL}^2) \geq P_f \geq 1 - \Phi(\beta_{HL}). \quad (29)$$

Shinozuka credits this result to an unpublished work of Hasofer. Hohenbichler and Rackwitz (1981) considered the problem of evaluation of  $\beta_{HL}$  when the basic random variables  $X_i$  are non-normal. The basic idea of their approach was to employ the Rosenblatt transformation (Ang and Tang 1984) on  $\mathbf{X}$  which results in a vector of standard normal random variables  $\mathbf{Y}$ . Implementation of this transformation requires the complete knowledge of the  $n$  dimensional joint probability density function of  $\mathbf{X}$ . Also, the transformation of basic random variables is accompanied by an associated transformation of the performance function from  $g(\mathbf{X})$  to a new function  $G(\mathbf{Y})$ . A critical assessment of first order and second moment method for reliability analysis was undertaken by Dolinski (1983). The author noted the following objections to this approach:

1. The solution is sensitive to the arrangement of random variables insofar as their dependence and non-normality is concerned.
2. More than one solution of the problem may exist and not the best solution may be found.
3. There is no possibility of error estimation unless we know that the transformed limit state surface is either wholly concave or convex to the origin.

When information beyond the second moments are available, rotationally symmetric standard space of random variables can be produced by employing a nonlinear transformation (Ditlevsen 1979a, Der Kiureghian and Liu 1986). Liu and Der Kiureghian (1991) have surveyed methods of constrained optimization from the view of their applicability in computing reliability indices using nonlinear finite element method. Specifically they have investigated the usefulness of five methods, namely, the gradient projection method, penalty method, augmented Lagrangian method, sequential quadratic programming approach, and the methods developed by Hasofer and Lind (1974) and Rackwitz and Fiessler (1978). They examined these methods from the points of view of generality (if there are any restrictions on nature of constraints and objective functions), need for the computation of the Hessian of the constraint, robustness (power of the method to solve the problem with a specified accuracy including issues on global convergence), efficiency (measured in terms of number of evaluations of the constraints and their gradients), and capacity (limits on maximum number of variables that the method can handle). Based on studies conducted on five specific examples, these authors conclude that the quadratic programming method, the gradient projection method and the modified Hasofer-Lind and Rackwitz-Fiessler method are robust techniques for use in structural reliability applications. The paper by Der Kiureghian and Dakessian (1998) proposes a method that successively finds the multiple design point of a component reliability problem. The idea here is to first determine FORM or SORM approximations at each

of the design point and employ a series system reliability analysis. Rackwitz (2000) has discussed in detail the algorithmic problems associated with FORM and other methods. He points out the problems associated with failure to arrive at global minima in computing the reliability indices. The usual problem here is to know whether multiple critical points exist at all for a given problem. There exists no algorithmic concept in mathematical programming that leads to acceptable solutions in all situations. Given the omnipresence of simulation tools as alternative options, the concern here is that the FORM must not consume computational time that is incommensurate with the accuracy realized in evaluating the reliability.

### 3.2 Second order reliability methods (SORM) and Asymptotic reliability analysis

In evaluation of  $P_f$ , the following four class of problems can be envisaged:

1.  $g(\mathbf{X})$  is linear in  $\mathbf{X}$  and  $\mathbf{X}$  is a vector of Gaussian random variables.
2.  $g(\mathbf{X})$  is linear in  $\mathbf{X}$  and  $\mathbf{X}$  is a vector of non-Gaussian random variables.
3.  $g(\mathbf{X})$  is nonlinear in  $\mathbf{X}$  and  $\mathbf{X}$  is a vector of Gaussian random variables.
4.  $g(\mathbf{X})$  is nonlinear in  $\mathbf{X}$  and  $\mathbf{X}$  is a vector of non-Gaussian random variables.

As has been already noted, the first of the above class of problems admits exact closed form solutions and the remaining class of problems generally need approximations. Furthermore, problems of class 2 and 4 can always be transformed to problems of class 3. In FORM,  $g(\mathbf{X})$  is replaced by a first order Taylor's expansion about a specified point thereby linearizing the performance function. This gives a reasonable approximation to the probability of failure provided  $g(\mathbf{X})$  is flat near the design point. Questions on how to improve upon this first order approximation has led to the development of second order reliability methods.

The basic idea of SORM is to replace the performance function  $g(\mathbf{X})$  by its second order Taylor's expansion around the check point  $\mathbf{x}^*$ , that is,

$$g(\mathbf{x}) = g(\mathbf{x}^*) + (\mathbf{x} - \mathbf{x}^*)^T \mathbf{g}_x(\mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \mathbf{G}_x(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*). \quad (30)$$

Here,  $\mathbf{g}_x$  is the vector of gradients of  $g(\mathbf{x})$  and  $\mathbf{G}_x$  is the  $n \times n$  matrix of second derivatives of  $g(\mathbf{X})$ . This results in a quadratic failure surface and the task on hand consists of evaluating the multi-normal probability integral over this surface. There are two strategies for fitting the quadratic surface: the first consists of matching the principal curvatures of the limit surface with those of the proposed quadratic surface at the check point; the second approach employs a point fitting strategy.

Fiessler *et al.*,(1979) introduced a transformation  $\mathbf{X} = \mathbf{TZ}$  where  $\mathbf{T}$  is the matrix of eigenvectors of  $\mathbf{G}_x$  with  $\Lambda$  being the diagonal matrix of eigenvalues of  $\mathbf{G}_x$ , so that  $\mathbf{T}^t \mathbf{G}_x \mathbf{T} = [\Lambda]$ . Given that  $\mathbf{G}_x^t = \mathbf{G}_x$ , it follows that  $\Lambda$  will be real. This results in quadratic surfaces in the standard form, given by

$$\sum_{i=1}^n \lambda_i (z_i - \delta_i)^2 = K_1 \quad (31)$$

where it is assumed that  $G_x$  is regular. An alternative standard form when  $G_x$  is singular with some of the  $\lambda_i$ 's being zero, is shown to be given by

$$\sum_{i=1}^m \lambda_i (z_i - \delta_i)^2 + \sum_{i=m+1}^n \tilde{z}_i g_{z_i} = K_2. \quad (32)$$

Here,  $(n - m)$  variables occurring only in linear forms are denoted by  $\tilde{z}$ . Thus, for the two cases of quadratic surfaces,  $P_f$  is given respectively by

$$\begin{aligned} P_f &= P[W > K_1] = 1 - P_W(K_1) \\ P_f &= P[V > K_2] = 1 - P_V(K_2) \end{aligned} \quad (33)$$

where

$$\begin{aligned} W &= \sum_{i=1}^n \lambda_i (z_i - \delta_i)^2 \\ V &= \sum_{i=1}^m \lambda_i (z_i - \delta_i)^2 + \sum_{i=m+1}^n \tilde{z}_i g_{zi}. \end{aligned} \quad (34)$$

Using theory of quadratic forms of normal random variables, Fiessler *et al.*,(1979) showed that

$$\begin{aligned} P_W(x) &= \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \frac{\sin \theta(u)}{u \rho(u)} du \\ P_V(v) &= \int_{-\infty}^\infty \phi(z_n) P_W(v - z_n) dz_n. \end{aligned} \quad (35)$$

Here,

$$\begin{aligned} \theta(u) &= \frac{1}{2} \sum_{j=1}^n [\tan^{-1}(\lambda_j u) + \delta_j^2 \lambda_j u (1 + \lambda_j^2 u^2)^{-1}] - \frac{1}{2} x u \\ \rho(u) &= \prod_{j=1}^n (1 + \lambda_j^2 u^2)^{\frac{1}{4}} \exp\left\{ \frac{\frac{1}{2} \sum_{j=1}^n (\delta_j \lambda_j u)^2}{1 + \lambda_j^2 u^2} \right\}. \end{aligned} \quad (36)$$

Numerical quadrature was suggested for evaluation of integrals in equation (35).

Breitung (1984) employed results from asymptotic analysis and studied the qualitative behavior of  $P_f$  for sequences of increasing safe domains. Using a paraboloid fit, Breitung showed that the probability of failure can be given approximately as

$$P_f = \Phi(-\beta_{HL}) \prod_{i=1}^n (1 + \beta_{HL} \kappa_i)^{-\frac{1}{2}} \quad (37)$$

in which,  $\kappa_i$ =principal curvature of the limit surface at the check point. It was shown that  $P_f$ , as given above, asymptotically approaches the true value of failure probability as  $1 \leq \beta_{HL} \rightarrow \infty$ , with  $\beta_{HL} \kappa_i$  remaining fixed. The discovery of this result marked an important development in theory of SORM. Furthermore, Breitung (1989) showed that when more than one point lie at equal distance from the failure surface, the failure probability is given in an asymptotic sense by

$$P_f = \Phi(-\beta_{HL}) \sum_{i=1}^k \left\{ \prod_{j=1}^{n-1} (1 - \beta_{HL} \kappa_{i,j}) \right\} \quad (38)$$

where  $k$  = number of points with  $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_k$  on failure surface with  $|\tilde{x}_i| = \beta_{HL}$  and  $\kappa_{i,j}$  is the  $i$ th curvature at  $\tilde{x}_j$ . The question on accuracy of the Breitung formula, when asymptotic conditions are not satisfied, has been considered by Hohenbichler and Rackwitz (1988). Using a Taylor's function for the probability distribution of Gaussian variate, these authors derived

$$P_f = \Phi(-\beta_{HL}) \prod_{i=1}^{n-1} \left\{ 1 - \frac{\kappa_i \phi(-\beta_{HL})}{\Phi(-\beta_{HL})} \right\} \gamma_0. \quad (39)$$



Here,  $\gamma_0$  is a correction that is proposed to be determined using simulations with importance sampling procedure (see section 6.1). The first two terms in the above formula for  $\beta_{HL} \rightarrow \infty$  approaches the Breitung's formula. In a subsequent study Breitung (1991) considered evaluation of  $P_f$  when the basic random variables are non-Gaussian in nature. The author noted the following problems with the use of transforming the basic non-Gaussian random variables into an equivalent standard normal space:

1. Since the probability distribution function of the basic random variables cannot be given in analytical form, the inverse transformations of the form

$$T^{-1}(u) = \{P_1^{-1}[\Phi(u_1)] \dots P_n^{-1}[\Phi(u_n)]\} \quad (40)$$

can only be computed numerically.

2. The transformation from non-Gaussian space to standard normal space also results in mapping of the original failure surface into a newer form. Clearly any simplicity that might exist in the original space, such as linearity, would be lost upon this transformation. Also, the new failure surface has no clear interpretation in terms of the original random variables.
3. The study of sensitivity of the failure probability with respect to the parameters of the original random variables becomes unduly complex.

Breitung considered the proposition, made earlier by Shinozuka (1983), that the check point be found by maximizing the log-likelihood function. Accordingly, he proposed that approximations to  $P_f$  can be constructed by finding the points where the log likelihood is maximal. He approximated the log likelihood function by a second order Taylor's expansion. This avoided the step of transforming the non-Gaussian random variables to standard normal space and offered newer interpretations to FORM/SORM.

Naess (1987) considered special class of quadratic forms of Gaussian random variables which admit closed form expressions for the distribution functions. By exploiting these special case solutions, bounding approximations to the failure probability of more general quadratic limit states in normal random variables were derived. Tvedt (1990) expressed the failure surface as a paraboloid in the rotated space, in which the direction of  $x_n$  coincides with the direction of the check point, as

$$\mathbf{y}_n = \beta_{HL} + \frac{1}{2} \mathbf{y}^t \mathbf{A} \mathbf{y} \quad (41)$$

where,  $\mathbf{A}$  is the  $(n-1) \times (n-1)$  second-derivative matrix and  $\mathbf{y}$  is the rotated space of random variables. Using procedures based on characteristic function of random variables, procedures for finding  $P_f$  were outlined. The following three formulae have been proposed by Tvedt to evaluate the failure probability

$$\begin{aligned} P_f = & \Phi(-\beta_{HL})[\det(\mathbf{I} + \beta_{HL}\mathbf{A})]^{-\frac{1}{2}} + [\beta_{HL}\Phi(-\beta_{HL}) - \phi(\beta_{HL})][(\det(\mathbf{I} + \beta_{HL}\mathbf{A}))^{-\frac{1}{2}} \\ & - [\det(\mathbf{I} + (\beta_{HL} + 1)\mathbf{A})]^{-\frac{1}{2}}] + (\beta_{HL} + 1)[\beta_{HL}\Phi(-\beta_{HL}) - \phi(\beta_{HL})][(\det(\mathbf{I} + \beta_{HL}\mathbf{A}))^{-\frac{1}{2}} \\ & - \text{Real}\{[\det \mathbf{I} + (\beta_{HL} + i)\mathbf{A}]^{-\frac{1}{2}}\}] \end{aligned} \quad (42)$$

$$P_f = \Phi(\beta_{HL}) \sum_{j=1}^k w_j \text{Real}\{[\det(\mathbf{I} + ((\beta_{HL}^2 + 2s_j)^{\frac{1}{2}} + i)\mathbf{A}))^{-\frac{1}{2}}](\beta_{HL}^2 + 2s_j)^{-\frac{1}{2}}\} \quad (43)$$

$$P_f = \frac{2}{\pi} \Phi(\beta_{HL}) \text{Real} \int_0^\infty \int_0^\infty \prod_{i=1}^{n-1} (r_p\{[1 + (\beta_{HL}^2 + 2s)^{\frac{1}{2}}\kappa_i + \sqrt{2}iu\kappa_i]^{-\frac{1}{2}} \exp(-s - u^2)\} ds du \quad (44)$$

In equation (43), the summation represents a  $k$ -point Gauss-Laguerre quadrature approximation with weights  $w_j$  and abscissas  $s_j$ . Similarly in equation (44),  $r_p\{\cdot\}$  denotes the root with positive real part. The above

three formulae are, respectively, known as Tvedt's three term formula, single integral formula and the double integral formula.

Der Kiureghian *et al.*,(1987) noted that the fitting of the paraboloid to the limit surface is based on the use of curvature at the check point. This requires the computation of the eigenvalues of the Hessian of the performance function at the design point. When the algorithm to compute  $g(\mathbf{X})$  is complicated, the evaluation of the Hessian involves approximate methods that introduce numerical noise into the calculations which can seriously endanger the accuracy of the calculations. Also, if the check point is a point of inflection, the curvatures are zero and the paraboloid reduces to the tangent hyperplane, thus, providing no improvement to the FORM results. Motivated by these considerations, Der Kiureghian *et al.*,(1987), proposed that the paraboloid be fitted by using values of  $g(\mathbf{X})$  at a set of discrete points in a segment in the neighborhood of the check point. The approximate paraboloid, in the rotated standard normal space, in which  $y_n$  axis coincides with the direction of check point, is taken to be of the form

$$y_n = \beta_{HL} + \frac{1}{2} \sum_{i=1}^{n-1} a_i y_i^2. \quad (45)$$

It may be noted that this representation is regardless of the principal direction of the limit surface. The fitting of the paraboloid involves approximating the limit state by weighted sum of two semi-parabolas which are tangential to limit surface at the check point and are curve fitted using a set of discrete points around the check point. In determining the coefficients of the fit, Breitung's formula (1984) (equation 37) is used to estimate the probability content of unsafe region. This approach to obtain SORM approximation to  $P_f$  is shown to be insensitive to noise in the evaluation of the limit surface. In a subsequent study, Der Kiureghian and Stefano (1991) proposed an iterative algorithm to compute principal curvatures of the limit surface without computing the Hessian matrix. The proposed algorithm used only the gradient of  $g(\mathbf{X})$ . The method is shown to be advantageous when dealing with large number of random variables. Suggestions on the use of perturbational approaches to compute the gradients, when  $g(\mathbf{X})$  is computed using finite element methods, are also made. Cai and Elishakoff (1994) expressed the limit surface in a transformed standard normal space in a form similar to equation (45) and expressed the reliability as

$$1 - P_f = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi(u_1)\phi(u_2)\dots\phi(u_{n-1}) \left\{ \int_{-\infty}^{\beta_{HL} + \frac{1}{2} \sum_{i=1}^{n-1} a_i y_i^2} \phi(u_n) du_n \right\} du_1 du_2 \dots du_{n-1} \quad (46)$$

The inner integral in the above equation was written in the form

$$\int_{-\infty}^{\beta_{HL} + \frac{1}{2} \sum_{i=1}^{n-1} a_i y_i^2} \phi(u_n) du_n = \Phi(\beta_{HL}) + \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\beta_{HL}^2}{2}\right\} \int_0^{\frac{1}{2} \sum_{i=1}^{n-1} a_i y_i^2} \exp\left\{-\beta_{HL}v - \frac{v^2}{2}\right\} dv \quad (47)$$

The integrand in the right hand side of the above equation was expressed in a Taylor's series around  $v = 0$  leading to

$$1 - P_f = \Phi(\beta_{HL}) + \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{\beta_{HL}^2}{2}\right\} (D_1 + D_2 + D_3 + \dots) \quad (48)$$

Here

$$D_1 = \sum_j a_j$$

$$D_2 = \sum -\frac{1}{2}\beta_{HL}(3\sum_j a_j^2 + \sum_{j \neq k} a_j a_k)$$

$$D_3 = \frac{1}{6}(\beta_{HL}^2 - 1)(15\sum_j a_j^3 + 9\sum_{j \neq k} a_j^2 a_k + \sum_{j \neq k \neq l} a_j a_k a_l) \quad (49)$$

The formulary was illustrated with reference to the reliability analysis of a shaft where the performance function was nonlinear and consisted of three Gaussian random variables. The results obtained were compared with exact results and it was concluded that the proposed method yields acceptable results even when the failure surface is not far from the origin and with large curvatures. Koyluoglu and Nielsen (1994) recast equation (46) in the form

$$1 - P_f = \Phi(\beta_{HL}) + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \phi(u_1)\phi(u_2)\dots\phi(u_{n-1})$$

$$[\Phi(\beta_{HL} + u) - \Phi(\beta_{HL})]du_{n-1}\dots du_2 du_1 \quad (50)$$

where

$$u = \frac{1}{2} \sum_{j=1}^{n-1} a_j u_j^2 \quad (51)$$

The authors considered the expansion

$$\Phi(\beta_{HL} + u) - \Phi(\beta_{HL}) = (1 - \Phi(\beta_{HL}))\{1 - \exp(-\frac{u}{c_{0,1}})(1 + c_{1,1}u + c_{2,1}u^2 + c_{3,3}u^3 + \dots)\} \quad (52)$$

Here  $c_{i,j}$  are the unknown coefficients that are determined by applying McLaurin series expansion of right and left sides of the above equation and requiring that both sides have identical first, second, third... order derivatives at  $u = 0$ . Zhao and Ono (1999a,b) examined the accuracy of the various SORM formulae for a large range of curvatures, number of random variables and first order indices. An empirical formula for SORM reliability index is proposed based on extensive case studies and the use of regression analysis. The method avoided the use of the eigenvalues of the rotated Hessian matrix. Their study also considered the identification when FORM is accurate enough, when SORM is needed and when both FORM and SORM fail to provide satisfactory results. In a further study, Zhao and Ono (1999c) noted that the development of methods that avoid computation of Hessian matrix represents a promising advance in the use of SORM. A point fitting strategy, to fit quadratic surface, that avoids the computation of the Hessian or the gradients of the performance function has been proposed. The failure probability is computed using an inverse fast Fourier transform algorithm on the characteristic functions. Hong (1999) has proposed a correction factor that improves the SORM estimates as developed by Hohenbichler and Rackwitz (1988).

Papadimitriou *et al*, (1997) considered reliability of randomly driven systems with stochastic system parameters. The basic premise of their study is that the failure probability conditioned on system parameters is obtainable at least in an approximate manner. These authors expressed the failure probability as

$$P_f = \int_{\theta} F(\theta)p(\theta)d\theta \quad (53)$$

where  $F(\theta)$ = probability of failure conditioned on the system parameters and  $p(\theta)$ = joint probability density function of the system parameters. Furthermore, with the substitution

$$l(\theta) = \ln F(\theta) + \ln p(\theta) \quad (54)$$

the failure probability was obtained as

$$P_f = \int_{\theta} \exp[l(\theta)]d\theta \quad (55)$$

An asymptotic approximation to  $P_f$  is obtained as

$$P_f = [2\pi]^{n/2} \frac{F(\theta^*)p(\theta^*)}{\sqrt{\det[L(\theta^*)]}} \quad (56)$$

where  $L(\theta)$  is the Hessian matrix given by

$$L_{ij}(\theta) = -\frac{\partial^2 l}{\partial \theta_i \partial \theta_j} \quad (57)$$

and  $\theta^*$  is the point at which  $l(\theta)$  is maximal. The authors illustrated their formulation on one & two degree of freedom systems that are driven by stationary random excitations. The use of asymptotic expansions, as in equation (56), has also been studied by Polidori *et al.*, (1999) in the context of SORM. These authors consider the  $P_f$  as per SORM to be of the form

$$P_f = \int_{R^{n-1}} \Phi\left\{-\beta_{HL} - \frac{1}{2} \sum_{i=1}^{n-1} a_i y_i^2\right\} \phi(y) dy \quad (58)$$

This form of the equation is recognized to be of the form of equation (53) and accordingly, the asymptotic approximation as given by equation (56) becomes applicable. A discussion on this study and the authors' reply is available, respectively, in Breitung (2001) and Polidori *et al.*, (2001).

## 4.0 Structural system reliability

Structural systems consists of many components. Each of these components can fail in more than one ways. Each of these ways in which a component can fail is termed as a failure element. Associated with each of these failure elements, a performance function can be defined. Violation of any single limit state, among this set of limit state functions, need not signal the failure of the structural system. For the structure to fail, a set of component limit state functions must be violated simultaneously. A particular combination of limit state violations that cause the structural failure is termed as a failure mode. Associated with a given structure there can exist several failure modes. Safety of the structure requires that the structure be safe against all these failure modes. Moses (1996) has noted the wide spread acceptance of reliability based methods in codified design of structural components. He points out the following two limitations of incorporation of reliability in design specifications :

1. The actual structure reliability may differ significantly from the individual component reliabilities. Thus, economic optimization of the structure cannot incorporate in the component design the economic consequences of failure. Furthermore, Bayesian calculation of the reliability formulation to update statistical parameters from observed performance is limited as long as the analysis deals with component risk and the observations are actually on the system risk.
2. Most of the reported actual failures in structures are not a consequence of the overload or understrength phenomenon checked in a specification. Rather, failures mostly occur because of accidents and human errors. Emphasis in recent years is on promoting structure safety with attention to structure redundancy and toughness as well as design review and inspection processes. These developments will benefit from a system risk analysis and optimum allocation of both costs and component risks to achieve a safe optimum structure.

The problem of reliability assessment of structural system needs attention to the following issues:

1. Idealization of the structure for the purpose of enumeration of failure modes and calculation of failure probability. This idealization is in addition to the mechanical modeling of the structure using procedures such as FEM. Individual structural components could be either ductile or brittle. Thus questions on post-failure behavior of structural components, including issues related to load re-distribution and load path dependency, need to be addressed.
2. Enumeration of the set of failure modes that are relevant to a given structure under specified loads. Identification of critical failure modes that significantly contribute to the failure probability.
3. Methods for calculation of failure probabilities taking into account more than one limit states. An additional complication would arise in dealing with correlation among the relevant safety margins. Such correlations normally arise because of common source of loading or resistance variables in a common material, common procedures used for strength assessment, fabrication, inspection and testing (Moses 1990). Questions on how to extend FORM/SORM, response surface methods (see section 7.0) and simulation methods (see section 6.0) to these problems need to be addressed.

These issues have been widely studied in the existing literature: see for example the works of Moses (1982, 1990, 1996), Thoft-Christensen and Murotsu (1986), Augusti *et al.*, (1984), Ditlevsen and Bjerager (1986), Hohenbichler *et al.*, (1987), Frangopol and Corotis (1990), Ditlevsen and Madsen (1996), Pandey (1998), and Melchers (1999). The paper by Grimmelt and Schueller (1983) reports on a benchmark study involving alternative approaches to system reliability assessment of a set of five skeletal structures. The following sections review some of the issues in this area of research.

#### 4.1 Series and Parallel systems

Consider a set of  $m$  failure elements with performance functions given by  $\{g_i(\mathbf{X})\}_{i=1}^m$ . These failure elements are said to be in series if failure of the system is signaled by failure of any one of these elements. A statically determinate truss is an example of a series system in which failure of any one of the truss members results in structural collapse. Conversely, the failure elements are said to be in parallel if structural failure can occur only if all the failure elements fail. Thus, in a statically indeterminate truss, failure of individual elements could potentially lead to re-distribution of stresses and more than one truss member may need to fail before the truss can collapse. The importance of redundancy in structural safety is often emphasized. Moses (1990) notes that merely counting the degree of redundancy may be irrelevant for a risk evaluation. If all members of a redundant system are designed against their behavior limit and if the load uncertainty overwhelms resistance uncertainty, as is often true in earthquake and wind loading, then, the existence of redundant elements does not give benefits. Furthermore, Moses also notes that the failure mode events for each component have high correlation and if one element fails, the other elements are also likely to fail. It is important to note that real engineering structures are seldom truly series systems nor truly parallel systems. The significance of series and parallel system models however lies in the fact that these models can serve as building blocks to construct reliability models for engineering structures. In fact, it can be shown that any system can be represented both as a series of parallel systems and as a parallel system of series systems (Ditlevsen and Madsen 1996).

