

New forms of extended Kalman filter via transversal linearization and applications to structural system identification

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Abstract

Two novel forms of the extended Kalman filter (EKF) are proposed for parameter estimation in the context of problems of interest in structural mechanics. These filters are based on variants of the derivative-free locally transversal linearization (LTL) and multi-step transversal linearization (MTrL) schemes. Thus, unlike the conventional EKF, the proposed filters do not need computing Jacobian matrices at any stage. While the LTL-based filter provides a single-step procedure, the MTrL-based filter works over multiple time-steps and finds the system transition matrix of the conditionally linearized vector field through Magnus' expansion. Other major advantages of the new filters over the conventional EKF are in their superior numerical accuracy and considerably less sensitivity to the choice of time-step sizes. A host of numerical illustrations, covering single- as well as multi-degree-of-freedom oscillators with time-invariant parameters and those with discontinuous temporal variations, are presented to confirm the numerical advantages of the novel forms of EKF over the conventional one.

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1. Introduction

Dynamic state estimation techniques offer an effective framework for time-domain identification of structural systems. These methods have their origin in the area of control of dynamical systems with the Kalman filter [19] being one of the well-known tools of this genre. The filter provides the exact posterior probability density function (pdf) of the states of linear dynamical systems with linear measurement model, with additive Gaussian noise processes. Recent research in this area is focused on extending the state estimation capabilities to cover nonlinear dynamical systems and non-Gaussian additive/multiplicative noises. Early efforts in dealing with nonlinear dynamical systems were focused on application of linearization techniques which have led to the development of the extended Kalman

filter. More recent developments have encompassed studies on evaluation of posterior pdf using Monte-Carlo simulation procedures. These methods, styled as particle filters, are computationally intensive but have wide-ranging capabilities in terms treating nonlinearity and non-Gaussian features (see, for example, [9,34]).

In the area of structural engineering problems, dynamic state estimation methods have been developed and applied in the context of structural system identification [49,15,10], input identification [18,26], and active structural control (see, for example, [40]). In these applications, the process equations are typically derived based on the finite element method (FEM) of discretizing structural systems. These equations represent the condition of dynamic equilibrium and often constitute a set of ordinary differential equations (ODEs). The measurements are typically made on displacements, velocities, accelerations, strains and/or support reactions. Furthermore, the process and measurement equations are often taken to be contaminated by additive Gaussian noise processes. These noises themselves are

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modeled as a vector of white-noise processes or, more generally, as a vector of filtered white-noise processes. In this context the classical Kalman filter itself has limited applications since interest is often focused on systems with nonlinear mechanical properties. Consequently, studies dealing with application of EKF methods to structural system identification have been undertaken [49,15,10]. Though in most of the studies, the parameters to be estimated are essentially time-invariant, time varying parameter identification problems have also been studied [43,24,48,7]. One of the important issues in the identification of structural parameters is the interplay of the various uncertainties in modeling and measurements and their effect on the variability of the identified parameters. The variance of the system parameters, conditioned on measurements, essentially captures this feature. If this variance itself is computed in an approximate manner, as is the case, for instance, when EKF or any suboptimal filtering strategy are used, satisfactory interpretation of the parameter variability becomes difficult. The works of Koh and See [22], Kurita and Matsui [23] and Sanayei et al. [39] contain discussions on related issues. Recently applications of particle filter techniques to structural system identification have been reported [29,7]) and control [38].

One of the main ingredients of filtering techniques applied to structural dynamic system is the discretization of the governing equations of motion into a set of discrete maps. In this context, it is of interest to note that the solution of equilibrium equations in the finite element framework has been widely researched (see, for example, [3]). Various methods such as central difference, Wilson-theta, Newmark-beta, etc., have been developed and are routinely available in commercial finite element packages. Questions on spectral stability and accuracy of these schemes have thus been discussed. Most studies that combine finite element structural modeling with dynamic state estimation methods employ either forward or central difference-based schemes for discretizing the governing equations (see, for example [6]). Alternatively, in the context of nonlinear structural mechanics problems, a closed-form expression of the state transition matrix (STM) is employed on the linearized equations of motion to obtain a discrete map for filtering applications (see, for example [49]). Thus it appears that the relationship between the methods of integrating governing equation of motion and the formulation of discrete maps for dynamic state estimation techniques has not been adequately explored in structural engineering applications. The present work is in the broad domain of addressing this issue.

In the recent years, the present authors have been involved in research related to the development of numeric-analytic solution techniques for nonlinear ODEs and stochastic differential equations (SDE) based on the concept of transversality, namely, locally transversal linearization (LTL) [37,35] and multi-step transversal linearization (MTrL) [36,12,13] methods. These methods completely avoid gradient (Jacobian) calculations at any

stage. In addition to the laborious and possibly inaccurate nature of such an exercise in the context of finite element applications, this assumes importance in systems having vector fields with C^0 continuity. Also, it has been validated by the authors [12] that, due to less propagation of local errors, these methods work for considerably higher step sizes. Moreover, a certain class of these methods (MTrL-I, [12,13]) qualifies as one of geometric integrators and hence may be elegantly applied to preserve the invariants of motion. Thus, they offer an effective solution for a more general class of problems than methods based on tangential (path-wise) linearization. Consequently, it is of considerable interest to explore the application of these integration schemes in the context of deriving discrete equations for process equations in dynamic state estimation problems arising in structural system identification.

The present work accordingly proposes two alternative forms of the extended Kalman filter on the basis of the theories of local and multi-step transversal linearizations. The LTL method, originally proposed as a numeric-analytic principle for solutions of nonlinear ODEs, can be considered as a bridge between the classical integration tools and the geometric methods. The essence of the LTL method is to exactly reproduce the vector fields of the governing nonlinear ODEs at a chosen set of grid points and then to determine the discretized solutions at the grid points through a transversality argument. The conditionally linearized vector fields may be non-uniquely chosen in such a way that their form precisely matches the given nonlinear vector field and are transversal to this field at least at the grid points. The recent multi-step extension of the LTL, referred to as the MTrL method, uses the solutions of transversally linearized ODEs in closed-form and hence distinguishable from other multi-step, purely numerical, implicit methods. Owing to the condition of transversality, the need for any differentiation of the original vector field is eminently avoided in this class of methods.

It is theoretically impossible to derive a time-invariant linear dynamical system whose evolution will, in general, precisely match that of a given nonlinear system over any finite time interval. Over a sufficiently small time interval, however, the nonlinear equations may be conditionally linearized which is presently accomplished using the still-unknown, instantaneous estimates to be determined at the grid points over the interval. Once these conditionally linearized equations are derived, a discrete Kalman filtering algorithm may be applied to get estimates of states of the system. It should be understood that, the LTL-based Kalman filter, dealing with conditionally linearized equations, results in implicit and nonlinear algebraic equations thereby necessitating the use of either a Newton–Raphson or a fixed-point iteration scheme to solve for the instantaneous estimates. The basic idea is that the constructed conditionally linearized estimate manifold is made to transversally intersect the nonlinear (true) estimate manifold at time instants where the estimate vectors need to be determined. While choices for conditionally linearized

LTL vector fields are non-unique (i.e., infinitely many), solutions are, more or less, insensitive to the specific chosen forms. The condition of intersection of manifolds at the grid points provides a set of constraint equations whose zeros are the instantaneous estimates. For the MTrL-based Kalman Filter, the linearization is implicitly achieved using an interpolating expansion of part(s) of the nonlinear vector functions in terms of the unknown estimates at p (for an order p MTrL) grid points. We emphasize that this scheme yields the estimates at p points simultaneously (via a single linearization step) and, to the authors' knowledge, it is not an aspect that has been explored earlier. However, the proposed filtering algorithms still use a sequential linear estimation theory in the form of the Kalman filter and thus complete information about the posterior probability density function of the states is not obtainable. We have used a weighted global iteration procedure (WGI) [41] with the proposed filters replacing the conventional EKF. Several numerical examples, covering both linear and nonlinear structural behavior, are considered to verify the potential and relative merits of the method. Along with a linear multi-degree-of-freedom (MDOF) shear building, a hardening Duffing oscillator and a hysteric oscillator (Bouc–Wen hysteresis model) have been used for numerical study. In each case, the forward system model is simulated on a computer using the stochastic Heun's method to generate synthetic noisy measurements, which are fed to the filtering algorithm to arrive at estimates of the model parameters.