The probability of failure of a series system is given by

$$P_{fs} = P[\cup_{i=1}^m \{g_i(\mathbf{X}) \leq 0\}] \quad (59)$$

and, for parallel systems, the failure probability is

$$P_{fp} = P[\cap_{i=1}^m \{g_i(\mathbf{X}) \leq 0\}] \quad (60)$$

Hohenbichler and Rackwitz (1983) obtained approximations to the above failure probabilities using FORM. Here the basic random variables were first transformed to the standard normal space and each of the resulting

performance functions were linearized around the design point to get

$$P_{fs} = P[\cup_{i=1}^m \{\alpha_i U_i + \beta_{HLi}\}] \quad (61)$$

$$P_{fp} = P[\cap_{i=1}^m \{\alpha_i U_i + \beta_{HLi}\}] \quad (62)$$

Defining  $Z_i = \alpha_i U_i + \beta_{HLi}$  as the linearized safety margins, it was shown that

$$\begin{aligned} P_{fs} &= 1 - P[\cup_{i=1}^m \{-Z_i \leq \beta_{HLi}\}] \\ &= 1 - \Phi_m(\tilde{\beta}_{HL}; \mathcal{R}_Z) \end{aligned} \quad (63)$$

$$P_{fp} = \Phi_m(-\tilde{\beta}_{HL}; \mathcal{R}_Z) \quad (64)$$

Here  $\tilde{\beta}_{HL}$  is the vector of reliability indices for the  $m$  performance functions,  $\mathcal{R}_Z$  is the matrix of correlation coefficients between the safety margins and  $\Phi_m$  is the  $m$ th order standard normal joint probability distribution function.

## 4.2 Evaluation of multi-normal integrals

As has been noted above, the use of FORM for series and parallel system reliability assessment requires the evaluation of the multi-normal integral of the form

$$\Phi_m(\mathbf{c}; \mathbf{R}) = \int_{-\infty}^{c_1} \int_{-\infty}^{c_2} \cdots \int_{-\infty}^{c_m} \frac{|\mathbf{R}|^{-\frac{1}{2}}}{(2\pi)^{\frac{m}{2}}} \exp\{-\frac{1}{2}\mathbf{x}^T \mathbf{R}^{-1} \mathbf{x}\} dx_1 dx_2 \cdots dx_m \quad (65)$$

Numerical evaluation of this integral is again unfeasible especially if  $m \geq 5$ . Consequently, approximate and bounding methods have been developed (Hohenbichler and Rackwitz 1983, Pandey 1998, Tang and Melchers 1987, Appendix C in the book by Melchers 1999). The evaluation of the multi-normal integral can be reduced to that of a single integral if the correlation structure is of the form  $r_{ij} = \lambda_i \lambda_j$  (Johnson and Kotz 1972, Pandey 1998):

$$\Phi_m(\mathbf{c}, \mathbf{R}) = \int_{-\infty}^{\infty} \phi(u) \prod_{i=1}^m \Phi\left\{\frac{c_i - \lambda_i u}{\sqrt{1 - \lambda_i^2}}\right\} du \quad (66)$$

For general forms of correlation matrix, Hohenbichler and Rackwitz (1983) developed a first order approximation that is referred to as first order multinormal method (FOMN). In applying this method, one begins by the transformation  $\mathbf{X} = \mathbf{B}\mathbf{U}$  where  $\mathbf{U}$  is the standard normal vector and  $\mathbf{B}$  is the lower triangular matrix obtained from the Choleski factorization of the correlation matrix,  $\mathbf{R}$ , such that  $\mathbf{R} = \mathbf{B}\mathbf{B}^T$ . Accordingly,

$$\Phi_m(\mathbf{c}, \mathbf{R}) = P[U_1 \leq c_1 \cap b_{21}U_1 + b_{22}U_2 \leq c_2 \cap \cdots \cap \sum_{j=1}^m b_{mj}U_j \leq c_m] \quad (67)$$

This is now re-written in an equivalent form

$$\Phi_m(\mathbf{c}, \mathbf{R}) = P[b_{21}U_1 + b_{22}U_2 \leq c_2 \cap \cdots \cap \sum_{j=1}^m b_{mj}U_j \leq c_m | U_1 \leq c_1] P(U_1 \leq c_1) \quad (68)$$

This can further be written as

$$\Phi_m(\mathbf{c}, \mathbf{R}) = \Phi_1(c_1) P[b_{21}\bar{U}_1 + b_{22}U_2 \leq c_2 \cap \cdots \cap \{b_{m1}\bar{U}_1 + \sum_{j=2}^m b_{mj}U_j\} \leq c_m] \quad (69)$$

Here  $\bar{U}_1$  is a non-Gaussian random variable whose pdf coincides with the pdf of  $U_1$  conditioned on  $U_1 \leq c_1$ . This non-Gaussian random variable now can be transformed into an equivalent Gaussian random variable through the transformation

$$\Phi(U_1) = \frac{\Phi_1(\bar{U}_1)}{\Phi_1(c_1)} \quad (70)$$

This leads to

$$P_F = \Phi(c_1)P[b_{21}\Phi_1^{-1}\{\Phi_1(c_1)\Phi_1(U_1)\} + b_{22}U_2 \leq c_2 \cdots \cap [\{b_{m1}\Phi_1^{-1}\{\Phi_1(c_1)\Phi_1(U_1)\} + \sum_{j=2}^m b_{mj}U_j\} \leq c_m] \quad (71)$$

Following spirit of FORM, the nonlinear terms in the event on the right side of above equation is linearized about the check point. This leads to an approximate relation of the form

$$\Phi_m(\mathbf{c}, \mathbf{R}) = \Phi(c_1)\Phi_{m-1}[\mathbf{c}^{(2)}, \mathbf{R}^{(2)}] \quad (72)$$

where  $\mathbf{c}^{(2)}$  is the modified vector of normal fractiles and  $\mathbf{R}^{(2)}$  is the modified correlation matrix. This leads to a sequential reduction of the number of distribution dimensions and one gets the approximation

$$\Phi_m(\mathbf{c}, \mathbf{R}) = \Phi_1(c_1)\Phi_1(c_2^{(2)}) \cdots \Phi_m(c_m^{(m)}) \quad (73)$$

Modifications to this procedure, with an aim to achieve improved accuracy, have been proposed by Tang and Melchers (1987) and Melchers (1999).

Terada and Takahashi (1988) introduced the idea of failure conditional reliability index. Here the authors formulated a failure conditioned reliability index of correlated multi-variate normal distribution to estimate failure probability and, finally, system reliability. An alternative approach, designated as the product of conditional marginals (PCM) method, has been proposed by Pandey (1998). The starting point in this approach is the representation

$$\Phi_m(\mathbf{c}, \mathbf{R}) = P[(X_m \leq c_m) | \cap_{k=1}^{m-1} (X_k \leq c_k)]P[(X_{m-1} \leq c_{m-1}) | \cap_{k=1}^{m-2} (X_k \leq c_k)]\Phi(c_1) \quad (74)$$

The essence of the idea here consists of approximating the conditional marginals appearing in the above equation by equivalent normal distributions. This leads to the approximation of the form

$$\Phi_m(\mathbf{c}, \mathbf{R}) = \prod_{k=1}^m \Phi(c_{k|k-1}) \quad (75)$$

The criterion to approximate a conditional marginal by an equivalent normal distribution is furthermore tackled using results from moments of truncated multi-normal random variables and approximate results on conditional probabilities.

### 4.3 Probability bounds

Consider  $k$  failure events denoted by  $\{E_k = [g_i(\mathbf{X}) \leq 0]\}_{i=1}^k$ . In evaluating system reliability interest is focused on determining the  $k$ -dimensional integral

$$P_F = \int_{\cup_{i=1}^k E_i} p_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \quad (76)$$

This integral is clearly more complicated than the one encountered in equation (1) and again can only be evaluated approximately. Alternatively, the failure probability could be characterized in terms of lower and

upper bounds. This line of work is widely pursued in the existing literature, see for example, the works of, Cornell (1967), Ditlevsen (1979b), Thoft-Christensen and Murotsu (1986), Karamchandani (1987), Greig (1992), Xiao and Mahadevan (1994) and Melchers (1999). Thus Cornell (1967) noted

$$\max_{i=1}^k [P(E_i)] \leq P_F \leq 1 - \prod_{i=1}^k [1 - P(E_i)] \quad (77)$$

Here the left and right bounds correspond to situations in which the failure modes are, respectively, fully dependent and mutually independent. Bounds that include probability of joint events  $E_i \cap E_j; i \neq j$  are given by (Ditlevsen 1979b)

$$\begin{aligned} P(E_1) + \sum_{i=2}^k \max\{P(E_i) - \sum_{j=1}^{i-1} P(E_i \cap E_j), 0\} \\ \leq P_F \leq \sum_{i=1}^k P(E_i) - \sum_{i=2}^k \max_{j < i} P(E_i \cap E_j) \end{aligned} \quad (78)$$

The study by Xiao and Mahadevan (1994) showed the equivalence of the bounds

$$P(\cap_{i=1}^k E_i) \leq 1 - \{P(E_1^c) + \sum_{i=2}^k \max[0, P(E_i^c) - \sum_{j=1}^{i-1} P(E_i^c \cap E_j^c)]\} \quad (79)$$

$$P(\cap_{i=1}^k E_i) \leq 1 - P(E_i^c) - \sum_{i=2}^n \max[0, \sum_{j=1}^{i-1} P(E_j \cap E_i^c) - (i-2)P(E_i^c)] \quad (80)$$

$$P(\cap_{i=1}^k E_i) \leq P(E_i) + \sum_{i=2}^n \{\min[0, i-2 + P(E_i) - \sum_{j=1}^{i-1} P(E_i \cup E_j)]\} \quad (81)$$

These bounds were proposed, respectively, by Ditlevsen (1979), Thoft-Christensen and Murotsu (1986) and Karamchandani (1987). The above bounds consider failure events according to different failure modes of failure for all possible loadings. An extension of the first order bounds to consider the effects of both the loading sequence and failure modes reads (Melchers 1999)

$$\max_i \{ \max_j [P(E_{ij})] \} \leq P_F \leq 1 - \prod_{i,j}^{mn} [1 - P(E_{ij})] \quad (82)$$

Here  $P(E_{ij})$  is the probability of failure in the  $i$ th mode under the  $j$ th load in the loading sequence,  $n$ =number of loads. Improvements to second order bounds by including higher order terms of the form  $P_{ijl} = P(E_i \cap E_j \cap E_l)$  lead to (Feng 1989, Greig 1992)

$$\begin{aligned} \sum_{i=1}^k [P(E_i) - \sum_{j < i} P(E_i \cup E_j) \max_{l < j} P_{ijl}]^+ \\ \leq P_F \leq \sum_{i=1}^k [P(E_i) - \sum_{j < i} \{P(E_i \cup E_j) - \sum_{l < j} P_{ijl}\}^+] \end{aligned} \quad (83)$$

Here the superscript  $[\ ]^+$  indicates that the term is to be included if it is positive. A feature of the bounds discussed above is that the bounds obtained depend upon the ordering of the events and optimal ordering of the events  $E_i$  is required to sharpen the bounds.



## 4.4 Failure mode enumeration

A large engineering structure typically would have numerous failure modes. Methods to search these failure modes, especially, those that have high failure probability of occurrence, constitute major tools of system reliability analysis. Related studies are found, for example, in the works of Moses (1982,1990,1996), Thoft-Christensen and Murotsu (1986), Bennet and Ang (1986), Thoft-Christensen (1990), Corotis and Nafday (1989), Karamchandani (1990), Zimmerman *et al.*, (1992), Dey and Mahadevan (1998), Shao and Murotsu (1999) and Melchers (1999). The approaches available for identifying dominant failure modes can be divided into three categories: enumeration approach, plasticity approach and simulation based approach (Shao and Murotsu 1999). In enumeration methods the approach to system failure *via* the failure of a set of failure elements is tracked in a sequential manner. In plasticity approaches attempt is made to identify dominant collapse mechanism by investigating failure by fundamental mechanisms and their linear combinations. Simulation based methods adopt sample function approach: these approaches being made computationally feasible by introduction of intelligent sampling strategies (see section 6.0). Some of the specific procedures available are incremental load method (Moses 1982), branch and bound method (Murotsu *et al.*, 1984, Thoft-Christensen and Murotsu 1986, Thoft-Christensen 1990), truncated mode enumeration (Melchers and Tang 1984), heuristic search techniques (Ranganathan and Deshpande 1987),  $\beta$ -unzipping method (Thoft-Christensen and Murotsu 1986, Thoft-Christensen 1990), methods based on the use of mathematical programming to search dominant failure modes (Nafday *et al.*, 1988, Corotis and Nafday 1989, Simoes 1990, Zimmerman *et al.*, 1992), and simulation based methods (Melchers 1999). Common threads in most of these methods lie in tracing failure paths starting from component failures with criterion to terminate those which are judged to have low probability of occurrence. The search strategies could be based on probabilistic arguments or employ heuristic/deterministic procedures. The former methods are time consuming while the latter offer no guarantee of not ignoring any important failure paths.

In dealing with post-yield behavior of members that fail, a common strategy that is used is to adopt member replacement method. Here, after an element fails, the element is removed from the structure and its effect is accounted for by applying a set of forces at the ends of the element. Thus, at every step, linear structural analysis methods are employed. In pursuing these steps it is noted that the formation of a mechanism is signaled by the stiffness matrix becoming singular. Karamchandani (1990) has investigated the limitations of these approximate procedures in the context of skeletal structural failure. The study by Wang *et al.*, (1997) reports on the use of system reliability concepts in the analysis of pile supported structures that are commonly used for dams. Shao and Murotsu (1999) have recently employed genetic algorithms to develop a selective search procedure for identifying dominant failure modes. Discussion on the use of response surface methods, stochastic FEM and simulation based methods for system reliability analysis have been presented elsewhere in this paper (see sections 6.0-8.0).

## 5.0 Time-dependent Reliability

Here one treats problems wherein the basic uncertainty variables are taken to evolve in time. These variables are modeled as random processes with time as the evolution parameter. Problems involving time variation of loads are commonplace. If the time variations in the loads are such that they produce appreciable inertial forces in the system, then the problem needs to be treated as a vibration problem. Otherwise, for slow variations of loads, such as in the case of variations in live loads, the problem becomes quasi-static in nature. In several problems, such as in fatigue related problems, the structural resistance also changes with time. Thus, in problems of time variant reliability the probability of failure can be written as

$$P_f = P[g\{\mathbf{X}(t)\} \leq 0; \forall t \in (0, T)] \quad (84)$$

The evaluation of this probability in terms of a multi-dimensional integration is not feasible since the size of

the integral here becomes intractably large. Also, the failure probability evaluated at any single time instant is of limited use, since, such a probability merely indicates the non-availability of structure at the time instant considered. Thus the treatment of time dependent reliability problems require alternative tools.

## 5.1 Methods based on Markov process theory

Here we consider randomly excited vibrating systems. When the inputs arise from Gaussian white noise processes, the response will be a diffusion process and the associated transitional pdf (tpdf) will satisfy the well known Kolmogorov equations. The governing equations of motion in these situations can be cast in the form of equations of the Ito type as follows:

$$d\mathbf{X}(t) = f(\mathbf{X}(t), t)dt + G(\mathbf{X}(t), t)d\mathbf{B}(t) \quad (85)$$

under the initial conditions

$$\mathbf{X}(t_0) = \mathbf{Y} \quad (86)$$

where,  $\mathbf{X}(t) = n \times 1$  response vector,  $f(\mathbf{X}(t), t) = n \times n$  matrix,  $G(\mathbf{X}(t), t) = n \times m$  matrix,  $\mathbf{B}(t) = m \times 1$  vector of the Brownian motion processes having the properties

$$E[\Delta_j(t)] = E[B_j(t + \Delta t) - B_j(t)] = 0 \quad (87)$$

$$E[\Delta_i(t)\Delta_j(t)] = 2D_{ij}\Delta t \quad (88)$$

and  $\mathbf{Y} = n \times 1$  vector of initial conditions independent of  $\mathbf{B}(t)$ . Here  $E[\cdot]$  represents the mathematical expectation operator. The above representation of equations of motion is fairly general in the sense that it allows for: (1) Multi-degree linear/nonlinear discrete systems, (2) external and parametric excitations, (3) nonstationary excitations, (4) nonwhite excitations, in which case, additional filters to model inputs as filtered white noise processes are to be appended to the system equations with a consequent increase in the size of the problem and (5) random initial conditions. The Kolmogorov equations satisfied by the response tpdf,  $p(\mathbf{x}, t|\mathbf{y}, t_0)$  are

- the Chapman-Kolmogorov-Smoluckowski (CKS) integral equation

$$p(\mathbf{x}, t|\mathbf{y}, t_0) = \int_{-\infty}^{\infty} p(\mathbf{x}, t|\mathbf{z}, \tau)p(\mathbf{z}, \tau|\mathbf{y}, t_0)d\mathbf{z} \quad (89)$$

- the forward equation or the Fokker-Planck-Kolmogorov (FPK) equation

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial t} = - \sum_{j=1}^n \frac{\partial}{\partial x_j} [f_j(\mathbf{x}, t)p(\mathbf{x}, t|\mathbf{y}, t_0)] + \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} [(G D G^T)_{ij} p(\mathbf{x}, t|\mathbf{y}, t_0)] \quad (90)$$

- the backward equation

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial t_0} = - \sum_{j=1}^n f_j(\mathbf{y}, t) \frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial y_j} - \sum_{i,j=1}^n [(G D G^T)_{ij} \frac{\partial^2 p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial y_i \partial y_j}]. \quad (91)$$

In these equations, the superscript T denotes the matrix transpose operation. The first of these equations represents the consistency condition for the response process to be Markov. Equation (91) is the adjoint of equation (90) and these two equations can be derived using equation (89) together with the equation of motion given by equation (85). It is of interest to note that the forward equation and backward equations are also satisfied by several other response probability functions of interest. Thus, for instance, the probability that first passage across a specified safe domain will not occur in the time interval  $t_0 - t$  for trajectories in the phase plane starting at  $\mathbf{y}$  at  $t = t_0$ , denoted by  $Q(t|\mathbf{y}, t_0)$ , can be shown to satisfy the backward

Kolmogorov equation. The formulation of these equations leads to the exact response characterization of a limited class of problems and helps in formulating strategies for approximate analysis for a wider class of problems. References to the details of the derivation of these equations along with a discussion on the initial conditions, boundary conditions, well posedness, eigenvalues and eigenfunctions and the existence, uniqueness and stability of stationary solutions can be found in the review paper by Manohar (1995).

As has been noted above, the Markov property of response can be used in the study of first passage probabilities. Here, either the forward or the backward Kolmogorov equation is solved in conjunction with appropriate boundary conditions imposed along the critical barriers. Alternatively, starting from the backward Kolmogorov equation, one can also derive equations for moments of the first passage time, which, in principle, can be solved recursively. Thus, denoting by  $\tilde{T}(\mathbf{y})$ , the time required by the response trajectory of equation (85) initiated at the point  $\mathbf{x} = \mathbf{y}$  in the phase space at time  $t = t_0$  to cross a specified safe domain, the moments  $M_k = E[\tilde{T}^k]$ ,  $k = 1, 2, \dots, N$ , of this random variable can be shown to be governed by the equation

$$-\sum_{j=1}^n f_j(\mathbf{y}, t) \frac{\partial M_k}{\partial y_j} - \sum_{i,j=1}^n (\mathbf{GDG}^T)_{ij} \frac{\partial^2 M_k}{\partial y_i \partial y_j} + k M_{k-1} = 0 \quad (k = 0, 1, 2, \dots) \quad (92)$$

with the condition  $M_0 = 1$ . These equations are referred to as the generalized Pontriagin-Vitt (GPV) equations in the literature. Although no exact analytical solutions exists for finding  $M_k$ , several approximations are available and they have been reviewed by Roberts (1986a). These methods include method of weighted residuals (Spanos 1983), random walk models (Toland and Yang 1971, Roberts 1978), finite difference method (Roberts 1986b), finite element method (Spencer and Bergman 1985) and cell mapping techniques (Sun and Hsu 1988). Cai and Lin (1994) have examined issues related to the specification of boundary conditions for the GPV equations. The book by Lin and Cai (1995) provides detailed account of the basic formulations. For systems that are driven by broad band random excitations, the method of stochastic averaging provides a means to approximate the response as a Markov process (Manohar 1995). Subsequently Markovian methods can be used to study first passage failures of such systems. This approach has been adopted in several studies, see for example, the recent studies by Noori *et al.*, (1995), Cai and Lin (1998) and Gan and Zhu (2001). A recursive scheme that uses the Green's function of the forward Kolmogorov equation for the study of first passage failure has been outlined by Sharp and Allen (1998).

## 5.2 Outcrossing approach

Here we consider the performance function as  $g[\mathbf{R}, b(\mathbf{Q}(t))]$  where,  $\mathbf{R}$  is the vector of random variables that model the structural parameter uncertainties,  $\mathbf{Q}(t)$  is the vector of random processes that model the time varying loads and  $b$  is the operator that transforms the load vector into load effects. Let  $N(T)$  denote the number of times the safe region is exited in the time interval  $[0, T]$ . The failure probability is the probability of the event that no failure at  $t = 0 \cup$  structure has not failed at  $t = 0$  and there is at least one outcrossing from the safe region in  $[0, T]$ . That is,

$$\begin{aligned} P_f(T) &= P_f(0) + P[N(T) \geq 1 | \text{structure has not failed at } t = 0][1 - P_f(0)] \\ &= P_f(0) + [1 - P_f(0)] \sum_{n=1}^{\infty} P[N(T) = n] \\ &\leq P_f(0) + [1 - P_f(0)] \sum_{n=1}^{\infty} n P[N(T) = n] \\ &= P_f(0) + [1 - P_f(0)] \langle N(T) \rangle \end{aligned} \quad (93)$$

If  $N(T)$  is modeled as being Poisson distributed, then

$$P_f(T) = P_f(0) + [1 - P_f(0)]\{1 - \exp[-\int_0^T \nu(\tau)d\tau]\} \quad (94)$$

where  $\nu(t)$  is the average rate of outcrossing from the safe region. Thus, the key to establishing bounds on  $P_f(T)$  lies in estimating the average rate of outcrossing of the response from the safe region. Furthermore, for problems involving random variable models for structural parameters, the failure probability can be calculated using

$$P_f(T) = \int_{\mathbf{r}} P_F(T|\mathbf{R} = \mathbf{r})p_{\mathbf{R}}(\mathbf{r})d\mathbf{r} \quad (95)$$

This calculation, in turn, can be carried out within the framework of FORM/SORM or the simulation methods discussed, respectively, in sections 3 and 6.

The problem of outcrossing of scalar and vector random processes across deterministic thresholds are widely studied in the literature. The average number of outcrossing from safe to unsafe region for a scalar random process  $x(t)$  across a time varying boundary  $\xi(t)$  can be derived based on the well known Rice's formula (1956) and can be shown to be given by

$$\nu^+(t) = \dot{\xi}(t) \int_{-\infty}^0 p_{X\dot{X}}[\xi(t), \dot{x}; t]d\dot{x} + \int_{\dot{\xi}(t)}^{\infty} [\dot{x} - \dot{\xi}(t)]p_{X\dot{X}}[\xi(t), \dot{x}; t]d\dot{x} \quad (96)$$

This formulation can be generalized to consider the average number of outcrossing of vector of random processes from safe region. Let  $G[\mathbf{X}(t), t]$  denote the performance function. The total number of zero crossing of this function can be shown to be given by

$$N(T) = \int_0^T \left| \sum_{i=1}^n \frac{\partial G}{\partial X_i} \dot{X}_i + \frac{\partial G}{\partial t} \right| \delta[G\{\mathbf{X}(t)\}] dt \quad (97)$$

With the substitution

$$Y = \sum_{i=1}^n \frac{\partial G}{\partial X_i} \dot{X}_i + \frac{\partial G}{\partial t} \quad (98)$$

the average number of outcrossings is obtained as

$$\nu(t) = \int_{G(x,t)=0} \int_{-\infty}^{\infty} |y| p_{XY}(x, y) dx dy \quad (99)$$

This, in turn, can be recast as a surface integral

$$\nu(t) = \int_{G(x,t)=0} E\{Y|X(t) = x\} p_X(x; t) dx \quad (100)$$

As might be expected, exact solutions here are possible only for limited class of problems. See the works of Veneziano *et al.*, (1979) on vector Gaussian random process outcrossing of elliptic, spherical and polyhedral safe regions. In the context of studies on outcrossing of vector random processes, Hagen and Tvedt (1991) and Hagen (1992) explored an earlier proposition by Madsen (unpublished) that the upcrossing rate of scalar stochastic processes as a particular sensitivity measure of the failure probability associated with a suitable defined parallel system. The recent paper by Rackwitz (1998) documents the results available on mean outcrossing rates for a set of random processes that include rectangular wave vector process, scalar and vector Gaussian random processes, Nataf and Hermite random processes. Further details are also available in the book by Melchers (1999). The recent study by Der Kiureghian (2000) offers a new perspective to problems of random vibration involving linear/nonlinear oscillators under Gaussian/non-Gaussian random excitations.