2. Process and measurement equations

Identification of any dynamical system requires, in general, two models. Firstly, a mathematical model describing the evolution of state with time, i.e., the process equations, to idealize the dynamics of the physical system, and, secondly, measurement (or observation) equations, relating the measured quantities to the state variables, should be available. Both these models include noise terms that serve to account for unmodeled dynamics in the process equations and measurement errors in the observation equations. It is common practice to model the noise terms as Gaussian white noise processes. In time-domain system identification problems, the parameters of the mathematical model are declared as additional states so that any acceptable algorithm for state estimation becomes applicable for these problems as well. In the general context of engineering dynamical systems, one generally starts with the following set of second order, nonlinear ODEs, derivable via a spatial projection technique (such as the Petrov–Galerkin) applied to the governing PDEs:

$$M\ddot{X} + C\dot{X} + KX + Q(X, \dot{X}, t) = \hat{F}(t) + \hat{G}(t)\xi_s(t) \quad (1)$$

with $X \in \mathbb{R}^n$, $t \in [t_0, T] \subset \mathbb{R}$, M being the mass matrix, C the viscous damping matrix, K the stiffness matrix, $Q(X, \dot{X}, t) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$ the nonlinear and/or parametric part of the vector field, and $\hat{F} : [t_0, T] \subset \mathbb{R} \rightarrow \mathbb{R}^n$

the external (non-parametric and deterministic) force vector. The components of the process noise vector $\xi_s(t) \in \mathbb{R}^m$ are taken to be independently evolving Gaussian white noise processes. \hat{G} is an $n \times m$ noise distribution matrix. The noise term in Eq. (1) represents the effects of modeling errors and errors in applying the test signals. Presently, we assume that elements of $Q(X, \dot{X}, t)$ and $\hat{F}(t)$ are C^k functions, $k \in \mathbb{Z}^+$, in X and t with $k \geq 1$. Define the new state vector as

$$Z(t) = \{Z_s(t)^T \quad Z_p(t)^T\}^T, \quad (2)$$

where $Z(t) \in \mathbb{R}^{2n+l}$ is the extended state vector at time t , $Z_s(t)$ is the vector of states of the original system, i.e., $Z_s(t) = \{X^T \dot{X}^T\}^T$, $Z_p \in \mathbb{R}^l$ is the vector of model parameters (e.g., stiffness, damping or any nonlinear stiffness or damping parameters to be identified). Furthermore, n is the number of degrees of freedom (DOF) and l is the total number of model parameters that need to be identified. For a general time-invariant parameter identification problem, the state space equations for the dynamical system in Eq. (1) has the following representation:

$$\dot{Z}_s(t) = \tilde{g}(Z_s(t), Z_p(t), t) + \tilde{F}(t) + \tilde{G}(t)\xi_s(t), \quad (3)$$

$$\dot{Z}_p(t) = 0 + \zeta_p(t), \quad (4)$$

where $\tilde{g} : \mathbb{R}^{2n} \times \mathbb{R}^l \times \mathbb{R} \rightarrow \mathbb{R}^{2n}$, $\tilde{F} : \mathbb{R} \rightarrow \mathbb{R}^{2n}$ and $\tilde{G} \in \mathbb{R}^{2n \times m}$ are respectively the augmented state-dependent drift vector, forcing vector and process noise distribution matrix with the last two being appropriately padded by zeros wherever needed. Note that the model parameters are expressed in terms of elements of $Z_p(t)$, which are declared as the additional states of the system. $\zeta_p(t) \in \mathbb{R}^l$ is a vector of independent white noise processes that are small random disturbances to states and may be interpreted as an artificial evolution of the model parameters in time [8,25]. It should be noted that once the system identification problem is formulated by declaring the unknown model parameters as additional states of the system, even a problem without any mechanical nonlinearity is transferred to the domain of nonlinear filtering.