Here, the excitation process is discretized using a series representation, such as, Karhunen-Loeve expansion or stochastic Fourier series. The geometric forms of system response in the space spanned by these discretized random variables are explored. In doing so, various forms such as vectors, planes, half-spaces, wedges and ellipsoids are encountered depending upon the response variable studied. The application of FORM/SORM in this context is explored. Let  $\mathbf{U}$  = vector of random variables after discretizing the excitations,  $\mathbf{X}(\mathbf{U}, t)$  = response process of an oscillator, and  $G(\mathbf{X}) \leq 0$  be the failure domain. The outcrossing rate from safe domain is expressed as

$$\nu(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} P[G\{\mathbf{X}(\mathbf{U}, t)\} > 0 \cap G\{\mathbf{X}(\mathbf{U}, t + \Delta t)\} \leq 0] \quad (101)$$

The evaluation of  $\nu(t)$  is proposed to be studied as a problem in reliability of a system in parallel with performance functions  $-G\{\mathbf{X}(\mathbf{U}, t)\}$  and  $G\{\mathbf{X}(\mathbf{U}, t + \Delta t)\}$ . Towards this end, the use of FORM/SORM is proposed.

### 5.3 Load combinations

In a quasi-static problem that involves only a single time varying load, the reliability of the structure, in a specified duration, could be assessed by replacing the time varying load by its maximum in the specified time interval. This replacement converts a time variant reliability problem into a time invariant problem. However, this classical approach would fail if more than one load act simultaneously on the structure. The difficulty here essentially stems from the fact that the highest values of the two loads need not occur at the same instant. Thus, the knowledge of the extreme value distribution of the individual loads would not suffice to obtain the extreme value distributions of their combination. Detailed account of techniques available to treat this problem is provided in the monograph by Wen (1990) and, more recently, in the papers by Rackwitz (1998) and Floris (1998).

For the sum of two random processes,  $Q(t) = Q_1(t) + Q_2(t)$ , the bounds on outcrossing rate  $\nu_Q^+(\xi)$  can be obtained by manipulating the integral

$$\nu_Q^+(\xi) = \int_0^\infty \dot{q} p_{Q\dot{Q}}(\xi, \dot{q}) d\dot{q} = \int_0^\infty \dot{q} \left\{ \int_{-\infty}^\infty \int_{-\infty}^\infty p_{Q_2\dot{Q}_2}(\xi - q_1, \dot{q} - \dot{q}_1) p_{Q_1\dot{Q}_1}(q_1, \dot{q}_1) dq_1 d\dot{q}_1 \right\} d\dot{q} \quad (102)$$

Accordingly, Madsen *et al.*, (1986) show that

$$\nu_Q(\xi) \leq \nu_{Q_1} * p_{Q_2} + \nu_{Q_2} * p_{Q_1} \quad (103)$$

Here \* denotes the convolution operation. Conversely,

$$\nu_Q(\xi) \geq \int_{q=-\infty}^\infty \nu_{Q_1}(q) p_{Q_2}(\xi - q) [1 - P_{\dot{Q}_2|Q_2}(0, q)] dq + \int_{q=-\infty}^\infty \nu_{Q_2}(\xi - q) p_{Q_1}(q) [1 - P_{\dot{Q}_1|Q_1}(0, q)] dq \quad (104)$$

Similarly, for the case of combination  $Q(t) = Q_1(t) + Q_2(t) + Q_3(t)$ , it has been shown that

$$\nu_Q(\xi) \leq \nu_{Q_1} * P_{Q_2} * p_{Q_3} + \nu_{Q_2} * P_{Q_1} * p_{Q_3} + \nu_{Q_3} * P_{Q_1} * p_{Q_2} \quad (105)$$

Madsen *et al.*, have provided details of these bounds for the cases of combinations of normal and normal, normal and unimodal renewal pulse process, renewal spike process and arbitrary process, renewal rectangular pulse process and arbitrary process, unimodal Poisson process and unimodal Poisson process, and filtered renewal rectangular pulse process and arbitrary process. It is concluded that the pair of functions  $[\nu_{Q_i}(\xi), p_{Q_i}(q)]$  provides sufficient information about each load process in a linear combination. The paper by Rackwitz (1998) provides details of outcrossing rates of combination of Gaussian vector processes, combination of differentiable processes and rectangular wave processes, combination of intermittent processes, combination of intermittent differentiable and intermittent rectangular wave processes, and combination of intermittent processes with

non-intermittent processes. The consideration in these results are based on asymptotic SORM concepts but modifications are introduced in order to obtain accurate results for non-asymptotic situations.

Problems of linear load combination using the idea of load coincidence has been expounded by Wen (1990). To illustrate this idea, consider the combination  $R(t) = S_1(t) + S_2(t)$ , where  $S_1(t)$  and  $S_2(t)$  are independent processes. For any given instant of time, four possibilities exist:  $S_1(t) = 0$  and  $S_2(t) = 0$ ,  $S_1(t) \neq 0$  and  $S_2(t) = 0$ ,  $S_1(t) = 0$  and  $S_2(t) \neq 0$ , and  $S_1(t) \neq 0$  and  $S_2(t) \neq 0$ . When the last condition, *i.e.*,  $S_1(t) \neq 0$  and  $S_2(t) \neq 0$ , occurs, ‘coincidence’ is said to have occurred. For this situation, a new process  $S_{12}(t) = S_1(t) + S_2(t)$  called the coincidence process is defined. When the condition  $S_1(t) \neq 0$  and  $S_2(t) \neq 0$  is not satisfied,  $S_{12}(t) = 0$ . Introducing  $R_1 = \max_{t \in [0, T]} |S_1(t)|$ ,  $R_2 = \max_{t \in [0, T]} |S_2(t)|$ ,  $R_{12} = \max_{t \in [0, T]} |S_{12}(t)|$ , and  $R_m = \max_{t \in [0, T]} |R(t)|$  one gets

$$P_{R_m}(r) = P[R_m \leq r] = P[R_1 \leq r \cap R_2 \leq r \cap R_{12} \leq r] \quad (106)$$

This is approximated as

$$P_{R_m}(r) = P_{R_1}(r)P_{R_2}(r)P_{R_{12}}(r) \quad (107)$$

which implies an *ad hoc* assumption that  $R_{12}$  is independent of  $R_1$  and  $R_2$ . However, it can be noted that  $R_{12}$  is positively correlated with  $R_1$  and  $R_2$ , and, therefore,  $P_{R_m}(r)$  as given above provides an overestimation of the probabilities. The book by Wen (1990) provides details of characterization of the coincidence for cases of combination of Poisson pulse processes, combination of intermittent continuous processes, and combination of pulse and intermittent processes. Discussion on extension to multiple load and load effect combinations is also provided.

Problems of nonlinear load effect combination arises when performance function is nonlinear or when the mechanical behavior of the system is nonlinear or when both of these nonlinearities co-exist. Approximate methods involving linearization of performance function lead to results that are akin to to results using FORM in the time invariant reliability problems. When  $X_i(t)$  are Gaussian, this approach immediately provides estimates of the outcrossing rates. The results so obtained are clearly dependent on the choice of the point  $X^*$  about which the performance function is linearized. The question on choice of linearization point has been considered by Breitung and Rackwitz (1982), Breitung (1984, 1988), Pearce and Wen (1985), and Wen (1990). Pearce and Wen (1985) consider three choices for this purpose. Thus, in a standard normal space and for a fixed  $t$ , the linearization point could be any one of the following:

1. the point closest to the origin,
2. the point of maximum local outcrossing rate, and
3. a stationary point of the mean outcrossing rate out of the tangent hyperplane as the tangent is varying over the limit surface.

Ditlevsen (1987) has shown that choice 1 and 3 have the property that they lead to asymptotically correct results for large safe domains. He further favors the third choice as the best, but argues that the point closest to the origin is the more acceptable choice, since this point can be determined in a simpler manner.

## 6.0 Simulation based methods

The analytical methods discussed in the previous sections, for reliability analysis of failure elements/structural systems for time varying/time invariant situations, are essentially approximate in nature. Methods such as SORM have strong theoretical underpinnings, since, these methods lead to asymptotically exact solutions for large safe domains. In practical situations, however, criteria for application of asymptotic results are

not always met which leads to errors in the analysis that can neither be specified nor controlled. From a mechanistic point of view, analytical methods are also handicapped by lack of exactness in dealing with structural nonlinearities and parametric excitations. Monte Carlo simulation based approaches essentially overcome these deadlocks of analytical methods. The essential idea here is to digitally generate ensemble of structural and loading systems that obey specified probabilistic laws and treat individual structural analysis problems of this ensemble by using deterministic procedures. This leads to the generation of ensemble of sample solutions to the problem which is further processed to derive the statistical description of desired response quantities. Thus, in principle, the method is applicable to any situation where

- (a) it is possible to generate ensemble of structural parameters and external loads that obey prescribed probability laws, and
- (b) it is possible to solve the sample deterministic structural analysis problem.

In contrast to analytical methods, these methods are computationally intensive. However, given

- (a) the progress that has been achieved in methods for digitally simulating random variables/random fields,
- (b) developments in finite element methods to treat geometric/structural complexities and structural nonlinearities, and
- (c) the increased availability of fast and cheap computational power,

the simulation methods form powerful alternatives to analytical procedures. The monograph by Marczyk(1999) provides a spirited account of merits of Monte Carlo simulations in engineering modeling.

In the context of reliability analysis, Monte Carlo simulation, in its elementary form, becomes computationally intensive, since, often it is required to estimate failure probabilities that are very small. Obviously, the simulation of these rare events requires large computational effort. To find failure probability, as given by equation (1), using Monte Carlo simulations, one begins by generating samples of  $\mathbf{X}$  that conform to the prescribed  $p_{\mathbf{X}}(\mathbf{x})$ . An estimate of  $P_f$  is obtained from an ensemble of  $N$  samples and is given by

$$\hat{P}_f = \frac{1}{N} \sum_{i=1}^N I[g_i(\mathbf{X}) \leq 0]. \quad (108)$$

Here,  $g_i(\mathbf{X})$  is the  $i$ th realization of the performance function,  $I[g(\mathbf{X})]$  is an indicator function taking values of unity if  $g(\mathbf{X}) \leq 0$  and 0 otherwise. It can be shown that this estimator is unbiased and has minimum variance given by

$$\sigma^2 = \frac{\hat{P}_f(1 - \hat{P}_f)}{N} \quad (109)$$

Thus, it follows that to reduce the estimate of variance to acceptable levels,  $N$  needs to be increased. This, in turn, implies the need to solve a large number of sample problems, each of which requires the analysis of a structure-load system. Typically, a sample size of about  $10^4$  might not succeed in simulating even a single realization in failure domain for limit surfaces with probability of failure in the range of  $10^{-8} < P_f < 10^{-4}$ . The recent research developments in Monte Carlo simulations for structural reliability analysis are focused on overcoming this impasse. This has lead to development of several specialized techniques that primarily aim to improve the estimates of failure probability without taking recourse to increasing the sample size. The paper by Ayyub and McCuen (1995) provides a brief overview of the currently available procedures that include Latin hyper cube sampling, importance sampling, stratified sampling, and method of conditional expectation. Among these methods, the method of importance sampling and many associated variants probably are most

general in their scope and are more widely used than the others. In view of this, the review presented here focuses on the developments in the method of importance sampling.

## 6.1 Importance sampling methods

This perhaps is the most widely used variance reduction technique. The basic idea here is, in fact, not new: the paper by Kahn (1956) is one of the early references in this area. In implementing this method, the failure probability, as given by equation (1), is first re-written as

$$P_f = \int_{-\infty}^{\infty} I[g(\mathbf{x}) \leq 0] p_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (110)$$

where  $I[g(\mathbf{x}) \leq 0]$  is the indicator function. This equation is now modified to read

$$P_f = \int_{-\infty}^{\infty} \frac{I[g(\mathbf{x}) \leq 0] p_{\mathbf{X}}(\mathbf{x})}{h_{\mathbf{Y}}(\mathbf{x})} h_{\mathbf{Y}}(\mathbf{x}) d\mathbf{x} \quad (111)$$

where  $h_{\mathbf{Y}}(\mathbf{x})$  is a valid  $n$  dimensional joint pdf such that  $h_{\mathbf{Y}}(\mathbf{x}) \neq 0$  whenever  $g(\mathbf{X}) \leq 0$ . This function, which is yet unspecified, is called the sampling pdf. This nomenclature follows from the fact that  $P_f$ , as given by the above equation, can now be interpreted as

$$P_f = \langle \frac{I[g(\mathbf{X}) \leq 0] p_{\mathbf{X}}(\mathbf{X})}{h_{\mathbf{Y}}(\mathbf{X})} \rangle_h \quad (112)$$

where  $\langle \cdot \rangle_h$  denotes expectation operation with respect to the pdf  $h_{\mathbf{Y}}(\mathbf{x})$ . An estimate of the above expectation is given by

$$\hat{P}_f = \frac{1}{N} \sum_{i=1}^N \frac{I[g^{(i)}(\mathbf{X}) \leq 0] p_{\mathbf{X}}^{(i)}(\mathbf{X})}{h_{\mathbf{Y}}^{(i)}(\mathbf{X})} \quad (113)$$

where the superscript  $(i)$  denotes that the functions are evaluated at the  $i$ th realization of the vector  $\mathbf{X}$  sampled from  $h_{\mathbf{Y}}(\mathbf{x})$ . It can be shown that the above estimator is unbiased with minimum variance given by

$$\text{Var}[\hat{P}_f] = \frac{1}{N-1} \left[ \frac{1}{N} \sum_{i=1}^N \left\{ \frac{I[g^{(i)}(\mathbf{X}) \leq 0] p_{\mathbf{X}}^{(i)}(\mathbf{X})}{h_{\mathbf{Y}}^{(i)}(\mathbf{X})} \right\}^2 - \hat{P}_f^2 \right] \quad (114)$$

Thus, the variance of the estimator depends upon the yet unspecified sampling density function  $h_{\mathbf{Y}}(\mathbf{x})$ . Thus, this function can be selected to reduce the variance of the estimator. In fact, if we select

$$h_{\mathbf{Y}}(\mathbf{x}) = \frac{I[g(\mathbf{x}) \leq 0] p_{\mathbf{X}}(\mathbf{x})}{P_f} \quad (115)$$

the variance of the estimator becomes zero. This result appears miraculous but it stands demystified if one notes that the construction of  $h_{\mathbf{Y}}(\mathbf{x})$  actually demands the knowledge of  $P_f$  -the very quantity that is being estimated. In this context the remarks by Kahn (1956) reads:

”The significance of the existence of zero variance estimates lies not in the possibility of actually constructing them in practice but in that they demonstrate there are no ”conservation of cost” laws. That is, if the designer is clever, wise, or lucky he may, in choosing from the infinite number of sampling schemes available, able to choose a very efficient one. This is in some contrast to the situation in ordinary numerical analysis. It is usually true there that once a fairly good method of doing a problem has been found, that further work or additional transformations do not reduce the cost very much, if at all. In Monte Carlo problems, however, we are assured that there is always a better way until we reach perfection.”



Procedures that estimate  $P_f$  with specifically chosen  $h_Y(x)$  as sampling density functions are called important sampling procedures. It is interesting to note that the ideal  $h_{\mathbf{Y}}(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x}|\mathbf{x} \in \{g(\mathbf{x}) \leq 0\})$  is non-Gaussian in nature such that the function is zero in the safe regions and non-zero in the unsafe regions. Clearly, a major step in implementing importance sampling in structural reliability analysis lies in choosing an appropriate importance sampling pdf. Some of the options that have been proposed in the existing literature are briefly outlined below.

1. Shinozuka (1983) suggested  $h_{\mathbf{Y}}(\mathbf{x})$  to be uniformly distributed over an appropriately defined multi-dimensional square/rectangular domain covering the region of high likelihood around the design point.
2. Harbiz (1986) considered  $(X_i)_{i=1}^n$  to be standard normal random variables and assumed that the knowledge of the reliability index  $\beta_{HL}$  and the check point  $\mathbf{X}^*$  is available. He proposed that the sampling region be restricted to values outside the  $\beta$ -sphere. Accordingly, he arrived at the sampling pdf that is  $\chi^2$  distributed with  $n$  degrees of freedom. Furthermore, he transformed the basic variables  $(X_i)_{i=1}^n$  into polar coordinates  $(R, \Theta_1, \Theta_2, \dots, \Theta_{n-1})$ . Here  $R$  denotes the length of  $\mathbf{X}$  in the  $X$ -space and  $\Theta_1, \Theta_2, \dots, \Theta_{n-1}$  define the direction of  $\mathbf{X}$ . When  $X_i$ 's are standard normal, it can be shown that  $R$  is  $\chi^2$  distributed and is independent of the variables  $\Theta_1, \Theta_2, \dots, \Theta_{n-1}$ . It was also noted that sampling a value from the joint pdf  $p_{\Theta}(\theta)$ , corresponds to sampling a random direction unit vector  $\vec{\alpha}_j = (\alpha_1, \dots, \alpha_n)_j$  in  $\mathbf{X}$ -space. This may be done by sampling  $n$  i.i.d.  $N(0,1)$  variables  $Y_{1j}, Y_{2j}, \dots, Y_{nj}$ :

$$\vec{\alpha}_j = \frac{(Y_{1j}, Y_{2j}, \dots, Y_{nj})}{\{\sum_{i=1}^n Y_{ij}\}^{\frac{1}{2}}} \quad j = 1, 2, \dots, n \quad (116)$$

Thus, sampling random value of  $R > \beta$  from the corresponding truncated distribution of  $R$ , and using the random direction sampled as above, corresponds to sampling  $X$  from outside the  $\beta$ -sphere. Ouy-pornpraset and Schueller (1987), in a discussion on Harbiz's paper, pointed out that the formulation need not be restricted to standard normal space and, also, that there could be numerical difficulties in dealing with limit state function with multiple design points. Rackwitz (1987), also while discussing Harbiz's paper, argued that the most effective way to use simulation techniques is by estimating the error of FORM/SORM results.

3. Schueller and Stix (1987) began by considering single failure modes with only one design point. Based on the observation that main contribution to failure mode comes from integral over a neighborhood of design point, these authors proposed that the sampling pdf be a normal pdf  $g_{\mathbf{Y}}(\mathbf{x})$  where  $Y_1, Y_2, \dots, Y_n$  are mutually independent. If the failure probability is being evaluated in standard normal space,  $Y_i$  are also proposed to be standard normal. On the other hand, if the problem is formulated in the original  $\mathbf{X}$ -space, the sampling pdf is taken to be normal with mean shifted to the design point  $\mathbf{X}^*$ . The authors noted difficulty in a proper choice of covariance matrix of  $g_{\mathbf{Y}}(\mathbf{x})$ . Furthermore, in the context of problems with  $k$  failure modes, these authors propose that the sampling density function be of the form

$$g_{\mathbf{Y}}(\mathbf{x}) = \sum_{i=1}^k \frac{\Phi(\beta_i)}{\sum_{j=1}^k \Phi(\beta_j)} g_{\mathbf{Y}}^i(\mathbf{x}) \quad (117)$$

where  $\beta_i$  and  $g_{\mathbf{Y}}^i(\mathbf{x})$  are, respectively, the reliability index and the sampling density function corresponding to the  $i$ th failure mode. Discussion of a similar strategy in the context of series and parallel system reliability is available in works of Melchers (1990,1999). For parallel system reliability, it is suggested that the points of intersections of the limit surfaces be taken as points about which the  $g_{\mathbf{Y}}^i(\mathbf{x})$  be centered.

4. Bucher (1988) proposed an adaptive sampling procedure that does not require a first order reliability analysis to be performed prior to initiation of sampling. He proposed that the sampling density function

be selected to be a multinormal pdf, with mean and covariance given respectively by

$$\{\mu\} = \langle \mathbf{X} | g(\mathbf{X}) \leq 0 \rangle \quad (118)$$

$$[\mathbf{C}] = \langle (\mathbf{X} - \mu)(\mathbf{X} - \mu)^t | g(\mathbf{X}) \leq 0 \rangle . \quad (119)$$

From an initial simulation run right hand side of the above equations can be determined. These values are used to adapt  $g_{\mathbf{Y}}(\mathbf{x})$  for the next run. The author proposed a sensitivity based procedure to start the solution.

5. Hohenbichler and Rackwitz (1988) considered the use of importance sampling methods to update the estimates of failure probability using SORM especially when asymptotic conditions are not met. Engelund and Rackwitz (1993) consider similar corrections to be applied to FORM results also. The failure probability is expressed as

$$P_f = P(A) \frac{P(\Omega_f)}{P(A)} = P(A)C \quad (120)$$

where  $\Omega_f$  is the failure region in the standard normal space,  $P(A)$  is the approximate probability of failure computed using FORM/SORM, and  $C$  is the correction factor. For FORM results

$$C = \frac{1}{N} \sum_{i=1}^N \frac{\Phi(-|u^* + b_i \alpha|)}{\Phi(-\beta)} \quad (121)$$

The above estimate would have small variance because part of the integral is evaluated analytically. The recent paper by Rackwitz (1998) provides further details on these concerns as applied to time variant reliability problems.

6. Karamchandani *et al.*, (1990) outlined an adaptive importance sampling scheme with a multi-modal importance sampling pdf. In implementing this method, an initial sampling pdf, which has the same form as  $p_X(x)$ , but with its mean shifted to a point  $X^0$  in the failure region, is selected. The choice of  $X^0$  is guided by engineering judgment (point of low resistance and high loads, for instance). A pilot simulation run is made to identify a set of points that lie in the failure region. These set of points are further clustered into  $k$  groups that are centered around  $k$  representative points. The procedure for identifying these  $k$  representative points involves the following steps: Let  $S_i$  be the set of all previously identified sample points in the failure domain. Select a cluster radius  $d_0$ . Identify the point in  $S_i$  at which the  $p_X(x)$  is the largest. Call it  $\hat{x}^{(1)}$ . In  $S_i$  eliminate the points that lie within  $d_0$  of  $\hat{x}^{(1)}$ , including  $\hat{x}^{(1)}$ . From the remaining points in  $S_i$  identify the point at which  $p_X(x)$  is the largest. Call it  $\hat{x}^{(2)}$ . Eliminate the points in  $S_i$  that lie within the distance  $d_0$  from  $\hat{x}^{(2)}$ . Repeat this until there are no points left in  $S_i$ . The points  $\hat{x}^{(1)}, \hat{x}^{(2)}, \dots, \hat{x}^{(k)}$  are the representative points. The multi modal sampling density that is used to generate  $i$ th sample point reads

$$h_Y^i(x) = \sum_{j=1}^k \hat{w}_i^j p_X^{(j)}(x) \quad (122)$$

in which

$$\hat{w}_i^j = \frac{p_X(\hat{x}^{(j)})}{\sum_{r=1}^k p_X(\hat{x}^{(r)})} \quad (123)$$

and  $p_X^{(j)}(x)$ =original pdf with mean shifted to  $\hat{x}^{(j)}$ . Based on this, the estimate of  $P_f$  and the associated coefficient of variation are computed using the  $i$  number of samples that are generated till this step. The process of sampling is continued till the coefficient of variation falls below a specified threshold. The implementation of the method requires decision to be made on magnitudes of the distance  $d_0$  and also the variance of the sampling density function.

7. Ang *et al.*, (1989) began by sampling  $K$  points from  $p_X(x)$ . If  $M$  of these points are such that  $\{x_i | g(x) \leq 0\}$ , then an estimate of  $\hat{h}_Y(x)$  is proposed to be

$$\hat{h}_Y(x) = \frac{1}{M} \sum_{i=1}^M \frac{1}{w\lambda_i} K\left[\frac{x-x_i}{w\lambda_i}\right] \quad (124)$$

where  $w$  is a window width such that

$$\hat{h}_Y(x) = \lim_{w \rightarrow 0} \frac{1}{2w} P[x-w < X \leq x+w], \quad (125)$$

$\lambda_i$  is a scaling factor and  $K[\cdot]$  is a kernel function such that

$$\int_{-\infty}^{\infty} K(y) dy = 1; \quad K(y) \geq 0 \forall y \in (-\infty, \infty) \quad (126)$$

The parameter  $w$  is selected to minimize the variance of the estimator of  $P_f$ . In a subsequent study Wang and Ang (1994) proposed an adaptive scheme similar to the one proposed by Bucher (1988) to start the initial run.

8. The approach adopted by Maes *et al.*, (1993) is based on Breitung's (1984) results on asymptotic methods for failure probability estimation. These authors construct a sampling pdf that approaches the ideal sampling pdf  $h_Y(\mathbf{x}) = p_{\mathbf{X}}(\mathbf{x} | \mathbf{x} \in \{g(\mathbf{x}) \leq 0\})$  as  $P_f \rightarrow 0$ . The formulation is presented in the original  $\mathbf{X}$ -space and the paper emphasizes the relationship between extreme value distributions and the problem of determining small probability of failure. The starting point in applying this method is the linear transformation  $\mathbf{X} \rightarrow \mathbf{Y}$ , such that

- origin is at the point of maximum likelihood on the limit surface,
- first co-ordinate points towards the direction of steepest descent and
- other directions point towards direction of principal curvatures.

It is shown that the important sampling pdf is given by a  $n$  dimensional random variable such that, the first variable is exponentially distributed with mean  $\frac{1}{|\nabla L|}$  and the remaining  $n-1$  random variables are normal distributed with zero mean and covariance matrix  $\mathbf{W}^{-1}$ . Here  $L = \ln p_Y(\mathbf{y})$  is the log-likelihood function of  $\mathbf{Y}$ , and the matrix  $\mathbf{W}$  is given by

$$\mathbf{W} = \left\{ \left[ \frac{\partial^2 L}{\partial y_i \partial y_j} - \frac{|\frac{\partial L}{\partial y}|}{|\frac{\partial g}{\partial y}|} \frac{\partial^2 g}{\partial y_i \partial y_j} \right]_{i,j=1,2,\dots,n} \right\} \quad (127)$$

This matrix is assumed to be such that  $|\mathbf{W}| \neq 0$ .