The two Eqs. (3) and (4) may be combined to obtain

$$\dot{Z}(t) = g(Z(t), t) + F(t) + G(t)\xi(t), \quad (5)$$

where $g : \mathbb{R}^{2n+l} \times \mathbb{R} \rightarrow \mathbb{R}^{2n+l}$; $F : [t_0, T] \subset \mathbb{R} \rightarrow \mathbb{R}^{2n+l}$; $G(t)$ is a $(2n+l) \times (2n+l)$ matrix and $\xi(t) \in \mathbb{R}^{2n+l}$. The observation equation may be written as

$$Y(t) = \tilde{h}(Z(t), t) + \eta(t). \quad (6)$$

Here $Y(t) \in \mathbb{R}^r$; $\tilde{h} : \mathbb{R}^{2n+l} \times \mathbb{R} \rightarrow \mathbb{R}^r$; $\eta(t) \in \mathbb{R}^r$ is a vector of independent white noise processes and $r \leq 2n$. The noise term $\eta(t)$ in Eq. (6) is used to model measurement errors and possible errors in relating the observed quantities with the state variables. The above representation of the observation equation is formal and observations, in practice, would be made at a set of discrete time instants so the measurement equation may be realistically represented as

$$Y_k = Y(t_k) = \tilde{h}(Z(t_k), t_k) + \eta(t_k), \quad (7)$$

where $t_k = k\Delta t$, Δt being the sampling interval and $\eta(t_k)$ represents a sequence of r independent Gaussian random variables with covariance matrix R_k . As has been mentioned in Section 1, the objective of the present study is to develop two new forms of EKF based on transversal intersection concepts, namely the LTL-based EKF and the MTrL-based EKF. As a prelude to that we briefly outline the steps involved in the estimation of unknown structural parameters by the conventional EKF.

3. Structural system identification by EKF

We begin by considering the process Eq. (1) and the measurement Eq. (7) and summarize the recursive algorithm involved in implementing the EKF [49]. Given an initial estimate $\widehat{Z}_{0|0}$ of the state and the error covariance $P_{0|0}$, start the loop.

For $k = 0, 1, 2, 3, \dots, m$ and $(t \in (0, t_m])$, perform the following steps:

1. Begin with the filter estimate $\widehat{Z}_{k|k}$ and its error covariance matrix $P_{k|k}$.
2. Evaluate the predicted state $\widehat{Z}_{k+1|k}$ and the predicted error covariance matrix $P_{k+1|k}$ by

$$\widehat{Z}_{k+1|k} = \widehat{Z}_{k|k} + \int_{t_k}^{t_{k+1}} \left\{ g(\widehat{Z}_{t|k}(t)) + F(t) \right\} dt, \quad (8)$$

$$P_{k+1|k} = \Phi(t_{k+1}, t_k) P_{k|k} \Phi^T(t_{k+1}, t_k) + Q_{k+1}, \quad (9)$$

where $\Phi(t_{k+1}, t_k)$ is the state transition matrix of the linearized system and is given by

$$\Phi(t_{k+1}, t_k) = \exp[(\nabla_k)(t_{k+1} - t_k)]. \quad (10)$$

Here

$$\nabla_k = \left[\frac{\partial g(Z, t)}{\partial Z} \right]_{Z(t)=\widehat{Z}_{k|k}, t_k} \quad (11)$$

is the $(2n + l) \times (2n + l)$ system Jacobian matrix (that obtains the first-order Taylor increment of the nonlinear function g) and

$$Q_{k+1} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, u) G(u) E[\xi(u)\xi^T(v)] \times G^T(v) \Phi^T(t_{k+1}, v) du dv, \quad (12)$$

where Q_{k+1} is the process error covariance matrix ($E(\cdot)$ denotes the expectation operator). Note that, following linearization over the time interval $(t_k, t_{k+1}]$, the approximate solution to Eq. (5) is given by

$$Z(t) = \Phi(t, t_k) \widehat{Z}_{k|k} + \int_{t_k}^t \Phi(t_k, u) [F(u) + G(u)\xi(u)] du. \quad (13)$$

3. The Kalman gain is given by

$$K_{k+1} = P_{k+1|k} H_{k+1}^T [H_{k+1} P_{k+1|k} H_{k+1}^T + R_{k+1}]^{-1}, \quad (14)$$

where R_{k+1} is the covariance matrix of the measurement noise

$$H_{k+1} = \left[\frac{\partial \tilde{h}(Z, t)}{\partial Z} \right]_{Z(t)=\widehat{Z}_{k+1|k}, t_{k+1}}. \quad (15)$$

4. Obtain the new estimate $\widehat{Z}_{k+1|k+1}$ of state and the associated error covariance matrix $P_{k+1|k+1}$ as

$$\widehat{Z}_{k+1|k+1} = \widehat{Z}_{k+1|k} + K_{k+1} [Y_{k+1} - \tilde{h}(\widehat{Z}_{k+1|k}, t_k)], \quad (16)$$

$$P_{k+1|k+1} = [I - K_{k+1} H_{k+1}] P_{k+1|k} [I - K_{k+1} H_{k+1}]^T + K_{k+1} R_{k+1} K_{k+1}^T. \quad (17)$$

Estimation of model parameters by EKF has been widely studied. Asymptotic convergence results are also available [27]. Since EKF is based on a linearization of the state and observation equations over each time-step (only up to first-order terms in the Taylor expansion are retained) and a subsequent usage of the linear estimation theory in the form of Kalman filter, the deviation of the estimated trajectory from the true trajectory increases with time and the significance of higher order terms in the Taylor expansion also increases. Due to the accumulation and propagation of local errors, there is a distinct possibility of divergence, especially if the EKF starts off with poor initial estimates (see, for instance, [44,14,4]). This also implies that an increased sensitivity of numerical solutions to step-sizes can potentially lead to biased estimates [6].

4. Formulations of LTL- and MTrL-based EKFs

We will presently employ the concepts of transversal intersection to develop two new forms of the EKF, which bear a resemblance to a collocation method in the sense that the filter estimates will be equal to the ‘optimal’ estimates at the points of discretization and not necessarily over the entire time interval. Nevertheless, we note that the estimate for a general nonlinear system via the proposed as well as a conventional forms of the EKF is not the mean square estimate, as obtainable (in principle) through a strictly nonlinear filter (such as a Monte-Carlo filter). Indeed the ‘optimal’ solution presently refers to a minimum mean square estimator (MMSE) based on the assumption of a Gaussian posterior probability density function.

4.1. The LTL-based EKF

The general idea of the LTL-based EKF for system identification is to replace the nonlinear vector field of the process equation by a time-invariant conditionally linearized vector field over a time-step and an application of Kalman filter based on the estimates at the right end of the time-step. Consider Eq. (5) corresponding to a nonlinear dynamical system. Let the time interval of interest $(0, T) \subset \mathbb{R}$ be divided into m sub-intervals with

$0 = t_0 < t_1 < t_2 \cdots < t_i < \cdots < t_m = T$ and $h_i = t_{i+1} - t_i$, $i \in \mathbb{Z}^+$. It is now required to replace the process equation by a suitably chosen set of m transversally linearized vector equations, wherein the solution to the i -th element of the set should be, in a sense, a representative of the nonlinear flow over the i -th time-step. Note that when the measurement relationship given by Eq. (6) is nonlinear, m additional sets of conditionally linearized algebraic vector equations have also to be chosen. We define $T_i := (t_i, t_{i+1}]$. For a given set of measurements with the process equations precisely known, the optimal estimate of the vector state $\epsilon_i(\cdot)$ (the sense of optimality has been enunciated earlier) may be thought of as a flow evolving on a compact manifold M such that