9. The evaluation of probability integral by transforming the basic variables  $\mathbf{X}$  into polar coordinates has been considered by several authors (Ditlevsen 1988, Bjerager 1988, Kijawatworawet *et al.*, 1998). The transformation of the standard normal variable  $\mathbf{X} \rightarrow (R, \Theta_1, \Theta_2, \dots, \Theta_{n-1}) = (R, \tilde{\Theta})$  leads to

$$P_f = \int_{g(r, \tilde{\theta}) \leq 0} p_{R, \tilde{\Theta}}(r, \tilde{\theta}) dr d\tilde{\theta} \quad (128)$$

Here  $R$  is a scalar random variable such that  $R^2$  is  $\chi^2$  distributed with  $n$  degrees of freedom, and  $\tilde{\Theta}$  is a vector of  $n-1$  random variables that are independent of  $R$  and distributed uniformly in a  $n$  dimensional unit sphere. The failure probability can also be expressed as

$$P_f = \int_{g(r, \tilde{\theta}) \leq 0} p_R(r | \tilde{\Theta} = \tilde{\theta}) p_{\tilde{\Theta}}(\tilde{\theta}) d\tilde{\theta}$$

$$= \int_{\Omega} P[g(R, \tilde{\Theta}) \leq 0 | \tilde{\Theta} = \tilde{\theta}] p_{\tilde{\Theta}}(\tilde{\theta}) d\tilde{\theta} \quad (129)$$

Here  $\Omega$  is the unit sphere in  $n$  dimensions. In terms of the direction cosines  $\mathbf{A}$ , this integral can be expressed as

$$P_f = \int_{\Omega} P[g(R\mathbf{A}) \leq 0 | \mathbf{A} = \mathbf{a}] p_{\mathbf{A}}(\mathbf{a}) d\mathbf{a} \quad (130)$$

In evaluation of  $P_f$  by using simulation methods, a sample of the vector  $\mathbf{A}$  is first generated by simulating Gaussian vector  $\mathbf{U}$  and using the relation

$$A_i = \frac{U_i}{\sqrt{\sum_{i=1}^N U_i^2}} \quad (131)$$

For this choice of  $\mathbf{A}$ , the equation  $g(R\mathbf{A}) = 0$  is solved for  $r$  and  $P(R\mathbf{A} \leq 0)$  is evaluated analytically using the known pdf of  $R$ . The strategy for important sampling can be introduced by writing

$$P_f = \int_{\Omega} \frac{P[g(R\mathbf{A}) \leq 0 | \mathbf{A} = \mathbf{a}] p_{\mathbf{A}}(\mathbf{a})}{h_{\mathbf{B}}(\mathbf{a})} h_{\mathbf{B}}(\mathbf{a}) d\mathbf{a} \quad (132)$$

where  $h_{\mathbf{B}}(\mathbf{a})$  is the importance sampling density function. Discussion on selection of  $h_{\mathbf{B}}(\mathbf{a})$  is available in the works of Bjerager (1988) and Ditlevsen and Bjerager (1989). Generalization of the concept of directional simulation in the original  $X$  space is discussed in the works of Ditlevsen *et al.*, (1990), Melchers (1990) and in the load space by Melchers (1992).

10. Gupta and Manohar (2001a,b) considered adaptive strategies to construct sampling densities in the non-Gaussian random variable space. The basic variables in their study were taken to be partially specified in terms of ranges, mean and covariance matrix. A maximum entropy model for the first order pdf together with Nataf's model were used to construct the multidimensional  $p_{\mathbf{X}}(\mathbf{x})$ . Initial Monte Carlo runs with smaller safe domains were made and a Nataf's model is constructed for the sampling density function using data in the failure region. Here the range of the sampling random variables were assumed to be identical to the original variables. An adaptive strategy is used to construct sampling pdf as the failure region is expanded in increments. The failure probability of a randomly parametered curved Timoshenko beam is studied in figure (2). The beam is assumed to have non-Gaussian structural properties and is taken to be driven by a point harmonic force. Failure is characterized by exceedance of specified displacement levels. Preliminary results on reliability of vibrating curved beams and simple sdof oscillators have shown drop in variance of estimates of the failure probability as compared to the analysis of problem in standard normal space using similar sample sizes. Figure (3) illustrates this point in which the coefficient of variation of the estimate of failure probability using Gaussian and non-Gaussian importance sampling pdf are compared. The system considered is same as the one studied in figure (2). It is observed from this figure that for higher levels of failure thresholds, the non-Gaussian sampling pdf produces lesser coefficient of variation.

The study by Engelund and Rackwitz (1993) reports on relative performance of some of the widely used importance sampling schemes. The basis for comparison are the criteria related to the following issues:

- Basic variables in the original  $\mathbf{X}$ -space or in the space of standard normal variates.
- Capabilities to handle equalities, unions and intersections.
- Continuity of limit state function and/or joint distribution function of  $\mathbf{X}$ .
- Efficiency and accuracy (convergence properties) especially with respect to space dimension, probability level and curvatures of the limit state function.

The following methods were included in the comparative study:

- (a) The direct method where a Gaussian distribution with the same covariance matrix as  $p_{\mathbf{X}}(\mathbf{x})$  is used as the sampling density (Schueller and Stix 1987, Melchers 1989).
- (b) Same as in (a), but the covariance matrix is multiplied with a factor.
- (c) The asymptotic method according to Maess *et al.*,(1993).
- (d) The updating method based on FORM (Hohenbichler and Rackwitz 1988).
- (e) The updating method based on SORM (Hohenbichler and Rackwitz 1988).
- (f) Spherical sampling (Ditlevsen *et al.*, 1990 and Bjerager 1988).

A set of performance functions reflecting factors such as number of random variables (up to 50), probability level ( $\beta$  up to 10), nonlinearity, multiple  $\beta$  points, series and parallel systems and noisy limit state functions were considered. Distribution types considered included normal, lognormal, Gumbel and exponential distributions. Several conclusions were reached based on the study: under ideal conditions (simple components and series systems, smooth failure surfaces, continuous distributions, not too high uncertainty spaces) failure probability could be computed equally well by all these methods; when these ideal conditions were not met, the methods showed remarkable differences in efficiency and robustness. The authors conclude that no preference to any single method is justified in a given practical situation.

The evaluation of multi-normal equations that are encountered in reliability analysis of series and parallel system (equation 65) has been studied by Ambartzumian *et al.*,(1998) using importance sampling concepts. These authors consider the multi-normal integral

$$P(Q) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} \phi(x_1, x_2, \cdots, x_n) dx_1 dx_2 \cdots dx_n \quad (133)$$

where  $\phi(x_1, x_2, \cdots, x_n)$  is the n-dimensional Gaussian density with mean vector  $\mathbf{m}$  and covariance matrix  $\mathbf{C}$ . The authors propose a sequential conditioned importance sampling algorithm that proceeds on the following steps:

1. A random variable  $X_1$  is sampled in accordance with the pdf

$$P_{X_1}(x_1) = \frac{\phi_1(x_1)}{\Phi_1([a_1, b_1])} I_{[a_1, b_1]}(x_1) \quad (134)$$

where  $\Phi_1([a_1, b_1]) = \Phi_1(b_1) - \Phi_1(a_1)$ , and  $I_{[a_1, b_1]}(x_1)$  is the indicator function that is unity if  $x \in [a, b]$  and zero otherwise.

2. In the  $k$ th step, after  $x_1, x_2, \cdots, x_{k-1}$  are sampled,  $x_k$  is sampled in accordance with

$$\frac{\phi_k(x_k | x_1, \cdots, x_{k-1})}{\Phi_k([a_k, b_k] | x_1, \cdots, x_{k-1})} I_{[a_k, b_k]}(x_k) \quad (135)$$

where  $\Phi_k([a_k, b_k] | x_1, \cdots, x_{k-1}) = \Phi_k(b_k | x_1, \cdots, x_{k-1}) - \Phi_k(a_k | x_1, \cdots, x_{k-1})$

3. The trial is terminated after we sample  $x_n$ .

A variable  $Y$  is now introduced such that

$$Y = \prod_{k=1}^n \Phi_k([a_k, b_k]|x_1, \dots, x_{k-1}) \quad (136)$$

The authors prove the proposition that  $P(Q) = \langle Y \rangle$ . Numerical results on integral dimension up to seven are presented on multi-normal integrals for which alternative solutions using one dimensional integration are available. In a series of recent articles by Mahadevan and Dey (1997), Dey and Mahadevan (1998) and Mahadevan and Raghoeamachar (2000), the problem of reliability of structural systems made up of brittle or ductile elements has been investigated using importance simulation techniques. These authors employ branch and bound method of failure mode enumeration and combine it with importance sampling strategies for reliability estimation.

The use of importance sampling methods in the first passage failure analysis of nonlinear frame structures under stationary random excitations has been demonstrated by Bayer and Bucher (1999). These authors adopt a Fourier representation for the external excitation of the form

$$X(t) = \sum_{i=1}^{N_f} A_i \cos(\omega_i t + \Phi_i) \quad (137)$$

Here  $A_i$  are Rayleigh distributed with specified mean and variances and  $\Phi_i$  are uniformly distributed in  $(0, 2\pi)$ . If  $Y(t)$  is the response of the given structure, the failure probability is given by

$$P_f = P[\max_{t \in (0, T)} Y(t) \geq y^*] \quad (138)$$

Clearly  $Y(t)$  is a function of the basic random variables  $(A_i, \Phi_i)_{i=1}^{N_f}$ . To construct the importance sampling pdf, the authors propose an initial Monte Carlo run that would help the identification of dominant frequencies in the structural response. These frequencies are expected to be dependent on the linear structure natural frequencies and also the nonlinearities present. Treating these frequencies as being significant, an importance sampling density for the amplitudes  $A_i$  at these frequencies are constructed. The form of the pdf is assumed to be Rayleigh but the mean of the distribution is modified. The choice of the importance sampling pdf is based on adapting the simulation densities of the amplitudes with respect to the power spectral density of the observed structural response. A fact that this study seems to overlook is that a nonlinear system driven by a band limited excitation can have significant response components at frequencies that are outside the input frequency bandwidth.

## 7.0 Response Surface Method

In many real life applications, the functional relationship between basic design variables  $\{X_i\}_{i=1}^n$  and the performance function  $g(\mathbf{X})$  would not be explicitly available. This situation arises, for instance, when a large scale engineering structure is analyzed using commercial finite element software. Similarly, in vibration problems, when the governing field equations are nonlinear and/or contain parametric excitation terms, the definition of  $g(\mathbf{X})$  can only be given implicitly. Inclusion of interaction effects such as soil-structure interactions and fluid-structure interactions also can result in similar difficulty. This difficulty makes the use of FORM/SORM unfeasible because of difficulty in computing the gradients of performance function. On the other hand, direct Monte Carlo simulations or simulations with variance reduction strategies, are, in principle, applicable to tackle this difficulty. However, given the difficulty in evaluating  $g(\mathbf{X})$ , it is of interest to ask if alternative methods that are computationally less intensive than simulations could be developed to treat this class of

problems. An answer to this lies in substituting the performance function  $g(\mathbf{X})$  by a *response surface*  $\tilde{g}(\mathbf{X})$  that is typically of the form

$$\tilde{g}(\mathbf{X}) = A + \mathbf{X}^t \mathbf{B} + \mathbf{X}^t \mathbf{C} \mathbf{X} \quad (139)$$

Here,  $A$  is a scalar constant,  $\mathbf{B}$  is a  $n \times 1$  array and  $\mathbf{C}$  is a  $n \times n$  matrix with  $\mathbf{C}^t = \mathbf{C}$ . The problem of determination of  $\tilde{g}(\mathbf{X})$  consists of two steps:

1. Design of an appropriate experiment that essentially help to locate the points in  $\mathbf{X}$ -space at which  $g(\mathbf{X})$  needs to be evaluated.
2. Estimation of undetermined parameters  $A$ ,  $\mathbf{B}$  and  $\mathbf{C}$  in  $\tilde{g}(\mathbf{X})$  using known values of  $g(\mathbf{X})$  at the design points.

Once a satisfactory procedure is evolved to determine  $\tilde{g}(\mathbf{X})$ , the analysis of reliability could be subsequently carried out using FORM/SORM or Monte Carlo simulation with variance reduction techniques. The success of the method essentially hinges on how efficiently quantities  $A$ ,  $\mathbf{B}$  and  $\mathbf{C}$  are computed. For the response surface method to be a meaningful substitute for Monte Carlo simulations, clearly the computational effort expended in finding these constants must be smaller than what a Monte Carlo simulation study would demand.

The idea of using response surfaces in statistics is known for a long time (Box and Wilson 1954). The idea was used originally by Box and Wilson to find the operating conditions of a chemical process at which some response was optimized. Subsequent generalization of the idea led to the proposal that these methods can be used to develop approximating functions that surrogate for long running computer codes. The book by Khuri and Cornell (1987) and the paper by Romero *et al.*, (2000) provide modern perspectives in light of emergence of computational power. In a series of two papers Wong (1984,1985) considered the use of response surface methods in statistical analysis of geotechnical engineering problems, such as dynamic soil-structure interactions and slope stability analysis. The author used factorial designs and regression methods to fit the response surface. Monte Carlo simulations, using the response surface so obtained, were subsequently used to evaluate the statistical measures of response quantities. Faravelli (1989,1992) used response surface method in the context of nonlinear finite element analysis. The author employed quadratic surfaces including cross terms in matrix  $\mathbf{C}$  (equation 139). The undetermined coefficients were assembled in a single array  $\mathbf{d}$  and the governing equation for  $\mathbf{d}$  was recast as

$$\hat{g}(\mathbf{X}) = D\mathbf{d} + \mathbf{e}. \quad (140)$$

Here,  $D$  consists of constant, linear, quadratic and cross-combinations of  $\mathbf{X}$ ,  $\hat{g}(\mathbf{X})$  is the vector of values of performance function evaluated at the experiment design points, and  $\mathbf{e}$  is the error vector. The solution

$$\langle \mathbf{d} \rangle = [\mathbf{D}^t \mathbf{D}]^{-1} \mathbf{D}^t \hat{g}(\mathbf{X}) \quad (141)$$

leads to the estimates of the undetermined coefficients  $\mathbf{d}$ . The author also discusses methods to delineate contributions to response uncertainties from different sources of uncertainties. The work by Breitung and Faravelli (1996) provides further details of response surface method. This study identifies the following steps in the implementation of the response surface method:

1. Choice of one of several families of functions, which appear to be suitable to approximate the unknown function  $g(X)$ .
2. Design of experiments, which give optimal estimates of the parameters of the function chosen in the last step.
3. Validation of the desired approximation by statistical tests or other methods.

4. Test the significance of the contribution of terms in the designed functional form, for example, in a quadratic polynomial, the significance of the square terms. If they are not significant, a simpler model without these terms might be used instead of the whole polynomial expansion.

Bucher and Bourgund (1987) proposed an adaptive interpolation scheme to arrive at the response surface parameters. These authors take the response surface to be of the form

$$\tilde{g}(\mathbf{X}) = a + \sum_{i=1}^n b_i X_i + \sum_{i=1}^n c_i X_i^2. \quad (142)$$

The neglect of cross quadratic terms may be noted. They adopt a fully saturated experimental design and evaluate  $g(\mathbf{X})$  at design points given by  $2n + 1$  combinations of  $\mu_i$ ,  $\mu_i + h_i \sigma_i$  and  $\mu_i - h_i \sigma_i$ , where  $\mu_i$  and  $\sigma_i$  are, respectively, the mean and standard deviation of  $X_i$ . Using this information, an estimate of the undetermined response surface parameters are determined. Corresponding to the surface  $\tilde{g}(\mathbf{X})$  so obtained, the Hasofer-Lind reliability index and the associated design point  $\mathbf{x}_D$  are determined. In doing so it is assumed that the random variables  $\{X_i\}_{i=1}^n$  are uncorrelated and Gaussian distributed. An update on the location of the experimental design point is now obtained as

$$\mathbf{x}_M = \mu + (\mathbf{x}_D - \mu) \frac{g(\mu)}{g(\mu) - g(x_D)}. \quad (143)$$

This helps to locate the center point closer to the limit surface  $g(\mathbf{X}) = 0$ . A new surface using  $\mathbf{x}_M$  as the center point is obtained and this is used as the final estimate of the response surface. Thus, this procedure requires  $4n + 3$  number of evaluations of the performance function  $g(\mathbf{X})$ . Rajashekhar and Ellingwood (1994,1995) examined issues related to the choice of experimental points and suggested modifications to the approach used by Bucher and Bourgund (1987). They questioned if a single cycle of updating, as was proposed by Bucher and Bourgund, is always adequate. In answer to this question, these authors detailed how subsequent updating could be satisfactorily carried out. Other issues examined by these authors include a discussion on selecting design points near tails of probability distributions of the basic random variables and also on including cross terms in the response surface fit. A procedure for dropping relatively unimportant terms as the iterations progressed was also outlined. The conclusions from their study show that there are no special advantages in selecting experimental points near the tails of the distributions. Besides, it may also be remarked that in problems of dynamics and buckling analysis, the tails of probability distributions may not always have a crucial role to play. Thus, for example, occurrence of resonance condition requires a matching of driving frequencies and system natural frequencies, and, as such, the tails of probability distributions have little special significance. A modification to Bucher and Bourgund approach, in which updating of the response surface parameters are continued till a convergence criterion is satisfied, has been proposed by Liu and Moses (1994). Bucher and Schueller (1994) note the appropriateness of using response surface method in the study of nonlinear systems, since, the sample solution of these problems demands significant computational time. These authors used response surface method in the study of reliability of elastoplastic frames in which failure is defined by the loss of stiffness in any part of the structure due to progressive plastification. In the context of reliability analysis of framed structures, Ditlevsen and Nielsen (1994) considered the problem of determining corrections that need to be applied on rigid-idealized plastic models so that the predicted responses are equivalent to more elaborate finite element models. These corrections were approximated to the zeroth order by a constant term or to the first order by an inhomogeneous linear function of the basic variables. The relationship between these correction factors and response surface models for reliability analysis was noted by these authors. Kim and Na (1997) utilized linear response surface functions of the form  $\tilde{g}(\mathbf{x}) = a_0 + \mathbf{a}^t \mathbf{x}$ . These authors propose a sequential updating technique to fit a response surface that guarantees convergence on reliability index. Zhao and Ono (1998) demonstrate the successful use of the response surface method in the study of reliability of elastoplastic framed structures. These authors use mathematical programming tools to define



the performance function that are independent of failure modes and load paths. Guan and Melchers (2000) report on a parametric study on  $P_f$  as the parameter  $h_i$  in fitting the response surface is varied. Their study shows that there is no unique choice of  $h_i$  and the choices made on  $h_i$  can significantly affect the computed  $P_f$ . Huh and Haldar (2001) consider the time domain analysis of randomly parametered inelastic structures to earthquake ground motions. The study combines concepts from stochastic FEM, response surface models, and FORM. These authors consider both serviceability and strength limit states. Response surfaces using quadratic polynomials with/without cross terms are employed in conjunction with an iterative scheme. The earthquake inputs in this study are modeled as recorded time histories that are scaled by random peak ground acceleration factors. This would mean that the short term uncertainties that are present to a significant level in earthquake loads are not adequately modeled in this study.

Problems of time variant reliability analysis using response surface methods have been studied by Brenner and Bucher (1995), Yao and Wen (1996), and Zhao *et al.*, (1999). Brenner and Bucher (1995) used response surface method for reliability analysis of large scale nonlinear dynamical systems (see section 8.5.3 for a further discussion on this reference). Yao and Wen consider nonlinear oscillators with random parameters subject to random excitations. They begin by assuming that the maximum response over a given duration, conditioned on system parameters, follow the Type I extreme value distributions. Response surfaces are then fitted to the conditional mean and coefficient of variation of this response. The unconditional response statistics are obtained by using a numerical integration scheme. Zhao *et al.*, also consider random hysteretic multi-degree of freedom (mdof) systems subject seismic inputs and employ equivalent linearization scheme and fit response surfaces to performance functions. The use of response surface methods in problems of multidisciplinary stochastic optimization has been reported by Oakley *et al.*, (1998). Here the objective is to maximize system mechanical performance, such as aerodynamic efficiency, while satisfying reliability based constraints.

## 8.0 Response of uncertain systems using SFEM

Problems of structural mechanics in which the structural elastic, mass, geometric and strength characteristics vary randomly in space have received notable research attention in the recent past. One of the major developments that has taken place has been the generalization of the traditional FEM to take into account the additional complication arising out of modeling spatially inhomogeneous randomness in structural properties (Ghanem and Spanos 1991, Kleiber and Hien 1992, and Haldar and Mahadevan 2000a). Here, in parallel to the discretization of the displacement/stress fields in traditional FEM, the structural property random fields also need to be discretized. This results in the replacement of these random fields by an equivalent set of random variables. Consequently, the structural matrices now become functions of these discretized random variables thereby making the resulting equilibrium equations to possess random parameters. This leads to questions on treatment of random matrix eigenvalues, random matrix inverse and solution of random matrix differential equations. The following sections briefly report on the study that have been carried out in this context.

### 8.1 Discretization of Random Fields

An important step in finite element modeling of randomly inhomogeneous structures is the replacement of the element property random fields by an equivalent set of finite number of random variables. This process constitutes the discretization of the random field and the accuracy with which the field gets represented primarily depends on the size of finite element used. The factors to be considered in the selection of mesh size which, in turn, influence the accuracy and efficiency of discretization are: stress and strain gradients, frequency range of interest, nature of the information available about the random field, correlation length of the random fields representing the element properties and the excitations, ability to model accurately the tails of the pdf especially when non-Gaussian models are being used, ability to model nonhomogeneous random

fields, stability of numerical inversion of the probability transformations, gradient of limit state function and the desire to introduce a minimum number of random variables into the analysis.

Several alternatives for discretization of random fields have been proposed in the existing literature. Thus, Vanmarcke and Grigoriu (1983) replaced the random field within a finite element by its spatial average and this lead to the concept of local averages, variance function and scale of fluctuations, using which, it was possible to describe mean and covariance of the elements of the structural matrices. This method of discretization has been used by several authors including Shinozuka and Deodatis (1988), Chang and Yang (1991), Zhu *et al.*, (1992) and Anantha Ramu and Ganesan (1992a,b, 1993a,b). A simple way of discretizing the random field would be by approximating the field by its value at the centroid of the finite element (Hisada and Nakagiri 1985, Der Kiureghian and Ke 1988, Yamazaki *et al.*, 1988). This method is particularly suited for discretizing non-Gaussian fields and, in this context, it has been used by Liu and Der Kiureghian (1991) in their studies on reliability of nonlinear systems with non-Gaussian uncertainties. In fact, when the Nataf model is discretized this way, the discretized random variables are completely described by the marginal probability density functions and the covariance matrix. Discretization of random fields in terms of a set of shape functions has been used by a few authors. Thus, Liu *et al.*, (1986) approximate the random field  $f(x)$  by

$$f(x) = \sum_{i=1}^q N_i(x) f_i \quad (144)$$

where  $f_i$  are the nodal values of  $f(x)$ . This representation is equivalent to interpolating  $f(x)$  within an element using  $N_i(x)$  and the nodal values of  $f(x)$ . Furthermore, the error of discretization is characterized in terms of total mean square difference between the covariance function of the discretized field and the exact covariance function. To achieve computational efficiency, the discretized random variables are transformed into a set of uncorrelated random variables, after which, it is expected that number of random variables which needs to be retained in subsequent analysis is significantly less than  $q$ . The authors have also noted that  $q$  need not be equal to the number of finite elements used to discretize the displacement field and  $N_i(x)$  need not coincide with the shape functions used for finite element discretization. Lawrence (1987), Spanos and Ghanem (1989), Iwan and Jensen (1993) and Zhang and Ellingwood (1994) have constructed series expansions for random fields in terms of a set of deterministic orthogonal functions multiplied by random variables and incorporated them into finite element formulations. The expansion used by Lawrence for the random field  $f(x)$  has the form

$$f(x) = \sum_j f_{0j} \psi_j(x) + \sum_j \sum_i f_{ij} e_i \psi_j(x) \quad (145)$$

where,  $e_i$ = orthogonal set of random variables with zero mean and unit variance,  $\psi_j(x)$ = a set of known orthogonal deterministic functions, such as, for example, Legendre polynomials over a line segment,  $f_{ij}$ = unknown deterministic constants to be found by a least square fit to the first and the second moments of  $f(x)$ . In the study by Spanos and Ghanem (1989), the expansion is based on the Karhunen-Loeve expansion, in which case, the set of orthogonal functions  $\psi_j(x)$  are obtained as the solutions of the eigenvalue problem

$$\int_L R(x, \xi) \psi_j(x) d\xi = \lambda_j \psi_j(x). \quad (146)$$

This expansion is mathematically well founded with the expansion guaranteed to converge and, also, the expansion is optimum in the sense that it minimizes the mean square error resulting from truncating the series at a finite number of terms. Studies on series expansions in terms of arbitrary set of orthogonal functions, in which, the associated random variables become correlated, have been considered by Zhang and Ellingwood (1994). These authors have also noted that their method is equivalent to solving equation (144) by a Galerkin approximation in terms of the arbitrarily chosen orthogonal functions. Li and Der Kiureghian (1993) have used optimal linear estimation procedures in representing the random field as linear combination

of nodal random variables and a set of unknown shape functions, that is, the random field  $f(x)$  is estimated by

$$\bar{f}(x) = a(x) + \sum_{i=1}^N b_i(x)f(x_i) \quad (147)$$

in which,  $N$  = number of nodal points,  $a(x)$ = scalar function of  $x$ ,  $b(x) = [b_i(x)]$ = vector function of  $x$  with element  $b_i(x)$ . The unknown functions  $a(x)$  and  $b(x)$  are found by minimizing the Variance $[f(x) - \bar{f}(x)]$  under the constraint  $E[f(x) - \bar{f}(x)] = 0$ . The efficiency of the method is further shown to improve by employing spectral decomposition of the nodal covariance matrix which effectively reduces the number of random variables entering the formulation. Gupta and Manohar (2001b) have shown that the shape functions  $b_i(x)$ , satisfy the relation  $b_i(x_j) = \delta_{ij}$ , although this condition is not explicitly imposed as a requirement in deriving  $b_i(x)$ . This property is illustrated in figure (4), where the first six shape functions are obtained for a random field with covariance function of the form  $R_{ff}(\tau) = \exp[-\alpha\tau^2]$ . As a consequence of this property, it follows that the first order pdf of  $\bar{f}(x)$  matches exactly with the corresponding pdf of  $f(x)$  at  $x = x_i$ , thus the non-Gaussian features of the random field are well represented in the discretized random field. This also implies that the mean square error becomes zero at  $x = x_i$ . The study by Gupta and Manohar demonstrate the use of this discretization scheme in reliability analysis of vibrating skeletal structures.