$$\epsilon_t : M \times \mathbb{R} \rightarrow M \quad (18)$$

is C^k ($k \geq 0$) on M . Then, for any t , $\epsilon_t(Z)$ is a C^k diffeomorphism $M \rightarrow M$. Let (Z_i, t_i) and $(Z_{i+1}, t_{i+1}) \in M$ denote a pair of points on the estimate manifold (with a local structure of \mathbb{R}^{2n+l+1}). Now one can construct a couple of $2n+l$ -dimensional cross-sections \sum_i and \sum_{i+1} transverse to the vector field at (Z_i, t_i) and (Z_{i+1}, t_{i+1}) , respectively, such that $Z_i \in \sum_i$ and $Z_{i+1} \in \sum_{i+1}$. At this point we can view the Kalman filter as a map $\tilde{K} : \sum_i \times \mathbb{R}^r \times \mathbb{R} \times \mathbb{R} \rightarrow \sum_{i+1}$ such that

$$Z_{i+1} = \tilde{K}(Z_i, Y_{i+1}, t_i, h_i), \quad (19)$$

where $Y_{i+1} \in \mathbb{R}^r$ is the observation vector at the $(i+1)$ th time instant. \tilde{K} may be thought of as a Poincare (stroboscopic) map as in the case of a nonlinear flow. The principle of an LTL-based EKF is thus to generate a set of conditionally linear equations which, when used with a Kalman filter, gives rise to a map \tilde{K} that should be (ideally) identical to \tilde{K} for the same initial condition (Z_i, t_i) and the same interval h_i . It is conceivable that the estimate manifold of the proposed filter has to transversally intersect the optimal estimate manifold at least at the grid points in $S = \{t_i | i = 0, 1, 2, 3, \dots\}$ and not necessarily elsewhere.

Remark 1: From a straightforward application of the weak transversality theorem [2], it follows that mappings transversal to M at a given instant t_i form an open and everywhere dense set in the space of smooth maps $\tilde{f}_i : M \rightarrow \bar{M}_i$, where \bar{M}_i denotes the set of all transversally intersecting smooth manifolds. Hence the derivation of a conditionally linear system and therefore, the estimates of the proposed filter are non-unique and infinitely many.

For instance, Eqs. (5) and (6) may give rise to the following LTL system (with conditionally constant coefficients) over the interval $(t_i, t_{i+1}]$ for purposes of state estimation using the Kalman filter:

$$\dot{\bar{Z}} = B(\hat{Z}_i, t_i)\bar{Z} + F(t) + G(t)\xi(t), \quad (20)$$

$$\bar{Y} = H(\hat{Z}_i, t_i)\bar{Z} + \eta(t) \quad (21)$$

provided that the vector functions g and \tilde{h} are decomposable as

$$g(Z, t) = B(Z, t)Z, \quad (22)$$

$$\tilde{h}(Z, t) = H(Z, t)Z \quad (23)$$

with $B(Z, t)$ and $H(Z, t)$ being finite quantities. Eqs. (20) and (21) may be viewed as being conditionally linear with constant coefficients, given the value of the still-unknown estimated quantities $\hat{Z}_i = \hat{Z}(t_i)$. Now the proposed filter may be summarized as follows:

Given initial estimates of state $\hat{Z}_{0|0}$ and the error covariance matrix $P_{0|0}$, loop over $k = 0, 1, 2, 3, \dots, m$ for $(t \in (0, t_m])$ using the following steps:

1. Start with filter estimates $\hat{Z}_{k|k}$ and its error covariance matrix $P_{k|k}$.
2. Evaluate the predicted state $\hat{Z}_{k+1|k}$ as the LTL-based solution of the following nonlinear differential equation at $t = t_{k+1}$ subject to the initial condition $\hat{Z}_{k|k}$ at $t = t_k$:

$$\dot{\hat{Z}}_{t|t_k} = g(\hat{Z}_{t|t_k}, t) + F(t), \quad (24)$$

which is the process Eq. (5) without noise. This is because the prediction stage basically provides the expected values of the state at the forward time-points and hence it is reasonable to ignore the additive zero-mean white noise at this stage.