The shape functions in finite element discretization are, most often, polynomials in spatial coordinates and, if these functions are used to discretize the random fields, integrals, typically of the form

$$W_n = \int_0^L x^n f(x) dx; \quad n = 0, 1, 2, \dots \quad (148)$$

appear in the expressions for the stiffness coefficients. Clearly,  $W_n$  are random variables, and, these have been termed as weighted integrals associated with the element. These integrals offer an alternative way of discretizing the random fields; related studies on static stiffness of one dimensional structural elements can be found in the papers by Shinozuka (1987), Bucher and Shinozuka (1988), Kardara *et al.*, (1989), Takada (1990a), Deodatis (1991), Deodatis and Shinozuka (1989,1991) and those on two dimensional elements in Takada (1990b) and Wall and Deodatis (1994). Bucher and Brenner (1992) have extended the weighted integral approach to dynamics problems. The number of weighted integrals resulting from discretizing a random field for an element depends upon the type of shape function used; if complete polynomials with maximum order  $n$  are used, then, the number of weighted integrals for one, two and three dimensional elements are given by  $(2n + 1)$ ,  $(2n + 1)(n + 1)$  and  $(2n + 1)(n + 1)(2n + 3)/3$ , respectively. Manohar and Adhikari (1998), Adhikari and Manohar (2000) and Gupta and Manohar (2001b) have generalized the concept of weighted integrals to cover problems of dynamics. These authors derive shape functions from dynamic equilibrium of the deterministic beam element. The resulting weighted integrals become dependent on frequency and also are complex valued. Figure (5) shows a typical plot of these shape functions. It is observed from the figure that these shape functions adapt their shape with frequency and are identical to the static shape functions, obtained from the Hermite polynomials, at  $\omega = 0$ . An advantage of this modeling is that the size of the finite element models are not dependent on frequency range of external forcing.

Deodatis(1991) has noted that the displacement field of a stochastic beam element subject to boundary displacements can be derived exactly in terms of the system Green function. This, consequently, leads to the definition of exact *stochastic* shape functions and the exact stochastic static stiffness matrix of the beam element. This would mean that random field discretization in this case would be error free. It can be noted that this result is valid whether or not the random field is Gaussian. Except for this exact solution, all other discretization procedures discussed above lead to error of discretization which, consequently, restricts the size of the finite element to a fraction of the correlation length of the random field. Zhu *et al.*, (1992) have observed that stochastic FEM based on local average discretization with fewer elements can yield the same accurate

results as those provided by the stochastic FEM based on mid-point discretization with more elements. The issue of selection and adaptive refinement of mesh size for random field discretization for reliability studies have been examined by Liu and Liu (1993). They have recommended that a coarse mesh must be used in areas where the gradient of the limit state function with respect to the discretized random variables is small and a fine mesh where the gradients are large.

When  $f(x)$  is non-Gaussian, it is clear that the weighed integrals are also non-Gaussian in nature. Given the moments of  $f(x)$ , the moments of the weighted integrals can, in principle, be computed. However, the determination of the pdf of the weighted integrals, in closed form, is, by and large, not possible. This feature of the weighted integral approach handicaps its use in reliability studies. Based on central limit theorem it could be argued that the weighted integrals of non-Gaussian random processes approach Gaussian distributions when certain conditions on correlation of  $f(x)$  and variation of the weighting functions are met. The usefulness of this argument however is not clear. It must be noted here that in SFEM applications, the random functions have bounded tails dictated by the constraints on positivity. Furthermore, failure probabilities are known to be sensitive to the tail regions of the distributions. This, again brings into focus the limitation of weighted integral approach. In this context, the attempts to study non-Gaussian characteristics of weighted integrals of non-Gaussian random processes by Ditlevsen *et al.*, (1996), Mohr and Ditlevsen (1996), Mohr (1999) and Choi and Noh (2000) are note worthy. While many authors have focused attention on errors arising due to the discretization of random fields, the influence of errors of random field discretization on the structural responses seem to have received less attention. In their studies on skeletal structures, Manohar and Adhikari (1999), Adhikari and Manohar (1999,2000) and Gupta and Manohar (2001), have analyzed the two sources of errors when approximate methods are used in response prediction. In these studies, the dynamic stiffness of inhomogeneous structural elements are computed by converting the associated boundary value problems into a larger class of equivalent initial value problems. An important aspect of this procedure is that it does not require discretization of random fields beyond what is required in step by step integration procedure. Treatment of multi-dimensional random fields in the context of continuum elements such as plates and shells, using similar approaches appears to have not been studied in the literature. Methods to simulate weighted integrals non-Gaussian random fields for simulation based response analysis has been discussed by Micaletti (2001). The proposed approach employs a Gaussian quadrature rule to evaluate an individual integral thereby reducing the computation of integral to problem of simulating a smaller vector of non-Gaussian random variables.

As a first step in the validation process of discretization errors using Monte Carlo simulations, an ensemble of the random fields/random variables obeying the prescribed joint probabilistic descriptions, needs to be digitally simulated. These simulation procedures are widely studied in the literature and, in this review, these issues are touched upon only briefly. Methods for simulation of vector of Gaussian random variables, univariate/multivariate, one-dimensional/multidimensional, stationary/nonstationary Gaussian random fields are well developed (Shinozuka and Deodatis 1991,1996, Samaras *et al.* 1985, Schueller 1997b, Spanos and Zeldin 1998) and are not elaborated here. The methods for simulating non-Gaussian random processes can be broadly classified under three categories: namely, memoryless translation, method of linear filters and method of nonlinear filters. A systematic way of constructing non-Gaussian field models  $f(x)$  is by making nonlinear memoryless transformations of a specified Gaussian field  $\nu(x)$ , through the relation  $f(x) = T[\nu(x)]$ , where  $T$  is a memoryless nonlinear function. Discussions on the digital simulation of non-Gaussian random processes using the method of memoryless transformation can be found in the works of Yamazaki and Shinozuka (1988), Grigoriu (1984,1995,1998) and Der Kiureghian and Liu (1986). These types of transformations enable characterization of  $f(x)$  in terms of mean and covariance of  $\nu(x)$ . Prominent among this type of models is the Nataf model, which can produce any desired marginal distribution of  $f(x)$  (see section 2.1.1.2). This model was employed by Liu and Der Kiureghian (1991) and Li and Der Kiureghian (1993) in their studies on reliability of stochastic structures. However, simulation of non-Gaussian random processes using memoryless

transformation has a limitation as it is not always possible to find a covariance function of the Gaussian process such that the simulated non-Gaussian process has the target covariance function. Liu and Munson (1982) proposed a scheme for generating non-Gaussian random processes using white Gaussian noise source input to a linear digital filter followed by a zero-memory nonlinearity. In this formulation, the zero-memory nonlinearity is chosen so that the desired distribution is exactly realized and the digital filter is designed so that the desired autocovariance is closely approximated. Lutes (1986) and Lutes and Hu (1986) considered filtered non-normal white noise processes through a linear filter such that the filtered signal would yield the desired power spectrum. While this approach is attractive, the non-normality prescribed for the initial white noise is affected by the filter characteristics. The mathematical difficulties encountered with non-normal white noise processes have been discussed by Grigoriu (1987). Alternatively, a Gaussian process with the prescribed power spectrum can be subjected to a memoryless functional transformation to obtain the non-Gaussian process. This approach has been used by Winterstein (1985), Bendat (1990), Smallwood(1996) and Merritt (1997). Gurley *et al.*,(1996) have used truncated Hermite polynomial transformation in simulating non-Gaussian random processes which closely match sample histogram, psd function, and central moments through fourth order. This study is carried out in the context of modeling of wind and wave loads. Method of conditional simulation and kriging have been used by Hoshiya *et al.*,(1998) and Noda and Hoshiya (1998) in their study on non-Gaussian random processes. Studies on log-normal random processes using polynomial chaos representations have been reported by Ghanem (1998).

## 8.2 Response Analysis: Random Eigenvalues

Eigensolutions constitute an important descriptor of the dynamics and stability of structural systems. Consequently, the study of probabilistic characterization of the eigensolutions of random matrix and differential operators has emerged as an important research topic in the field of stochastic structural mechanics. In particular, several studies have been conducted on both self adjoint and non-self adjoint (usually encountered with systems involving follower forces, aerodynamic damping, and gyroscopic couples) eigenvalue problems. Other issues include multiplicity of eigenvalues and related problems. A systematic account of perturbation approaches to random eigenvalue problems is well documented in a research monograph by Scheidt and Purkert (1983).

The majority of recent studies employed the mean centered first/second order perturbation approach to estimate the first and the second order statistics of eigenvalues and mode shapes. For example, Nakagiri *et al.* (1987) studied the statistics of natural frequencies of simply supported fiber reinforced plastic plates whose stacking sequence is subjected to random fluctuations. They presented case studies on the statistics of first natural frequency of square and rectangular plates using triangular finite elements to discretize the domain into 60-70 elements. It was concluded that the variability in natural frequencies increased with decreasing correlation among member lengths. Mironowicz and Sniady (1987) used a first order perturbational approach to study the vibration of a machine foundation block with random geometry and mass density. They used a resonance index given by

$$\beta = \frac{(\bar{\omega} - \omega_e)}{\sqrt{(\sigma_{\omega}^2 + \sigma_e^2)}} \quad (149)$$

to characterize the resonance characteristics, where  $\bar{\omega}$  and  $\sigma_{\omega}$  are the mean and standard deviation of the natural frequency, respectively, and  $\omega_e$  and  $\sigma_e$  are the mean and standard deviation of harmonic driving frequency, respectively. This index is akin to the reliability index in structural reliability problems. Nordmann *et al.* (1989) investigated the eigensolution variability of vessel and piping systems in the context of seismic response analysis using response spectrum-based approaches. Zhu *et al.* (1992) used the method of local averages to discretize random fields, in conjunction with a perturbational approach, to study the statistics of the fundamental natural frequency of isotropic rectangular plates. Their formulation also allows for multiplicity of

deterministic eigenvalues. The latter issue was also addressed by Zhang and Chen (1991). Song *et al.* (1995) outlined a first order perturbational approach to find the moments of the sensitivity of random eigenvalues with respect to the expected value of specified design variables. Bucher and Brenner (1992) employed a first order perturbation and, starting from the definition of Rayleigh's quotient for discrete systems

$$E[\lambda_i] = \frac{\mathbf{x}_{i0}^T \mathbf{K}_0 \mathbf{x}_{i0}}{\mathbf{x}_{i0}^T \mathbf{M}_0 \mathbf{x}_{i0}} \quad (150)$$

they showed that

$$\sigma_{\lambda_i}^2 = \epsilon^2 \lambda_{i0}^2 E\{(\mathbf{x}_{i0}^T \mathbf{K}_r \mathbf{x}_{i0})^2 + 2(\mathbf{x}_{i0}^T \mathbf{K}_r \mathbf{x}_{i0})(\mathbf{x}_{i0}^T \mathbf{M}_r \mathbf{x}_{i0}) + (\mathbf{x}_{i0}^T \mathbf{M}_r \mathbf{x}_{i0})^2\} \quad (151)$$

where  $\epsilon \ll 1$ , the random eigenvalue  $\lambda_i = \lambda_{i0} + \lambda_r$ ,  $\mathbf{x}_i = \mathbf{x}_{i0} + \epsilon \mathbf{x}_{ir}$ ,  $\mathbf{K} = \mathbf{K}_0 + \epsilon \mathbf{K}_r$  and  $\mathbf{M} = \mathbf{M}_0 + \epsilon \mathbf{M}_r$ . Subscripts containing 0 denote deterministic quantities and subscripts with  $r$  denote random quantities. Fang (1995) combined transfer matrix methods with first order second moment approach to analyze the natural frequencies and mode shapes of uncertain beam structures. A computational algorithm based on transfer matrices to compute natural frequencies of a fixed-fixed string with a set of intermediate random spring supports was given by Mitchell and Moini (1992). The spring constants in this study were modeled as a set of independent two-state random variables. Random eigenvalue problems arising in structural stability were studied by Anantha Ramu and Ganesan (1992a,b, 93a,b), Sankar *et al.* (1993), Zhang and Ellingwood (1995) and Ganesan (1996) using perturbational approaches. Koyluoglu *et al.* (1995b) used Monte Carlo simulation technique in conjunction with weighted integral method of random field discretization. Anantha Ramu and Ganesan and Sankar *et al.* considered several problems associated with stability of beams/rotors with randomly varying Young's modulus and mass density. These include systems with non-self adjoint eigenvalue problems. For example, the determination of whirling speeds of a stochastic spinning shaft is associated with the eigenvalue problem

$$[\mathbf{K} \bar{+} \mathbf{K}_r]\{x_0\} = \omega^2 \{\mathbf{M} \bar{+} \mathbf{M}_r + (\Omega/\omega)[\mathbf{G} \bar{+} \mathbf{G}_r]\}\{x_0\} \quad (152)$$

where  $\mathbf{x}_0$  is the eigenvector,  $\mathbf{K}$  is the stiffness matrix,  $\mathbf{M}$  is the mass matrix,  $\mathbf{G}$  is the gyroscopic matrix,  $\Omega$  is the shaft rotational speed,  $\omega$  is the whirling speed and a bar denotes the expected value. The perturbation of the eigenvalue  $\omega^2$  is shown to be given by

$$d\omega_i^2 = \sum_{j=1}^n \sum_{s=1}^n \frac{\partial \omega_i^2}{\partial k_{js}} dk_{js} + \sum_{j=1}^n \sum_{s=1}^n \frac{\partial \omega_i^2}{\partial m_{js}^*} dm_{js}^* \quad (153)$$

where the symbol of matrix  $[\mathbf{M}^*]$  (whose elements are  $m_{js}^*$ ) is used in place of the term  $\mathbf{M} \bar{+} \mathbf{M}_r + \frac{\Omega}{\omega}(\mathbf{G} \bar{+} \mathbf{G}_r)$ . The expressions for gradients of  $\omega^2$  with respect to mass and stiffness terms were obtained using the expressions given by Plaut and Huseyin (1973)

$$\frac{\partial \omega_i^2}{\partial k_{js}} = \{\mathbf{y}_i\}^T \left[ \frac{\partial (\mathbf{K} \bar{+} \mathbf{K}_r)}{\partial k_{js}} \right] \{\mathbf{x}_i\} \quad (154)$$

$$\frac{\partial \omega_i^2}{\partial m_{js}^*} = -\omega_i^2 \{\mathbf{y}_i\}^T \left[ \frac{\partial (\mathbf{M} \bar{+} \mathbf{M}_r)}{\partial k_{js}} \right] \{\mathbf{x}_i\}. \quad (155)$$

Here  $\mathbf{x}_i$  and  $\mathbf{y}_i$  are, respectively, the right and left eigenvectors defined through the conditions:

$$[\mathbf{K} \bar{+} \mathbf{K}_r - \omega_i^2 (\mathbf{M} \bar{+} \mathbf{M}_r + (\Omega/\omega)\{\mathbf{G} \bar{+} \mathbf{G}_r\})]\mathbf{x}_i = 0; \quad \mathbf{y}_i^T [\mathbf{K} \bar{+} \mathbf{K}_r - \omega_i^2 (\mathbf{M} \bar{+} \mathbf{M}_r + (\Omega/\omega)\{\mathbf{G} \bar{+} \mathbf{G}_r\})] = 0. \quad (156)$$

Similarly, the perturbations of elements of  $\mathbf{x}_i$  and  $\mathbf{y}_i$  take the form

$$dx_{ki} = \sum_{j=1}^n \sum_{s=1}^n \frac{\partial x_{ki}}{\partial k_{js}} dk_{js} + \sum_{r=1}^n \sum_{s=1}^n \frac{\partial x_{ki}}{\partial m_{js}^*} dm_{js}^* \quad (157)$$

where the gradients with respect to stiffness and mass coefficients are available in terms of whirl speeds,  $\mathbf{x}_i$  and  $\mathbf{y}_i$ . The covariance structure of the eigensolutions has been obtained using the expressions of  $d\omega^2$  and  $d\mathbf{x}_i$ . The flutter of uncertain laminated plates using a perturbation stochastic finite element formulation was studied by Liaw and Yang (1993). They used a 48 dof rectangular plate element. The modulus of elasticity, mass density, thickness, fiber orientation of individual lamina, geometric imperfection of the entire plate and in-plane loads were treated as random variables. The aerodynamic pressure due to supersonic potential flow was modeled using quasi-steady first order piston theory. The governing equation in this case was of the form

$$\mathbf{M}\ddot{\mathbf{X}} + [\mathbf{K}_T + q\mathbf{D}]\mathbf{X} = 0, \quad (158)$$

where  $q$  is the aerodynamic pressure parameter,  $\mathbf{D}$  is the associated matrix to aerodynamic pressure,  $\mathbf{M}$  is the mass matrix, and  $\mathbf{K}_T$  is the tangential stiffness matrix which introduces nonlinearity into the problem. This leads to the eigenvalue problem

$$\left[ \alpha^2 \mathbf{M} + \mathbf{K}_T + q\mathbf{D} \right] \Delta \mathbf{X} = 0. \quad (159)$$

An iterative solution scheme was used to determine the critical aerodynamic pressure which subsequently led to the determination of flutter boundaries. The combined effects of parameter uncertainties in the modulus of elasticity, mass density, thickness, fiber orientation, geometric imperfection of the plate, and in-plane load ratio on the structural reliability boundaries were studied. Here reliability is defined as the probability of the critical aerodynamic pressure being greater than a specified aerodynamic pressure. It was found that the random compensation effects among the six parameters with zero correlation tended to increase the reliability. It was found that the random compensation effects among the six parameters with zero correlation tended to increase the reliability.

Iyengar and Manohar (1989) and Manohar and Iyengar (1993, 94) extended the work of Iyengar and Athreya (1975) and studied the free vibration characteristics of systems governed by a second order stochastic wave equation. They considered the eigenvalue problem

$$\frac{d}{dx} \left[ \{1 + \delta g(x)\} \frac{dy}{dx} \right] + \lambda^2 [1 + \epsilon f(x)] y = 0 \quad (160)$$

$$y(0) = 0; \quad y(1) = 0 \quad (161)$$

The solution of this stochastic boundary value problem is sought in terms of solutions of an associated inhomogeneous initial value problem which consists of finding the solution of equation(160) under the initial conditions at  $x = 0$  given by  $y^* = 0$  and  $\frac{dy^*}{dx} = 1$ . Denoting by  $Z_n(\lambda)$ , the  $n$ th zero of  $y^*(x, \lambda)$ , the eigenvalues of equation(160) can be defined as being the roots of the equation  $Z_n(\lambda) = 1$ . The study of  $Z_n(\lambda)$  is facilitated by the coordinate transformation

$$y(x) = r(x) \sin[\lambda x + \phi(x)] \quad (162)$$

$$[1 + \delta g(x)] \frac{dy}{dx} = r(x) \lambda \cos[\lambda x + \phi(x)]. \quad (163)$$

This leads to a pair of nonlinear coupled differential equations in  $r(x)$  and  $\phi(x)$ . The probability distribution of the eigenvalues  $\lambda_n$  is shown to be related to  $\phi(x)$  through the identity (Iyengar and Athreya 1975)

$$P[\lambda_n \leq \lambda] = P[Z_n(\lambda) \leq 1] = P[n\pi \leq \phi(1, \lambda)] \quad (164)$$

and similarly, the joint probability density function of the  $n$ th eigenvalue  $y_n(x)$  and  $n$ th eigenfunction was expressed in the form (Manohar and Iyengar 1994)

$$p_{y_n, \lambda_n}(y, x, \lambda) = \frac{p_{y^*, \phi | \lambda_n}[y, x; n\pi, 1 | \lambda_n = \lambda] p_{\lambda_n}(\lambda)}{p_\phi(n, \pi, 1, \lambda)}. \quad (165)$$

This would mean that probabilistic characterization of eigensolutions requires the solution of a pair of nonlinear stochastic equations in  $r(x)$  and  $\phi(x)$ . Extension of this formulation to consider other types of boundary conditions, including random boundary conditions, was presented by Manohar and Iyengar (1993). Exact solutions were shown to be possible only under special circumstances (Iyengar and Manohar, 1989, Manohar and Keane, 1993) and, consequently, approximations become necessary. For specific types of mass and stiffness variations, Iyengar and Manohar (1989) and Manohar and Iyengar (1993, 94) have developed solution strategies based on closure, discrete Markov chain approximation, stochastic averaging methods and Monte Carlo simulations. These combined schemes have been employed to estimate probability density functions of the eigensolutions. Based on the study of the distribution of zeros of random polynomials, Grigoriu (1992) examined the roots of characteristic polynomials of real symmetric random matrices. These roots identify the most likely values of eigenvalues and the average number of eigenvalues within a specified range. Brown and Ferri (1996) noted the cost effectiveness of component mode synthesis for Monte Carlo simulation of the dynamics of large scale structures. They treated the substructure dynamical properties as the primary random variables and combined the residual flexibility method of component mode synthesis with probabilistic methods. Adhikari (2000) has studied the problem of stochastic characterization of complex normal modes in random discrete structures. This author employs a first order perturbation procedure to achieve this goal. Gupta and Joshi (2001) study the analysis of seismic response of randomly parametered structures using perturbation and simulation methods. The analysis is presented in the framework of response spectrum based method and the structural responses are characterized in terms of covariance of the modal properties of the structure.

## 8.3 Forced Response

### 8.3.1 Linear Single-Degree-of-Freedom Systems

The behavior of simple uncertain systems can be studied in terms of damped single-degree-of-freedom systems subjected to stationary white noise excitation. Udawadia (1987a,b) studied the dynamical characteristics of linear sdof systems with random mass, stiffness and damping properties under free and forced vibration states. It was assumed that the probabilistic description of the system parameters is only partially available. Some results on response characteristics were obtained in closed forms. Wall (1987) evaluated the mean and variance of exceedance rate response of random sdof oscillators subjected to earthquake excitations with Kanai-Tajimi power spectra whose parameters were also assumed to be random. The statistics of the average number of level crossings, average number of maxima and departure from normality were computed by Kotulski and Sobczyk (1987) for the case of random oscillator under white and randomly filtered white noise inputs. Spencer and Elishakoff (1988) investigated the effect of system randomness on first passage failure of linear and nonlinear oscillators. They utilized a discretization of state space of the system random variables and subsequently solved the associated backward Kolmogorov equation using a finite element method to evaluate the first passage statistics.

Jensen and Iwan (1991) proposed an expansion technique to analyze random oscillators with random natural frequencies described by the equation

$$\ddot{x} + 2\eta(\bar{\omega} + \mu\omega_r)\dot{x} + (\bar{\omega} + \mu\omega_r)^2x = f(t) \quad x(0) = 0; \dot{x}(0) = 0 \quad (166)$$

where  $\eta$  is the damping ratio,  $\mu$  is a deterministic coefficient,  $\omega_r$  is a random variable, and  $f(t)$  is a random process in time  $t$ . The response  $x(t)$  was expressed in a series form

$$x(t, \omega_r) = \sum_{j=0}^n x_j(t) H_j(\omega_r) \quad (167)$$



where  $n$  is the order of approximation,  $x_j(t)$  is an unknown deterministic function of time and  $H_j(\omega_r)$  is a set of orthogonal polynomials. Substituting equation (167) into equation (166), then multiplying equation (166) by  $H_j(\omega_r)$  and using orthogonality and recursion relations satisfied by the polynomials yields an equivalent set of deterministic equations with external random excitations. Koyluoglu *et al.* (1995a) developed similar approach based on transforming the equation with random coefficients to one with deterministic coefficients and random initial conditions. Subsequently, the evolution of the probability density function of the extended response vector can be described by the well-known Liouville equation or by the Fokker-Planck equation. Thus when  $f(t)$  is modeled as a white noise process, the transitional pdf of the extended response vector would satisfy the Fokker Planck equation which, in turn, would enable the formulation of equations governing the response moments. However, the equation for  $m$ th moment gets coupled to the  $(m + 1)$ th moment which rules out exact solutions. Koyluoglu *et al.*, employed cumulant neglect closure scheme (Ibrahim, 1985) and solved for the first four moments. They obtained an approximate transient solution for a sdof system with random spring and damping coefficients subjected to a nonstationary modulated white noise process. The results were in a good agreement with the exact solution.

### 8.3.2 Linear Multi-Degree-of-Freedom Systems

The treatment of mdof systems is more involved than sdof systems. The starting point of this discussion is the matrix differential equation

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{F}(t); \quad \mathbf{X}(0) = \mathbf{X}_0; \quad \dot{\mathbf{X}}(0) = \dot{\mathbf{X}}_0 \quad (168)$$

where, at least, one of the matrices,  $\mathbf{M}$ ,  $\mathbf{C}$  or  $\mathbf{K}$ , is a function of a set of random variables. This equation results from the finite element and random field discretization of continuous structural models. The set of random variables entering the matrices  $\mathbf{M}$ ,  $\mathbf{C}$  and  $\mathbf{K}$  can be taken to be uncorrelated with zero means. The frequency domain representation of the above equation, when admissible, is given by

$$[-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]\mathbf{U}(\omega) = \mathbf{P}(\omega). \quad (169)$$

Singh and Lee (1993) used a direct product technique to estimate the statistical frequency response of a damped vibratory system. The solution procedure in the time domain consists of analyzing equation (168) using either modal expansion technique or direct numerical integration. In the frequency domain, it involves the inversion of the stochastic matrix  $\mathbf{H}(\omega) = [-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}]$ . Both approaches were studied in conjunction with other techniques such as perturbation, Neumann expansion, optimal series expansions, optimal linearization, and digital simulation methods. Lee and Singh (1994) assumed the amplitude of the excitation  $\mathbf{F}(t)$  to be randomly distributed which is multiplied by a time history process. This process can be taken as a deterministic function such as impulse or sinusoidal.

Different versions of perturbation formulations, including those based on the Taylor series expansion and sensitivity vector method have been used in the literature. These methods convert the given equation with stochastic coefficients into a sequence of deterministic equations. Other methods based on perturbations associated with a small parameter  $\epsilon \ll 1$  lead to a sequence of deterministic equations with deterministic operators and random right-hand sides.

Kleiber and Hien (1992) presented a systematic discussion of generalization of the principle of minimum potential energy and the Hamilton principle by including the effect of system stochasticity within the framework of mean-based, second moment, second-order perturbation techniques. The perturbation methods are applicable to a wide range of problems; however, they may be less accurate and suffer from lack of computational efficiency and convergence. This is true especially when the system is highly nonlinear, and when the parameters have skewed distributions with high levels of uncertainty. In addition, these methods lack invariance with

respect to the formulation of the problem (Igusa and Der Kiureghian 1988, Madsen *et al.*, 1986). Difficulties associated with these methods and their application, especially for transient dynamic problems are discussed in the literature; see, for example, Liu *et al.* (1992), Kleiber and Hien (1992) and Katafygiotis and Beck (1995). Remedial measures to overcome this limitation have also been suggested (Kleiber and Hien 1992). Chen *et al.* (1992) used a perturbation approach to assess the relative importance of uncertainty in excitations, geometrical, and material properties by considering examples of randomly driven truss and beam structures. Chang (1993) and Chang and Chang (1994) studied the transient response statistics and reliability of beams with stochastically varying elastic foundation modulus and Young's modulus. The dynamic response of an infinitely long, randomly damped beam resting on a random Winkler's foundation and excited by a moving force was examined by Fryba *et al.* (1993). They utilized a perturbational approach by introducing a new independent variable whose origin moves with the force.

The use of dynamic stiffness matrix in structural dynamics offers a few advantages particularly for skeletal structures (Manohar and Adhikari 1998). For instance, the method does not employ normal mode expansions: this not only avoids the need to compute the normal modes, but also does not introduce any modal truncation errors. This makes the method specially suitable for high frequency applications. Furthermore, the formulation permits greater flexibility in modeling structural damping. Motivated by these considerations, Manohar and Adhikari (1998), Adhikari and Manohar (1999,2000) and Gupta (2001a,b) have studied the use of dynamic stiffness matrices in analyzing randomly parametered skeletal structures. Frequency domain finite element procedures are used in deriving dynamic stiffness matrix for stochastic beam elements. The discretization of the beam displacements is based on the use frequency dependent shape functions. The study by Manohar and Adhikari (1998) provide the basic formulations which are then extended to skeletal structures (Adhikari and Manohar 1999) and transient problems (Adhikari and Manohar 2000). Methods to invert the structure dynamic stiffness matrix based on Neumann expansions and a novel eigenvector expansion scheme have been studied by these authors. In transient response analysis, extensive use is made of FFT algorithm to derive time domain description of the moments of the response. Figure (6) shows the time history of the mean and standard deviation of displacement response of a randomly parametered beam on a stochastic elastic foundation when it is traversed by a load moving at a constant velocity. The beam flexural rigidity, mass density and elastic foundation modulus are modeled as a vector of homogeneous random fields. The figure shows acceptable agreement between theoretical predictions and results from Monte Carlo simulations. Manohar and Bhattacharyya (1999) extended this work to quantitatively examine the influence of different non-Gaussian probabilistic models on the response of the system to harmonic and transient excitations. The response statistics of randomly inhomogeneous Timoshenko beams under stationary excitations were computed by Gupta and Manohar (2001b) based on the dynamic stiffness approach. The results were compared using analytical, combined analytical-simulation based methods and full scale simulations.

Methods based on orthogonal series expansions for both the system property random fields and response fields were used by Ghanem and Spanos (1990, 91a,b), Jensen and Iwan (1992) and Iwan and Jensen (1993). These methods usually lead to a set of algebraic equations. The size of these equations depends on the finite element discretization of the displacement field, the number of random variables entering the formulation, and the order of expansion used in representing the response field. The response power spectra of a beam mounted on a stochastic Winkler's foundation and subjected to a stationary Gaussian random excitation is computed using Karhunen-Loeve expansions. In their study, a spectral expansion of the nodal random variables is introduced involving a basis in the space of random variables. The basis consists of the polynomial chaos that are polynomials orthogonal with respect to the Gaussian probability measure; (it may be noted in this context that the term "polynomial chaos" denotes the fact that the polynomial in question is in terms of random variables and this terminology has no relation to the more modern usage of the term "chaos" as is used in nonlinear vibration literature). The results have been shown to compare favorably with Monte Carlo simulations. On the other hand, it has been illustrated that Neumann expansion based methods compared

poorly with simulation results, especially, at frequencies near the system natural frequencies. Computational aspects of SFEM that embody Karhunen-Loeve expansions, polynomial chaoses and Monte Carlo simulations have been discussed by Ghanem (1998,1999).

Jensen and Iwan (1992) considered the case when  $\mathbf{C}$  and  $\mathbf{K}$  are functions of a set of zero mean uncorrelated random variables  $\{\omega_{ri}\}_{i=1}^n$  and  $\mathbf{F}(t)$  to be a vector of nonstationary excitations. They studied the evolution of the nonstationary covariance matrix in time domain. This was achieved by expanding the response covariance matrix in terms of a set of known orthogonal multidimensional polynomials in  $\omega_{ri}$ . The resulting deterministic differential equations were integrated numerically. The procedure was illustrated by considering a five-degree-of-freedom system with uncertainty in stiffness/damping parameters and subjected to seismic base excitations. The influence of system uncertainty was shown to be significant in the analysis of tuning and interaction between primary-secondary modes and also in the reliability of secondary modes. Iwan and Jensen (1993) generalized the analysis to continuous stochastic systems. They obtained a set of deterministic ordinary differential equations in time which are integrated numerically. The method was illustrated by considering the response of a stochastic shear beam to seismic base excitation. It was observed that the variability in response was about half of the maximum mean, thereby indicating the importance of accounting for the system uncertainties in response calculations. Mahadevan and Mehta (1993) discussed matrix condensation techniques for stochastic finite element methods for reliability analysis of frames. They also computed the sensitivity of the response to the basic random variables by analytical differentiation as applied to the deterministic analysis.

Grigoriu (1991) developed an equivalent linearization approach to study the static equilibrium equation

$$\mathbf{K}(\omega_r)\mathbf{U} = \mathbf{S}(\omega_r) \quad (170)$$

where  $\omega_r$  is a vector of random variables. The displacement vector was described by the approximate expression  $\hat{\mathbf{U}} = \alpha\omega_r + \beta$ , where  $\alpha$  is a matrix and  $\beta$  is a vector with unknown deterministic elements which are determined by minimizing the error  $e = E\|\mathbf{S} - \mathbf{K}\hat{\mathbf{U}}\|^2$ . The process yielded a set of deterministic linear algebraic equations in the unknowns  $\alpha$  and  $\beta$ . The determination of these unknowns requires the knowledge of probabilistic description of  $\omega_r$  beyond the second moment. Thus, when  $K$  and  $S$  are linear functions of  $\omega_r$ , the determination of  $\alpha$  and  $\beta$  requires description of  $\omega_r$  up to the fourth order moments.

The application of stochastic averaging based techniques to problems involving the determination of dynamic stiffness coefficients of structural elements was studied by Manohar (1997). These systems are usually described by a stochastic wave equation. For example the field equation for the axial vibration of a non-homogeneous viscously damped rod element can be written as

$$\frac{d}{dx}[\{1 + \delta_1 f_1(x)\} \frac{dy}{dx} + i\beta_1\{1 + \delta_2 f_2(x)\} \frac{dy}{dx}] + \lambda^2[1 + \delta_3 f_3(x)]y - i\beta_2[1 + \delta_4 f_4(x)]y = 0 \quad (171)$$

for two sets of inhomogeneous boundary conditions

$$y(0) = y_0; \quad y(L) = y_L \quad (172)$$

and

$$\left. \frac{dy}{dx} \right|_{x=0} = \frac{P_1}{AE(0)}; \quad \left. \frac{dy}{dx} \right|_{x=L} = -\frac{P_2}{AE(L)} \quad (173)$$

where  $i^2 = -1$  which appears as a result of using complex algebra in solving the original partial differential equation of the rod.  $P_1$  and  $P_2$  are the end loads at  $x = 0$  and  $x = L$ , respectively. The functions  $f_1(x)$  and  $f_3(x)$  represent random components of the stiffness and mass, respectively,  $f_2(x)$  is the random component of the strain rate dependent viscous damping, and  $f_4(x)$  is the random component of the velocity dependent viscous damping. The parameters  $\beta_i$  and  $\lambda^2$  are functions of the system parameters and driving frequency.

It must be noted that the solutions of the above equations do not have Markovian properties even when the stochastic fields  $f_i(x)$  arise as filtered white noise processes. This is due to the fact that the solution trajectories have to satisfy boundary conditions at  $x = 0$  and  $x = L$ . The approach consists of expressing the solution of the above stochastic boundary value problem as a superposition of two basis solutions which are obtained by solving the field equation (171) under a pair of independent initial conditions. This subsequently enables the application of the Markov process-based approaches to construct the basis solutions. The solutions take into account the mean and power spectral density matrix of the system property random fields.

The problem of seismic wave amplification through stochastic soil layers was studied by Manohar (1997). Numerical results on the spectra of mean and standard deviation of the amplification factor were found to compare well with corresponding digital simulation results. Sobczyk *et al.* (1996) studied the harmonic response of undamped beams with stochastically varying inertial/elastic foundation moduli. They expressed the governing differential equation of motion by a random integral equation. The integral equation, in turn, was solved using the method of successive approximation. This method avoids the need to compute the stochastic free vibration analysis, and also permits estimation of the error of approximation.

In the analytical methods discussed so far, characterization of transmission of uncertainties from excitation to system parameter sources is accomplished by the use of approximate analytical procedures such as perturbations and Neumann series expansions. The accuracy of these methods can neither be easily estimated, nor improved. Moreover, these methods by themselves, have limited scope in so far as evaluation of probability of failure is concerned. Analytical methods used for inverting the structural operators, such as Neumann series expansions and other perturbation approaches, introduce errors into the solution due to the approximate nature of these solutions. The approximations that enter into the solution are because of series truncations and convergence problems. Improvements in the accuracy of the solution require the consideration of a large number of terms in the series, which makes the problem unwieldy. Stochastic finite element methods, combined with Monte Carlo simulations, offer a powerful alternative for response variability characterization and reliability analysis (Yamazaki and Shinozuka (1988), Adhikari and Manohar (1999), and Gupta and Manohar (2001b)). A comparison of spectrum of statistics of side sway in randomly parametered portal frame obtained using combined analytical-simulation method and full-fledged Monte Carlo simulations is shown in figure (7). The frame structure is taken to have beams with randomly inhomogeneous flexural rigidity and mass density. Yamazaki *et al.* (1988) showed that, within the framework of Monte Carlo simulations, the convergence criteria for the Neumann expansion can always be ensured irrespective of the magnitude of fluctuations, by expanding the Neumann series with reference to a scaled deterministic component of the random matrix. The scaling factor is determined from eigenvalue analysis and is a function of the maximum eigenvalue of  $\mathbf{K}^{-1}\Delta\mathbf{K}$ , where,  $\mathbf{K}$  is the deterministic part of the matrix to be inverted and  $\Delta\mathbf{K}$  is the random part. However, it must be noted that this scaling is possible for individual samples of random matrix and consequently, the procedure cannot be used within an analytical framework. Adhikari and Manohar (1999) and Manohar and Bhattacharyya (1999) digitally simulated samples of  $\Delta\mathbf{K}$ , and numerically inverted each sample matrix. Gupta and Manohar (2001b) directly simulated the random variables representing the discretized random field. The structure dynamic stiffness matrix was then formulated using these random variables and the response is obtained by numerically inverting the sample dynamic stiffness matrix.

In the context of statically determinate stochastic beams, Ren *et al.*, (1997) and Elishakoff and Ren (1999) have proposed a variational scheme that enables a direct treatment of mean and covariance of the response. Efforts to derive exact solutions on mean and covariance for static response of stochastic beams under randomly distributed loads have been reported by Elishakoff *et al.*, (1995,1999). Based on the survey of literature conducted by the present authors, it appears that there exist no discussion on exact solutions to vibration problems involving randomly parametered structures barring the case reported by Iyengar and Manohar (1989) on eigenvalues of stochastic strings and the study on frequency response of an axially vibrating stochastic rod

by Manohar and Keane (1993). Further work is clearly needed in exploring if exact solutions on response variability and pdf of randomly parametered continuum structures under static/dynamic loads can be derived.

In a recent study by Rahman and Rao (2001) ideas on using meshless methods for solving boundary value problems in linear elasto-statics that involve random material properties have been presented. The treatment of discretization of material random fields requires the introduction of nodes which need not co-incide with the meshless nodes associated with characterization of displacement fields. The authors outline a perturbational procedure to treat system randomness and to derive the mean and covariance of response quantities.

### 8.3.3 Nonlinear mdof uncertain Systems

The study of structural systems including the effects of system nonlinearity in the presence of parameter uncertainties presents serious challenges and difficulties to designers and reliability engineers. Recent developments in the mathematical theory of random processes and stochastic differential equations have promoted the study of response and stability in structural systems driven by random excitations (Ibrahim 1991,1995, Manohar 1995). The questions of structural nonlinearity are particularly important while addressing the problem of failures and safety assessments, especially, since the nonlinear response is, at times, radically different from the one obtained using a simplified linear model.

However, there is no unique theory that can be generalized to analyze any nonlinear system. Each method has its own limitation with respect to the nature of the excitation, the type of nonlinearity, and the number of degrees of freedom. Moreover, nonlinear modeling allows the designer to predict a wide range of complex response characteristics, such as multiple solutions, jump phenomena, internal resonance, on-off intermittency, and chaotic motion. These phenomena have direct effects on the reliability and safe operation of structural components. Accordingly, the designer must estimate the reliability of nonlinear systems subjected to Gaussian/non-Gaussian random excitations. In this case the engineer has to deal with both the catastrophic type and fatigue type failures. The former is related to the distribution of extreme values of the system response, and the latter is related to the crossing rates at different levels of the system response.

Socha and Soong (1991), Ibrahim (1991, 95) and Manohar (1995) presented overviews of methods, limitations, and experimental results of treating nonlinear systems under random excitations. Socha and Soong highlighted on the method of statistical and equivalent linearization and its applications to nonlinear systems subjected to stationary and nonstationary random excitations. Some controversies were reported regarding different results obtained by different methods for the same system. In such cases, experimental tests are valuable in providing complex phenomena not predicted by the available methods, and can provide guidelines to refine theory. Ibrahim (1995) reported a number of difficult issues and controversies encountered in the development of the nonlinear theory of random vibration. The analysis of nonlinear structural systems with parameter uncertainties is very limited to special cases such as static problems and numerical simulations.

Static problems involving system nonlinearities and stochasticity were studied by Liu *et al.*, (1986, 88). Liu *et al.* (1987) considered the dynamic response to a step input of an elasto-plastic beam with isotropic hardening whose plastic modulus was modeled as a Gaussian random field. Both displacement and random fields were discretized using 32 elements and, for this purpose, the same finite element shape functions were used. Furthermore, using orthogonalization of random variables, the 32 random variables were replaced by nine transformed random variables. Connor and Ellingwood (1988) considered response of randomly parametered hysteretic systems to an ensemble of recorded earthquake ground motions and investigated the statistics of damage related response quantities using simulation methods. A mean-centered second-order perturbation method in conjunction with direct numerical integration in time was used to study the time evolution of response moments. The transient response of a transversely loaded stochastically inhomogeneous plate on a

random nonlinear elastic foundation was studied by Deodatis and Shinozuka (1988). They used Monte Carlo simulation in conjunction with finite element discretization and time integration techniques. They examined the influence of the stochasticity of the elastic modulus and/or stochasticity of a nonlinear foundation, the support conditions and degree of nonlinearity of the foundation on the coefficient of variation of the maximum deflection. Furthermore, Gaussian and lognormal distributions were shown to provide good fits to the maximum plate deflection. A similar approach was considered by Brenner (1994) to study harmonic forced excitation of a three-dimensional skeletal model of a transmission line tower.

Chang and Yang (1991) considered large amplitude vibration of a beam with randomly varying material and geometric properties. They analyzed the free and forced response to harmonic and random excitations. Their analysis involved discretization of the random field using the method of local averages and a second order perturbation scheme. The forced nonlinear random vibration response was obtained using the equivalent linearization technique. Satisfactory comparisons of analytical results with Monte Carlo simulations were demonstrated. Liu and Der Kiureghian (1991) treat the reliability analysis of geometric nonlinear structures within the framework of FORM and SORM. The expressions for gradients of the nonlinear response with respect to the basic variables needed in the reliability analysis are newly developed. The paper presents the first applications of FORM and SORM reliability methods in conjunction with non-Gaussian random fields and with system reliability analysis. Tiegen *et al.*, (1991a,b) detail nonlinear finite element analysis of statically loaded concrete structures with stochastic parameter variations. The proposed formulation accounts for randomness in loads, material properties, and structural geometry and nonlinearities in material and geometry. The loads are applied gradually in increments till collapse occurs. The method is based on Taylor's expansion around an arbitrarily specified value of the basic variables. Haldar and Zhou (1992) employ advanced FOSM to estimate reliability of geometrically nonlinear structures. These authors employ a stress field finite element method. The numerical results show a comparison of reliability indices for linear structure, nonlinear structure and nonlinear structure with flexible connections. For stability related structures, the nonlinearity is shown to have a notable influence on structural reliability and the reliability will be very different when the flexible connections are taken into account. Studies on randomly excited uncertain nonlinear vibrating trusses have been reported by Chreng and Wen (1994a,b). Both large deflection and inelastic deformations are considered. Similarly, both random field and random variable models for material and loading are included. The response analysis is based on a combination of equivalent linearization and perturbations. Response statistics conditioned on system property random variables are first evaluated and the unconditional statistics are subsequently evaluated by using an integration scheme. Koyluoglu *et al.*, (1995c) employed weighted integral method of random field discretization in the development of a nonlinear stochastic finite element formulation for stochastic plane frame analysis. The stiffness and damping properties were taken to be random in nature, and the excitations were modeled as stationary random processes. They employed mean centered second order perturbation method to treat system uncertainties and the Gaussian closure method to handle the nonlinearities. Klosner *et al.* (1992) considered the response of nonlinear sdof and two-dof systems with stochastic parameters under white noise excitation. For more general classes of problems possessing no exact solutions, Klosner *et al.* employed a statistical linearization technique to determine the conditioned response statistics. It should be noted that statistical linearization gives satisfactory results only if the system does not involve secular terms which give rise to internal resonance conditions.

Brenner and Bucher (1995) considered a conglomerate of computational procedures for reliability analysis of large nonlinear mdof systems with uncertain properties and loads. The steps involved in their study consists of the following:

1. Use stochastic finite element method to discretize the displacement fields and the system property random fields. The random fields are substituted by their values at the integration points. The time varying random loads are discretized using Fourier series representation.

2. Perform sensitivity analysis and rank the random variables in order of their importance (see section 9.0). The performance functions considered by the authors included local failure criterion.
3. Fit a response surface to the relevant limit state function in terms of random variables that are deemed to be important.
4. Employ adaptive importance sampling to evaluate failure probability.

Thus, this study employs a wide ranging battery of tools that are currently available and deal with problem of reliability analysis of large randomly parametered nonlinear systems under random excitations. Specific application problem discussed by the authors include the study on a wing structure with 372 dofs. The results show that the estimated probability of failure with two most important random variables was  $4.08 \times 10^{-7}$  and with three random variables it was  $7.34 \times 10^{-7}$ . If the structural uncertainties were ignored, the failure probability was  $1.22 \times 10^{-8}$  which lead to the conclusion that the structural uncertainties indeed play a significant role in reliability analysis of the wing structure considered. Qiao *et al.*, (2001) have considered the dynamics of nonlinear beams with joint parameter uncertainties. Two models of joint stiffness uncertainties are considered. The first represents the uncertainty by a random variable and the second considers the relaxation process of the joint under dynamic loading. The source of excitation was taken to be through support motions. The governing equations were discretized using Galerkin's method and the resulting equations for the generalized co-ordinates were solved using Monte Carlo simulations. Under sinusoidal excitation, it is shown that the relaxation process of the joints may result in bifurcation of the response amplitude, when even all excitation parameters are fixed. Zhang and Ellingwood (1996) study reliability indices of skeletal structures with bi-linear stress-strain characteristics using stochastic FEM. The structures are taken to be loaded by spatially varying random loads. The spatial variability of material characteristics are observed to be significant only when scales of correlations are less than the lengths of structural members. Whether the inclusion of random spatial fluctuation increases or decreases the estimated reliability depends upon the limit state considered. Iwan and Huang (1996) studied nonlinear dynamic systems with polynomial form of nonlinear restoring forces that are driven by earthquake like excitations. These authors extend the earlier work of Jensen and Iwan (1991) and employ a series expansion in terms of a set of multi-dimensional orthogonal polynomials. Grundmann and Waubke (1996) considered nonlinear structures subjected to harmonic and random excitations. They employed Karhunen-Loeve expansions to represent the system randomness and used the method of equivalent linearization to obtain approximate solutions. An account of equivalent linearization analysis of randomly parametered inelastic framed structures with a view to compute aseismic structural reliability is provided in the CEB publication (CEB 1998). Manohar and Gupta (2000) investigated the steady state random response of stochastically parametered beams supported on nonlinear springs subjected to differential support motions. This study employed inversion of the random dynamic stiffness of the structure after linearizing the system using equivalent linearization. Imai and Frangopol (2000a,b,c) describe computational details of combining FORM/SORM and nonlinear FEM based on total Lagrangian formulation for the analysis of reliability of geometrically nonlinear static structures. Detailed examples on system reliability of a nonlinear truss are presented. Studies on static behavior of randomly parametered skeletal structures with inelastic members have been carried out by Papadopoulos and Papadrakakis (1998) by using Monte Carlo simulations in conjunction with weighted integral method of random field discretization. Studies on nonlinear analysis of deformation of stochastically parametered earthen embankments during construction process have been reported by Mellah *et al.*, (2000). These authors also report on practical difficulties in identifying characteristics of the nonlinear parameters of the constitutive laws.

## 9.0 Probabilistic model reduction using importance measures

The computational effort expended in reliability assessment using FORM, SORM, RSM or simulation techniques increase with increases in number of random variables that enter the reliability computation. Thus, it is

important to adopt the most parsimonious representation of randomness in load and structural uncertainties as is possible. Towards achieving this end, it is essential that measures of importance associated with each of the candidate random variables entering the reliability calculations be established. Studies on sensitivity of reliability index with respect to parameters of the distribution and parameters of the performance function have been conducted by Hohenbichler (Madsen *et al.*, 1986). Similar studies on sensitivity of the direction cosines  $\alpha_i$  of limit surface evaluated at the check point have been carried out by Bjerager and Krenk (1989). The asymptotic behavior of these gradients as  $\beta \rightarrow \infty$  was also discussed. Discussion on calculations of sensitivities in the context of FORM and SORM is also available in the paper by Karamchandani and Cornell (1992). These results are clearly of value in determining the importance of distribution parameters in reliability assessment. However in the context of model development or reliability calculations, these quantities in themselves are of little use.

The concept of omission sensitivity factor to characterize the relative error in the reliability index when a basic variable is replaced by a deterministic number was introduced by Madsen (1988). He noted that the gradient of the reliability index with respect to a design variable  $u_i$  in standard normal space is given by

$$\frac{\partial \beta}{\partial u_i} \Big|_u^* = \alpha_i \quad (174)$$

where

$$\alpha_i = - \frac{\frac{\partial G}{\partial u_i}}{\sqrt{[\sum_{i=1}^n \frac{\partial G}{\partial u_i}]^2}} \quad (175)$$

are the direction cosines such that  $\sum_{i=1}^n \alpha_i^2 = 1$ . Here,  $G(u)$  is the performance function defined in terms of the standard normal variables. Thus  $\alpha_i$  denotes the measure of the sensitivity of the reliability index to inaccuracies in the value of  $u_i$  at the design point. He defined the omission sensitivity factor  $\gamma_i(z_i^f)$  for a basic random variable  $X_i$  as the inverse ratio between the value of the first-order reliability index and the first order reliability index with  $X_i$  replaced by a deterministic value. When  $X$  has non-Gaussian distribution, he showed that

$$\gamma_j(x_j^f) = \frac{1 + [(J^{-1})\alpha]_j [(x^* - Ju^*)_j - x_j^f]/\beta}{\sqrt{1 - \alpha^T J^{-1} (JJ^T)^j (J^{-1})^T \alpha}} \quad (176)$$

where  $J$  is the Jacobian matrix with  $J_{ij} = \frac{\partial x_j}{\partial u_i}$ . It was noted that if the value  $\alpha_i$  in absolute value is less than 0.14, then the relative error of the reliability index is less than one percent if  $X_j$  is replaced by its median. It was suggested that to reduce the computation work involved in FORM only the basic variables with large  $\alpha$ -values should be retained in the iterations. These variables are defined after the first step. At each further step in the iteration a variable  $U_j$  with  $|\alpha_j| < \alpha_t$  is replaced by the value  $\beta_m \alpha_{j,1}/2$ . Here,  $\beta_m$  = estimate of  $\beta$  in the  $m$ th iteration and  $\alpha_t$  is the threshold value that is decided based on accuracy requirements. The value of  $\beta$  is updated at each step, but the  $\alpha$  value is taken from the first iteration. This choice for the deterministic values is selected to minimize the error in the reliability index as the omission sensitivity factor for this choice is given by

$$\gamma_j\left(\frac{\beta \alpha_j}{2}\right) = \frac{1 - \alpha_j^2/2}{\sqrt{1 - \alpha_j^2}} = 1 + O(\alpha_j^4) \quad (177)$$

The calculations are performed with the reduced  $u$  vector and after satisfactory convergence is achieved in this reduced space, a final iteration step with full  $u$  vector is recommended. Further discussion on omission sensitivity factors with details of importance measures related to input parameters in series/parallel system reliability is presented in the book by Ditlevsen and Madsen (1996).



In their study on SFEM based reliability assessment of nonlinear structures under random dynamic loads, Brenner and Bucher(1995) propose a framework to omit the unimportant random variables in reliability calculations. These authors begin by discretizing the excitation random process using a Fourier representation and discretize the system property random fields by replacing them by their values at the integration points. The basic set of random variables thus arising are further grouped into three categories, namely, load random variables, structure random variables which influence the structural behavior only in its linear domain and structural random variables which come into play when nonlinear structural behavior is modeled. The importance of load random variables were determined by performing a linear random vibration analysis with all the structural parameters fixed at their deterministic values. The random variables associated with frequencies that produced higher stresses were deemed as being important. To assess the relative importance of the linear structural random variables, simulation methods were employed. The simulation is restricted to linear behavior of the structure under the applied random loads. Here, to derive the importance of a basic variable  $x_j$ , first a simulation run with all structure parameters fixed at their deterministic values is carried out. This is followed by another run in which only the  $i$ th variable  $X_i$  is fixed at mean plus one standard deviation. The difference in the magnitudes of the response produced during these two runs provides a measure of importance of the variable  $X_i$ . To assess the importance of the basic nonlinear variable  $x_{nj}$ , similar simulation runs are made on nonlinear static behavior of the structure. These three sets of calculations enable ranking of all the basic variables in order of their importance thereby providing a means to eliminate the less important variables in subsequent calculations. A notable feature of this study is that the importance measures are arrived at basically by computing the sensitivities at the mean of the basic variables. This is questionable since there is no way of knowing if the ranking of the random variables remains valid if the sensitivities were to be computed near the failure point. Furthermore, interaction amongst random variables from different groups would introduce an unknown error into the calculations of importance measures. Also relative ranking amongst random variables from different groups remains an unresolved issue. Notwithstanding these limitations, the approach can be viewed as a heuristic approach based on engineering intuitions.

## 10.0 Critical excitation models, convex models of uncertainty and robust reliability

A basic assumption that is made in structural response analysis is that adequate probabilistic information on the inputs and the structural parameters is available so that a desired response analysis can be carried out. In practical situations, however, this basic assumption may not be suitably satisfied. Thus, information on either the input or the structural parameters could only be partially available which, in itself, would not be adequate to carry out the desired response/reliability analysis. It is relevant in such situations to ask the following questions:

1. Consistent with the partial information available on inputs/system parameters, a class of admissible inputs/system parameters could be defined. Which member amongst this class of inputs/system parameters produce the least favorable response?
2. How large can be the uncertainty in input/structural parameters so that the structure continues to perform satisfactorily?

Problems of this type can be thought of as constituting a class of inverse problems in structural response modeling. These problems are of significant interest to structural analyst, since, answers to the above questions helps one to envisage the worst case scenarios.

### 10.1 Critical excitation models

Questions of the first type have been considered by engineers in the field of electrical circuit theory where optimal wave forms subject to constraints on total energy and/or peak values and which produce maximum

response in a linear system have been determined (Tufts and Shnidman 1964, Papoulis 1964,1970). In the context of earthquake engineering problems, the study of such problems was initiated by Drenick (1970) who termed the optimal inputs as critical excitations. For linear sdof systems he applied the Schwarz inequality on the Duhamel integral to show that among all inputs having the same total energy, the one which produces the highest absolute response is proportional to the time reversed impulse response function. This view point of modeling seismic inputs offers a counter point to the traditional load modeling using response spectrum approaches.

The subsequent research in this area has been mainly directed towards evolving this method as an alternative aseismic design philosophy. The critical excitations as obtained by Drenick had a deterministic structure and a frequency content that hardly compared with earthquake signals. Thus, additional constraints on the inputs, reflecting properties of earthquake accelerograms, such as, frequency content and nonstationary trend, were incorporated into the analysis (Shinozuka 1970 and Iyengar 1970). Drenick (1973) proposed that the class of allowable inputs should include all the earthquake accelerograms that have been previously recorded at the given site or on other geologically similar sites. The set of accelerograms was termed as basis earthquakes. Based on this idea, the concept of subcritical excitations has been developed by Wang *et al.*,(1978). Here a possible future earthquake is taken as a linear combination of basis accelerograms and the coefficients are determined by minimizing the mean square difference between critical excitation of the structure and the assumed earthquake. This method has been employed in seismic safety assessment of high rise buildings (Wang *et al.*, 1978) and chimneys and nuclear reactor vessels (Abdelrahman *et al.* 1979). Later Drenick and Yun (1979) have determined the unknown coefficients in the above expansion by directly maximizing the absolute response. In this approach the future earthquake acceleration  $a(t)$  is taken to be of the form

$$a(t) = \sum_{i=1}^N c_i a_i(t) \quad (178)$$

where  $\{a_i(t)\}_{i=1}^N$  are the set of basis accelerograms and  $\{c_i\}_{i=1}^N$  are the undetermined constants that are optimized so that a given structural response variable is maximized under the constraint on total intensity. Wang and Yun (1979) used this technique and proposed critical design spectra valid for different site conditions. Additional constraints on the input in terms of the peak velocity and acceleration have been imposed by Drenick *et al.*,(1984) and the critical excitations are determined by using nonlinear programming methods. Bedrosian *et al.*,(1980) have examined the practical applicability of the method in terms of conservatism and comparison with other standard methods of seismic analysis. The above method has been extended to take into account inelastic response by Phillippacopoulos and Wang (1984) and Pirasteh *et al.*,(1988). The idea of representing a future earthquake in terms of a linear combination of past recorded accelerograms raises serious mathematical questions on convergence and completeness of the representation. Recently Baratta *et al.*,(1998) re-visited this problem and have proposed that the future earthquake be expanded in the form

$$a(t) = \sum_{i=1}^N c_i u_i(t) \quad (179)$$

Here  $\{u_i(t)\}_{i=1}^N$  are a set of deterministic orthogonal functions such as sinusoidal functions. The ground motion is taken to satisfy the constraint

$$\int_0^T a^2(t)dt \leq E_o^2 \quad (180)$$

where  $E_o$  is a specified measure of intensity and  $T$  is the duration of the ground acceleration. An additional condition that the Fourier amplitude of spectrum of  $a(t)$  does not deviate more than a specified amount from a central spectrum is also considered. This leads to the additional constraint

$$\sum_{i=1}^N (c_i - c_{io})^2 \leq \theta^2 \quad (181)$$

where  $c_{oi}$  = coefficients of central spectra and  $\theta$  is the specified bound of variation. The undetermined coefficients  $c_i$ , ( $i = 1, \dots, N$ ) are now determined so that a given response variable of a specified structure is maximized subject to constraints given by equations (180 and 181). Abbas and Manohar (2001) represent the critical input in the form

$$a(t) = A_0[\exp(-\alpha t) - \exp(-\beta t)] \sum_{i=1}^n \{A_i \sin \omega_i t + B_i \cos \omega_i t\} \quad (182)$$

where the quantity  $A_0[\exp(-\alpha t) - \exp(-\beta t)]$  represents a known enveloping function that imparts the requisite transient nature to the input, and  $\{A_i, B_i\}_{i=1}^n$  are the undetermined coefficients. From the study of recorded earthquake accelerograms from geological sites that are similar to the site of the structure under consideration, the following set of constraints are established.

$$\begin{aligned} E_1 &= \max_{0 < t < \infty} |a(t)| \\ E_2 &= \max_{0 < t < \infty} |v(t)| \\ E_3 &= \max_{0 < t < \infty} |d(t)| \\ E_4(\omega) &\leq |A(\omega)| \leq E_5(\omega) \end{aligned} \quad (183)$$

Here,  $v(t)$  = ground velocity,  $d(t)$  = ground displacement,  $A(\omega)$  = Fourier transform of  $a(t)$  and the quantities  $E_i$  ( $i=1,5$ ) are the bounds. The authors determined the optimal values of the undetermined coefficients  $\{A_i, B_i\}_{i=1}^n$  that maximized a chosen response variable of a given structure subject to the constraints listed in equation (183). Their study showed that the constraint on lower bound of the Fourier amplitude spectra was crucial in arriving at realistic critical excitation models.

On a mathematical footing, the problem of finding critical excitations for nonlinear systems has been tackled by a few authors. Iyengar (1972) considered inputs having known total energy and obtained critical excitations for a class of nonlinear systems in terms of impulse response of corresponding linear system. He also treated the input total energy as random variable and obtained the worst possible distribution of the critical response. Drenick and Park (1975) pointed out that the procedure used in the above analysis enforces additional constraints on the input involving the system response. By linearizing the given nonlinear equation around the critical excitation-response pair, Drenick (1977) has obtained critical excitations in terms of impulse response of linearized equations. Westermo (1985) has defined critical response in terms of input energy to the system and has found critical excitations for linear elasto-plastic and hysteretic single degree of freedom systems using calculus of variations. The equation governing the critical excitation is similar to the original inhomogeneous equation and the excitations are obtained as the periodic solutions of this equation. A discussion on the use of critical excitation as an alternative to shock spectra in the context of vibration testing of inelastic structures has been discussed by Chang *et al.*, (1985).

Characterization of critical excitations as a Gaussian random process has been proposed by Iyengar and Manohar (1987). These authors have developed critical power spectral density models for earthquake inputs which maximize variance of a given system under a constraint on the meansquare value of the input. The formulation is based on a series representation for the square root of the critical power spectral density function and the solution is obtained by solving an algebraic eigenvalue problem. Iyengar (1989) and Varadharajan (1992) have studied this approach further by incorporating an additional constraint on bandwidth of the excitation. Srinivasan *et al.*, (1992) have modeled the critical seismic excitation as a nonstationary filtered shot noise process and the parameters of the model are optimized to yield the highest meansquare response of a given structure under a specified energy constraint. Manohar and Sarkar (1995) reconsidered the problem

studied by Iyengar and Manohar (1987) and employed linear programming methods to compute the critical psd models. The use of linear programming methods enabled the imposition of additional constraints involving zero crossing rates. These authors also proposed the use of entropy rate of the seismic time histories as a means to impart requisite levels of disorder into the critical excitations. Towards this end, methods on calculus of variations and linear programming methods were developed and applied to seismic response analysis of nuclear containment structures and an earth dam structure. The study by Abbas and Manohar (2001) demonstrates how the entropy rate could be estimated from recorded ground motions and how this further could be included in computing nonstationary random critical excitation models. Takewaki (2000) considered the problem of design of structures for critical excitations. He defined the sum of the mean square inter-storey drifts as the response variable to be maximized and analyzed the problem of optimal placement of dampers. He set the input variance and also the highest value of the input psd function as constraints that the critical psd function need to satisfy. In a subsequent study the same author (Takewaki 2001) extended the formulations to include the nonstationary nature of the inputs.

In the context of multi-component and multi-support earthquake load modeling, the concept of critical excitations has been introduced by Sarkar and Manohar (1996,1998). These authors considered the earthquake loads as a vector of stationary Gaussian random processes. The complete description of these loads is possible in terms of the psd matrix. These authors considered a family of critical psd matrix models covering the following cases:

1. The knowledge of the inputs is limited to the variance and zero crossing rates of the individual components.
2. The auto-psd functions are known while the cross psd functions are partially known. Depending on the nature of the knowledge available on the cross psd functions, further cases arise:
  - (a) Phase spectra associated with the cross psd functions are known while the amplitude spectra are unknowns.
  - (b) Both phase and amplitude spectra are unknowns.

Sarkar and Manohar outlined the different contexts in which these cases become relevant and also have illustrated the application of the models to land based and secondary structures. Their study showed that the highest response is produced by excitations that are neither fully correlated nor mutually independent but by excitations that have specific cross-correlation properties which are dependent on the system characteristics. This point is illustrated in figure (8) wherein the critical response psd function in a multi-supported piping structure subjected to differential random support motions is shown. These critical cross-correlations were determined by a mathematically exact procedure. Ravi (1997) and Srinivas (1999) extended the study by Sarkar and Manohar to cases of nonlinear multi-supported structures. They employed the method of equivalent linearization and developed approximations to the critical cross psd functions. Srinivas (1999) also extended the formulations to include non-stationary nature of the earthquake inputs.

## 10.2 Convex models for uncertainty and Robust Reliability

Alternative definitions of structural reliability, within the framework of non-probabilistic models for uncertainties, have been investigated by Ben Haim and Elishakoff (1990), Elishakoff (1995a,98) and Ben Haim (1996,97,98). In these studies it is assumed that the information available on uncertain quantities is fragmentary in nature and this permits the specification of only certain bounds on the variability. Depending on the choice of these bounds a range of convex models of uncertainty gets defined. Thus, for a given uncertain

vector of functions  $\mathbf{u}(t)$  a cumulative energy bound model consists of family of functions

$$\mathcal{U}(\alpha) = \{\mathbf{u}(t) : \int_0^T \mathbf{u}^T(t) \mathbf{W} \mathbf{u}(t) dt \leq \alpha\} \quad (184)$$

Here  $\mathbf{W}$  is a given positive definite matrix. Similarly an instantaneous energy bound model consists of

$$\mathcal{U}(\alpha) = \{\mathbf{u}(t) : \mathbf{u}^T(t) \mathbf{W} \mathbf{u}(t) \leq \alpha\} \quad (185)$$

An envelope bound model for an uncertain function  $u(t)$  on the other hand reads

$$\mathcal{E}(u_1, u_2) = \{u(t) : u_1(t) \leq u(t) \leq u_2(t)\} \quad (186)$$

For a vector of uncertain variables, denoted by  $\mathbf{v}$ , one could define an ellipsoidal bound given by

$$\mathcal{V}(\alpha) = \{\mathbf{v} : (\mathbf{v} - \bar{\mathbf{v}})^T \mathbf{W} (\mathbf{v} - \bar{\mathbf{v}}) \leq \alpha\} \quad (187)$$

The reliability analysis aims to determine the greatest value of  $\alpha$  that is consistent with no failure of the system. This quantity is taken to be the measure of reliability of the system. A reliable system will perform satisfactorily in presence of great uncertainty. Such a system is termed as being robust. The determination of the robust reliability of a system consists of three components:

1. a mechanical model describing the physical properties of the system,
2. a failure criterion, specifying the conditions which constitute failure of the system, and
3. an uncertainty model, quantifying the uncertainties to which the system is subjected.

A wide range of structural problems have been studied by Ben Haim and Elishakoff using these concepts. These include analysis of statically loaded structures, effect of geometric imperfections in axially loaded shells, analysis of fatigue failure and analysis of vibrating systems. The use of convex models for uncertainty in design problems has been explored by Pantelides and Ganzerli (1998) and Pantelides and Tzan (1996) and Tzan and Pantelides (1996). The recent study by Zingales and Elishakoff (2000) explore the relationship between probabilistic and non-probabilistic models of uncertainties in the context of dynamics of an axially loaded beam structure.

## 11.0 Discussion and future directions

An overview of wide ranging issues related to modeling and evaluation of structural reliability has been presented in this paper. Both analytical and simulation methods, as applied to time variant/time invariant reliability analysis of components/structural systems, have been considered. Mathematical issues underlying modeling of structural parameters and loads have been covered. The range of structural behavior discussed encompass static and dynamic behaviors. The choice of topics covered in this paper is biased by the authors' current interest in the subject: many important topics such as reliability based design, code calibration, reliability based structural optimization, reliability analysis of existing structures and structural behavior such as fracture, fatigue and deformation control are not included in this review. Also not included are issues related to alternative modeling of structural and load uncertainties using non-probabilistic methods. The focus of the review has been on methods of reliability analysis rather than on specific applications.

The first order and second order reliability methods are now well developed and have the capability to handle nonlinear performance functions and mutually dependent non-Gaussian vector random variables. These approaches convert the structural reliability problems into problems of constrained nonlinear optimization

for which wide ranging tools are available in mathematical programming literature. These methods generally require an explicit definition of performance function and an ability to compute its gradient vector. A major upshot of first order reliability method is the definition of reliability index which side steps the difficult problem of evaluating a multidimensional integral over a possibly irregular domain. The FORM in standard normal space offers simple geometric interpretations that is appealing to engineers. However, transformation to standard Gaussian space is not mandatory for the application of FORM provided, one forgoes the benefit of geometric interpretations of reliability index and check point. Efforts have been made in the recent past to cope up with the problem of presence of multiple critical points in FORM applications. However, there still exists scope for further development in computational tools that deal with this issue. The FORM and SORM, as they stand, can handle linear/nonlinear static behavior structures. Recent research has revealed their potential for extension to linear/nonlinear random vibration problems (Der Kiureghian, 2000). The second order reliability methods have strong theoretical underpinnings by virtue of their ability to asymptotically lead to exact probability of failure for large safe domains. Methods to deal with non-asymptotic cases have been developed; here, again, further developments are possible.

An alternative to analytical procedures, which is becoming increasingly attractive in recent times, has been the simulation methods. The developments in these methods have been spurred by advances in simulation methods of Gaussian/non-Gaussian, scalar/vector, random variables/random fields, and the increased availability of cheap and fast computers. Methods for treating partially specified non-Gaussian variates have also been developed. A perceived weakness of these methods could be that these are computationally intensive, especially, when dealing with simulation of rare events, such as structural failures. While this objection is apparently justifiable, however, it no longer remains tenable given the development of intelligent simulation procedures, such as, the various versions of importance sampling methods and other variance reduction techniques. Many of these methods build upon the results from FORM and SORM.

Whereas the initial progress in structural reliability studies were focused on single performance functions associated with individual structural components, more recent developments have recognized the fact that characterization of structural failure requires the study of simultaneous violation of several limit state functions. Accordingly, several techniques for modeling structural system failures have been developed. While many of these methods are inspired from system reliability research in other branches of engineering, the methods, however, have been modified to take into account peculiarities of structural failure such as, post failure behavior of individual components, redistribution of load carrying capacities subsequent to individual member failures and load path dependencies. Methods such as failure mode enumeration and  $\beta$ -unzipping are built upon basic ideas from FORM/SORM, as applied to individual structural components. Another source of complexity in structural reliability analysis pertains to time dependency of load and/or resistance variables. Here, problems involving linear mechanical systems and Gaussian loads are amenable for analytical treatment. A wide ranging array of results that employ Markov based approach and outcrossing theory are available. While the areas of system reliability and time varying reliability have witnessed development of several effective procedures, the authors feel that there exists scope for further work that combine recent advances in nonlinear finite element procedures with reliability analysis, especially, of structures having continuum elements.

The emergence of stochastic finite element methods and response surface methodology constitute major developments in structural reliability modeling. The stochastic finite element method enables the treatment of spatially varying stochastic structural inhomogeneities. This is of fundamental importance, for example, in treatment of geotechnical problems, in problems of elastic stability and structural problems involving high frequency vibrations. At present the best strategy to handle this class of problems consists of combining random field discretization schemes, such as optimum linear expansion method and Karhunen Loeve expansions, with intelligent simulation methods, such as, importance sampling strategies. The response surface method offers a

vehicle to efficiently combine the powerful finite element methods with reliability related problems. Methods have also been developed to rationally reduce the size of the probabilistic models using importance measures. The study by Brenner and Bucher (1995) demonstrate how some of these developments in reliability analysis could be combined to tackle realistic engineering problems.

In summary, the authors feel that the major developments in recent years in the field of structural reliability are the following:

- Placing of FORM/SORM on sound theoretical footing by relating them to asymptotic reliability concepts. Formulation of these methods either in standard normal space or in the space of the original non-Gaussian variables.
- Development of methods for describing, simulating and discretizing non-Gaussian random fields. Development of frameworks to deal with partially specified multi-dimensional non-Gaussian variates.
- Generalization of powerful deterministic procedures such as FEM to deal with distributed structural uncertainties.
- Emergence of effective simulation tools such as importance sampling procedures.
- Development of response surface methods which have potential to model reliability of very complex systems.

While the subject of structural reliability has seen these developments, several issues still remain open to debate. A recent opinion survey by Elishakoff (2000) on possible limitations of probabilistic methods provides a succinct summary of views of several researchers in this area. One of the major problems, where currently no solutions appear to be available pertain to lack of data to make probabilistic models, especially, in the tail regions of the probability distributions and in characterizing joint pdf of the vector random variates. Some other problems, such as models for human error and accidents, and lack of usable approaches in defining acceptable levels of failure probability, could be overcome by further research, see for example the recent book by Nathwani *et al.* (1997).

In the perspective of the authors, the following problems require further research attention:

1. The characterization of stress and strain tensors obviously is a fundamental problem in solid mechanics. Notwithstanding this, there are not many studies that discuss probabilistic modeling of stress and strain tensors. Thus, for example, no results seem to be available on the characterization of principle values of stress and strain tensors in random vibration analysis. A few studies in this line are those by Segalman *et al.* (2000a,b) and a brief discussion in the book by Madsen *et al.*,(1986). There exists scope for further research in arriving at descriptors of stress and strains, especially, those suited for study of structural failures in vibrating environment.
2. A central problem in system reliability studies lies in identification of dominant failure modes. This difficulty is accentuated when dealing with dynamics of inelastic structures as is encountered in earthquake engineering problems. An alternative way of describing structural system reliability, especially, with reference to seismic safety of structures, would be to define reliability with respect to acceptable levels of global damage as characterized by several indices that are currently available (Williams and Sexsmith 1995, CEB 1998). There appears to be much potential for further development.
3. The method of critical excitations provides an attractive alternative to earthquake load modeling. Most of the currently available procedures do not take into account reliability based concepts in defining critical excitations, nor do many studies allow for nonlinear mechanical models. Further work is warranted to remedy this situation.

4. In treatment of structural parameter uncertainties within the framework of finite element procedures, several problems pertaining to algebra of random matrices are encountered. Thus, several methods exist to deal with matrix inversion, eigenvalue analysis and matrix products. In dealing with large scale finite element models, often reduction schemes require characterization of pseudo inverses and singular value decompositions of random structural matrices. Currently, there exist no studies that deal with these problems. Furthermore, there exists scope for developing model reduction strategies from a probabilistic point of view.
5. Robust reliability methods, as proposed by Ben Haim and Elishakoff, provide alternative tools for modeling structural uncertainties. An attempt to discover common logical threads in alternative models of uncertainty has been reported recently by Langley (2000). He has used probabilistic models, models using interval analyses, convex models and fuzzy set theory as alternative settings to study structural safety. He has demonstrated the possibility of deriving common mathematical algorithms to study structural reliability using these alternative rules. These aspects require further research especially in context of reliability based structural optimization and in treatment of mechanical nonlinear models.
6. Data to generate probabilistic models for loads and structural resistances are the lifelines of probabilistic methods of reliability analysis. Given the variations across the regions of the world in construction materials and practices and external environments that generate structural loads, no universal probabilistic models/model parameters for these quantities are feasible. Thus, from the Indian national perspective, wide spread efforts are needed in arriving at probabilistic models for loads, such as, for example, earthquake loads in the inter-plate regions of Himalaya and intra-plate regions of Peninsular India, and guideway unevenness in roads, railways and bridge decks, and properties of construction materials. While efforts are being made in India to achieve this objective, still there is a need for large scale concerted efforts in data gathering and processing.
7. There exists a large body of work in computer systems reliability modeling, importance sampling, large deviations theory and statistical design tolerance in the field of computer networking and telecommunications, which appear to be strongly relevant to the area of structural reliability analyses. Establishing this relevance requires a close look at delineating features common to computer system reliability modeling and structural reliability analyses. Furthermore, developments in the field of computational complexity (see, for example, the book by Papadimitrou, 1994) could provide important insights into the relative merits of the various algorithms that are currently available for structural reliability analyses. Another area of structural reliability research, which to the best of the authors' knowledge seems to have received little attention, is the assessment of structural analysis software reliability.

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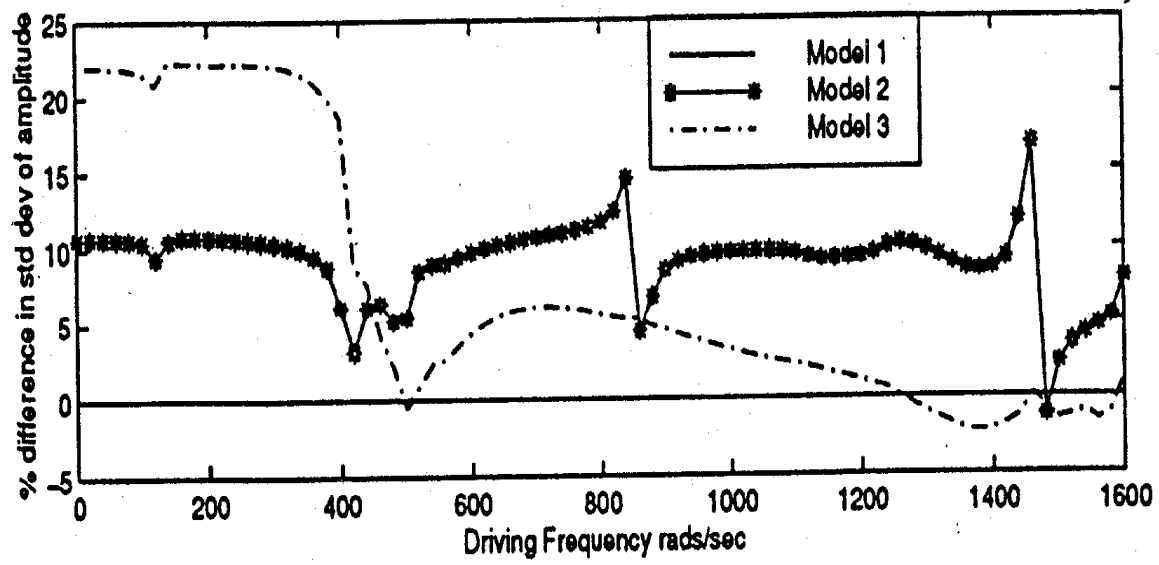
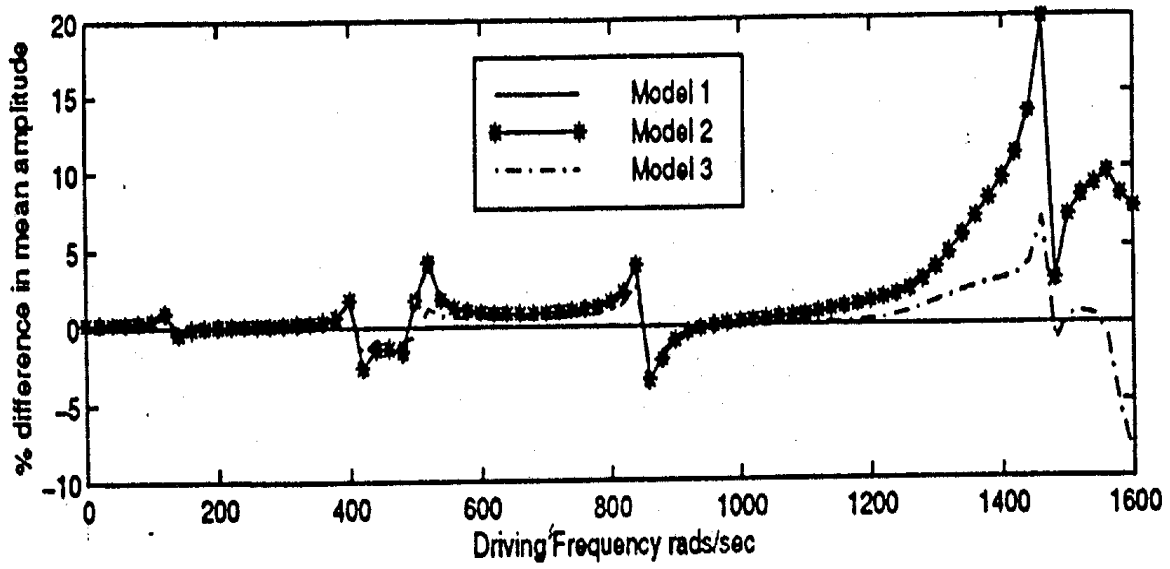


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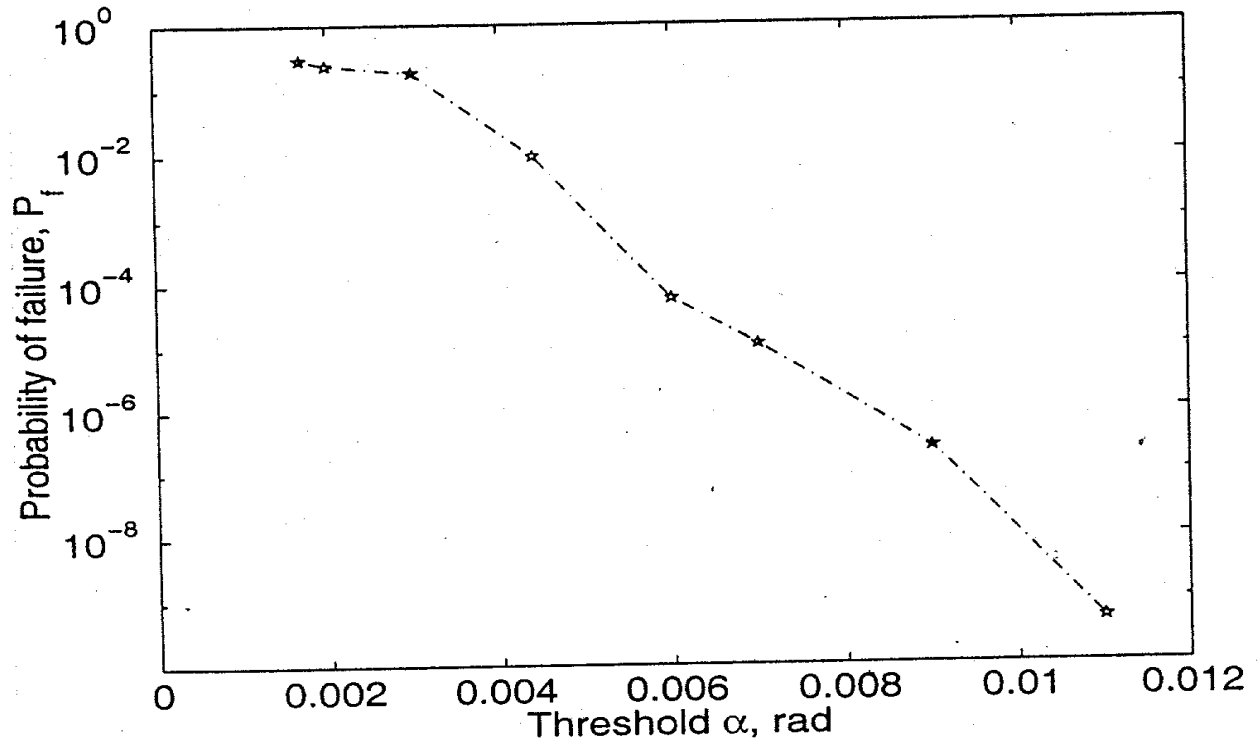


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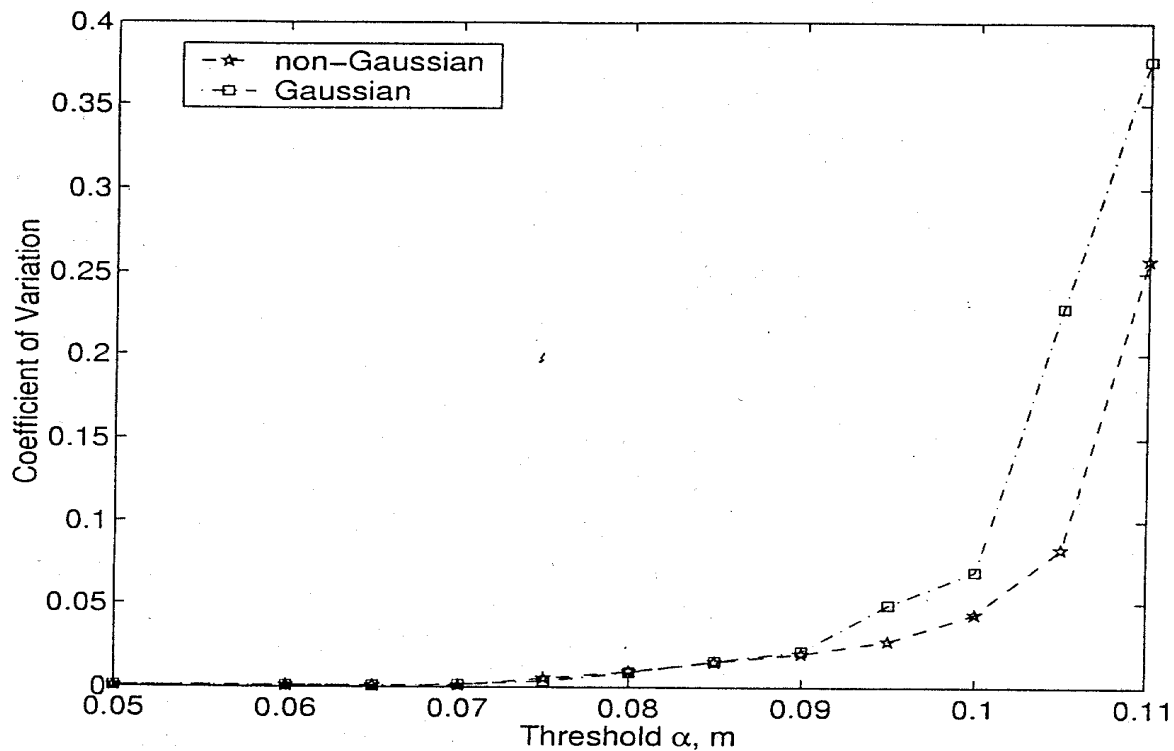


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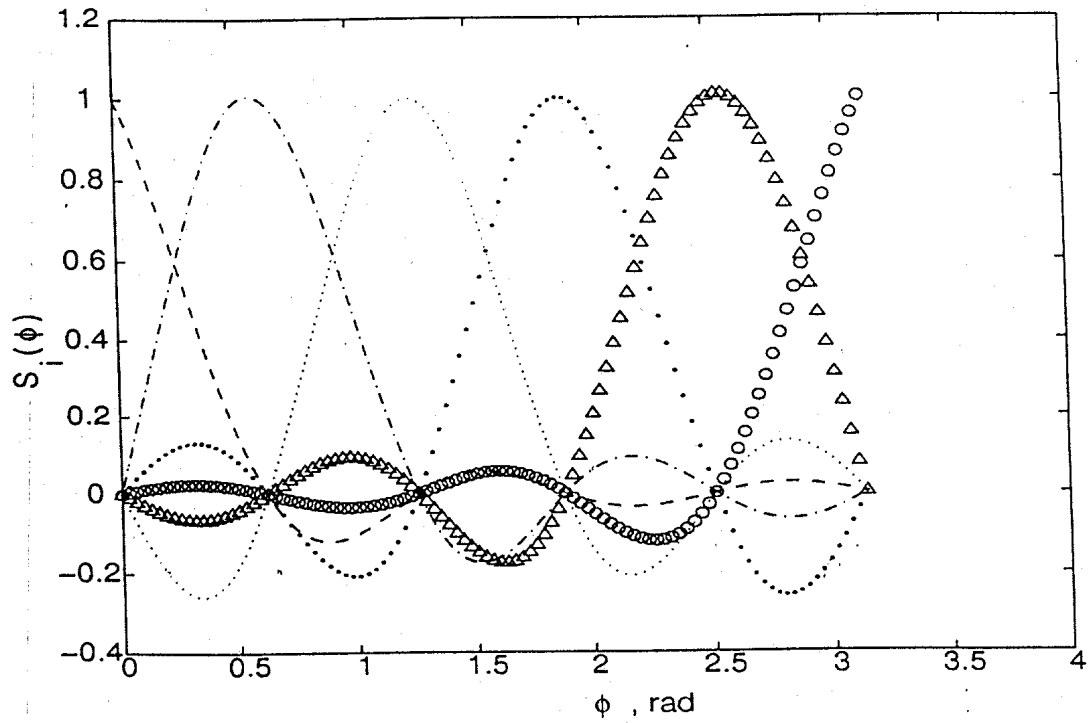


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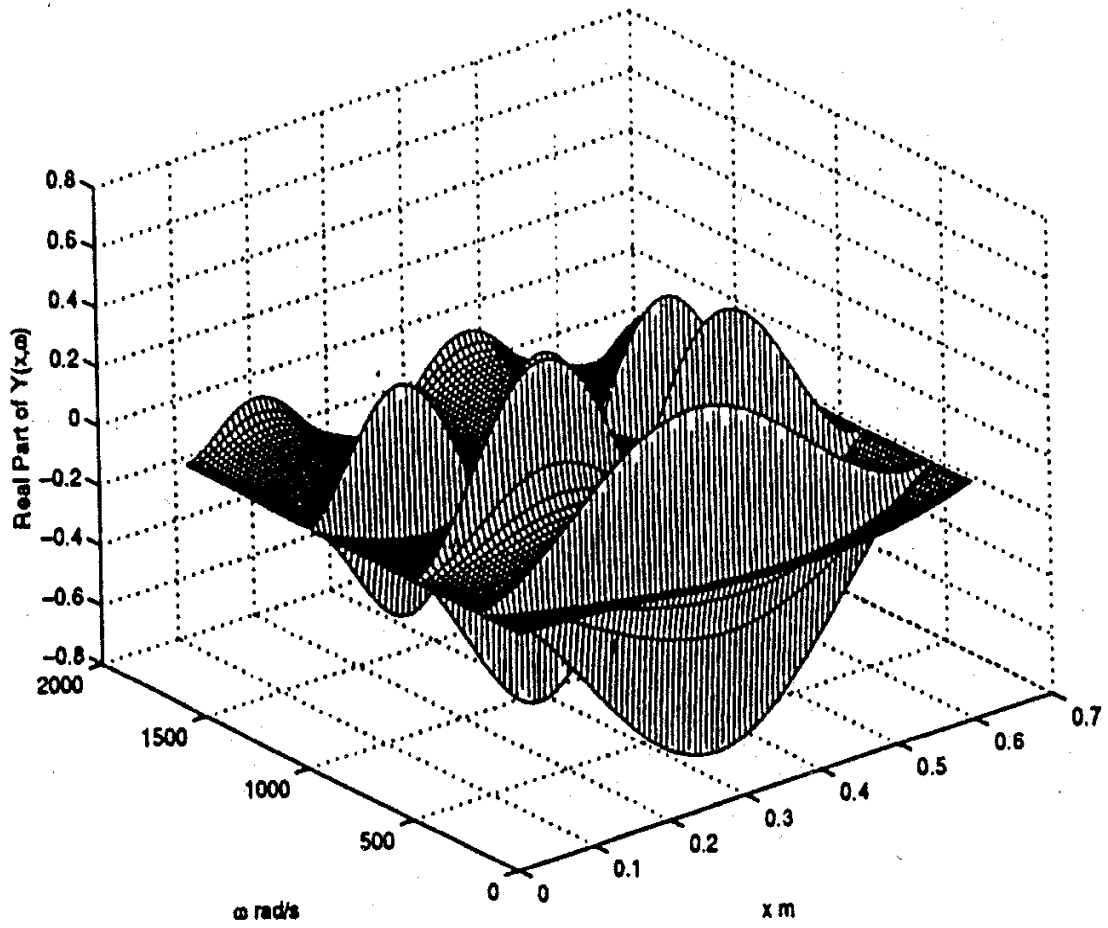


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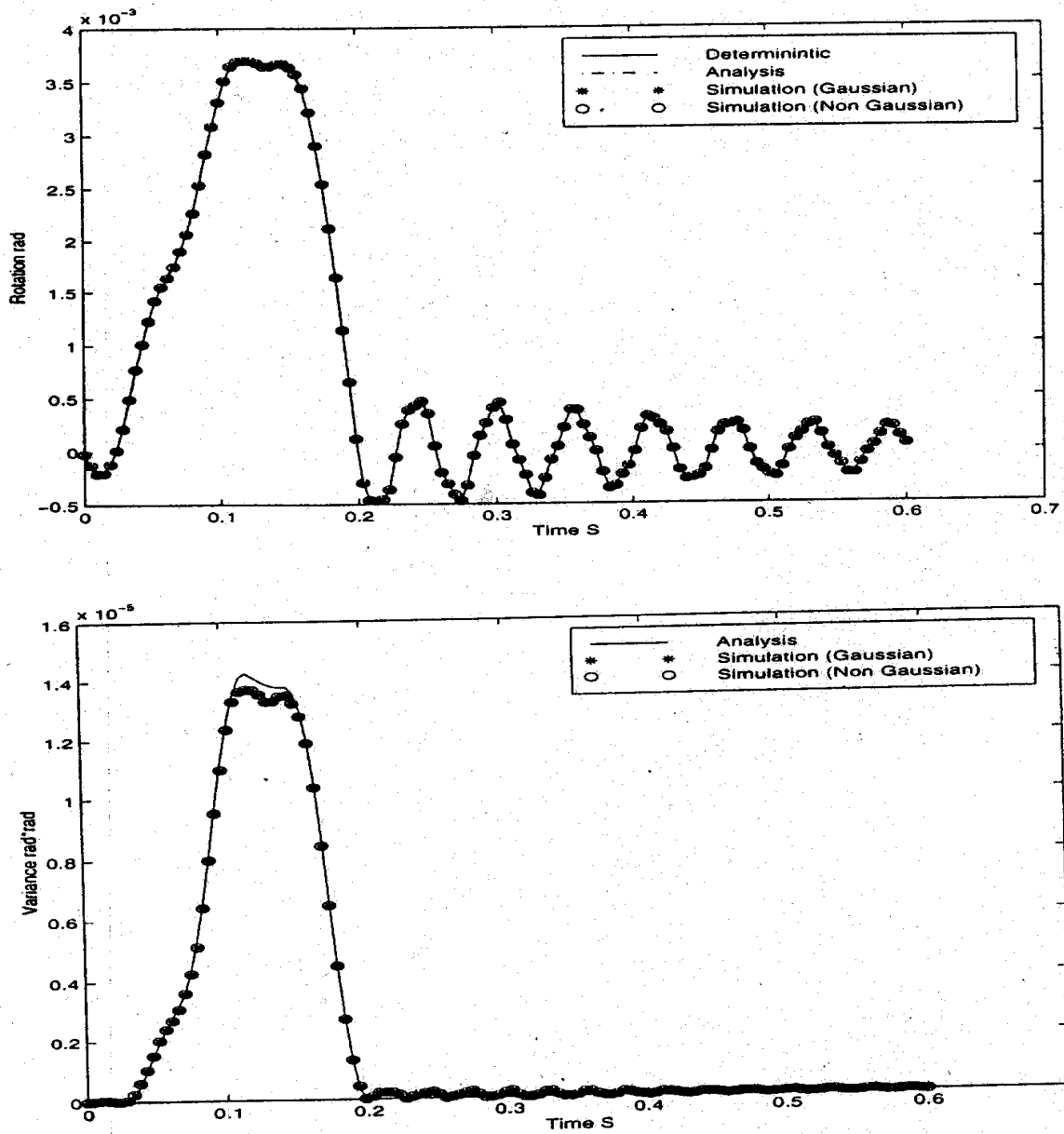


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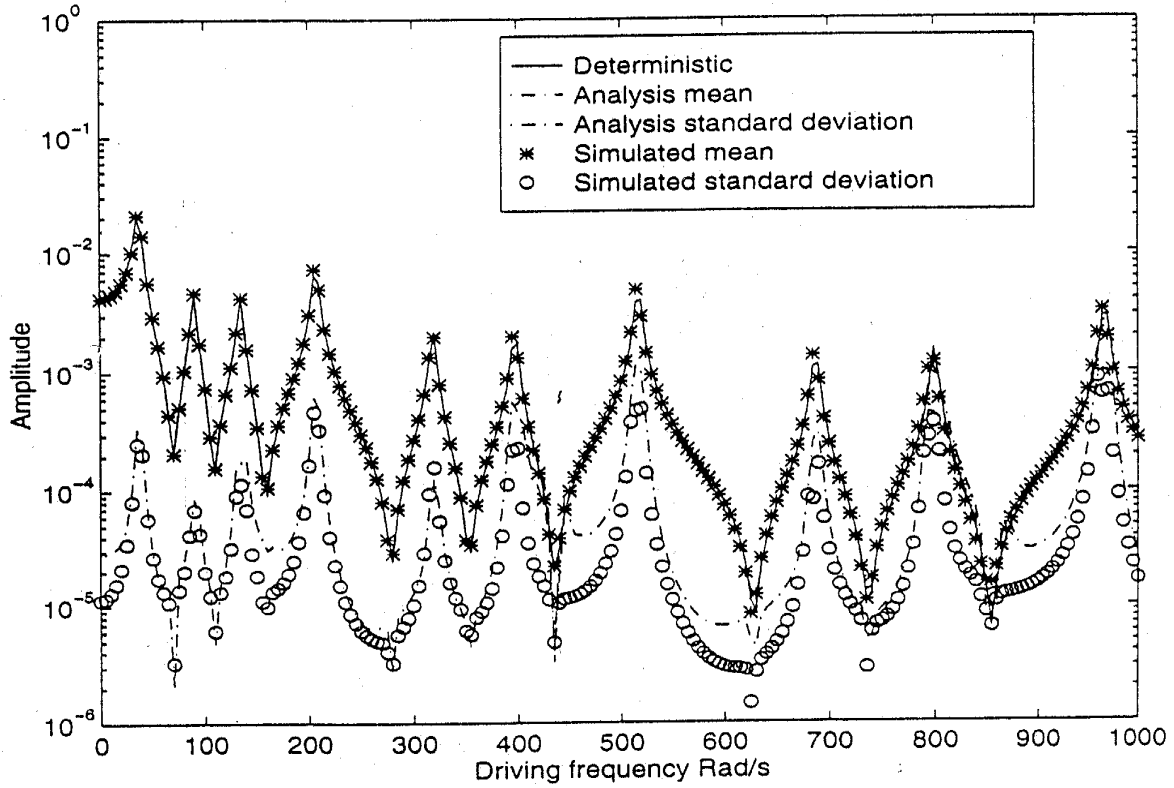


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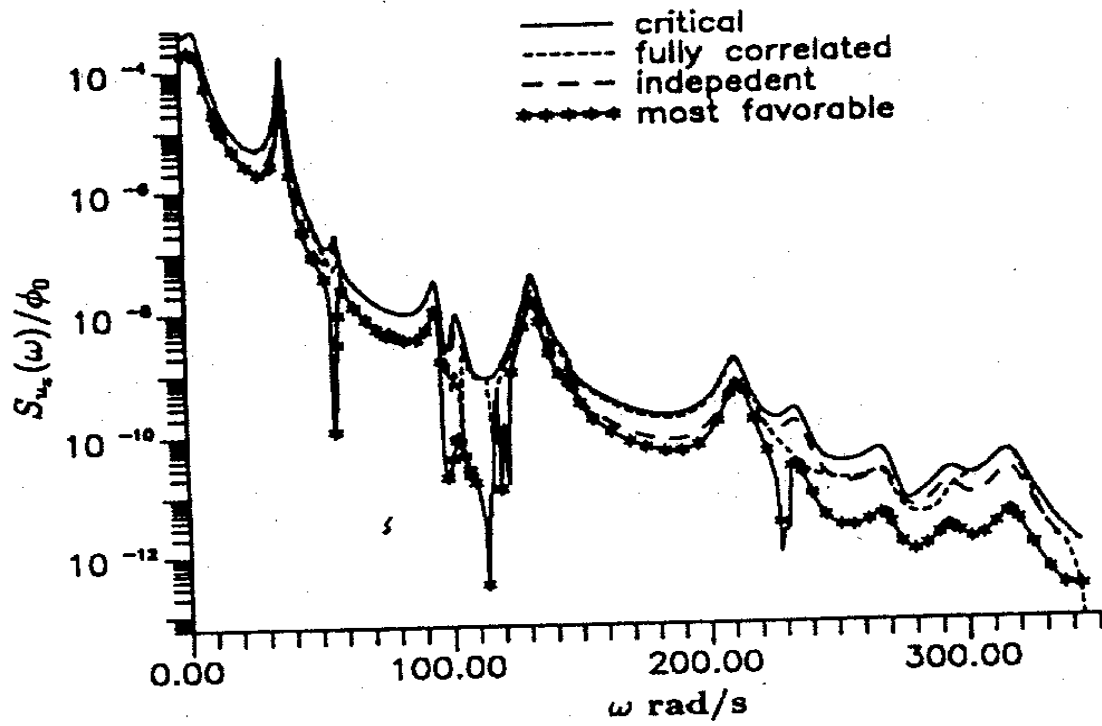


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