3. Write down the exact solution of the conditionally linearized equation as

$$\begin{aligned} \bar{Z}(t) = & \Phi(t, t_k, \hat{Z}_{k+1|k+1})\hat{Z}_{k|k} + \int_{t_k}^t \Phi(\hat{Z}_{k+1|k+1}, t_k, u) \\ & \times [F(u) + G(u)\xi(u)]du, \end{aligned} \quad (25)$$

where $t \in (t_k, t_{k+1}]$ and $\Phi(t, t_k, \hat{Z}_{k+1|k+1})$ is the state transition matrix (or the fundamental solution matrix) of the linearized system. The latter is given by

$$\Phi(t, t_k, \hat{Z}_{k+1|k+1}) = \exp[B(\hat{Z}_{k+1|k+1}, t_i)(t - t_k)]. \quad (26)$$

4. Estimate the predicted error covariance matrix $\bar{P}_{k+1|k}$ by

$$\begin{aligned} \bar{P}_{k+1|k}(\hat{Z}_{k+1|k+1}) = & \Phi(t_{k+1}, t_k, \hat{Z}_{k+1|k+1})P_{k|k}\Phi^T(t_{k+1}, t_k, \hat{Z}_{k+1|k+1}) \\ & + \bar{Q}_{k+1}(\hat{Z}_{k+1|k+1}). \end{aligned} \quad (27)$$

Here the process error covariance matrix \bar{Q}_{k+1} is given by

$$\begin{aligned} \bar{Q}_{k+1} = & \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \Phi(t_{k+1}, u, \hat{Z}_{k+1|k+1})G(u)E[\xi(u)\xi^T(v)] \\ & \times G^T(v)\Phi^T(t_{k+1}, v, \hat{Z}_{k+1|k+1})dudv. \end{aligned} \quad (28)$$

5. The Kalman gain is given by

$$\begin{aligned} \bar{K}_{k+1}(\hat{Z}_{k+1|k+1}) = & \bar{P}_{k+1|k}(\hat{Z}_{k+1|k+1})H_{k+1}^T(\hat{Z}_{k+1|k+1}, t_{k+1}) \\ & \times [\bar{P}_{k+1|k}(\hat{Z}_{k+1|k+1})H_{k+1}(\hat{Z}_{k+1|k+1}, t_{k+1})\bar{P}_{k+1|k} \\ & \times (\hat{Z}_{k+1|k+1})H_{k+1}^T(\hat{Z}_{k+1|k+1}, t_{k+1}) + R_{k+1}]^{-1}, \end{aligned} \quad (29)$$

where

$$\bar{Y}_{k+1} = H_{k+1}(\hat{Z}_{k+1|k+1}, t_{k+1})\bar{Z} + \eta(t). \quad (30)$$

6. Obtain the new estimate of the state $\hat{Z}_{k+1|k+1}$ via

$$\hat{Z}_{k+1|k+1} = \hat{Z}_{k+1|k} + \bar{K}_{k+1}[\bar{Y}_{k+1} - \tilde{h}(\hat{Z}_{k+1|k}, t_k)]. \quad (31)$$

Now the conditionally linearized estimate and the optimal estimate will become instantaneously identical at t_{k+1} if a root ($\hat{Z}_{k+1|k+1}$) of Eq. (31) can be found such that

$$\hat{Z}_{k+1|k+1} = \hat{Z}_{k+1|k+1}. \quad (32)$$

In other words, the constraint condition i.e. $\hat{Z}_{k+1|k+1}, \hat{Z}_{k+1|k+1} \in M \cap \bar{M}_i$ is automatically satisfied if a real and physically admissible root of the following transcendental equation can be found:

$$\hat{Z}_{k+1|k+1} - \hat{Z}_{k+1|k} + \bar{K}_{k+1}[\bar{Y}_{k+1} - \tilde{h}(\hat{Z}_{k+1|k}, t_k)] = 0. \quad (33)$$

This can be done through any standard root-finding algorithm as the fixed-point iteration, Newton–Raphson method, etc.

7. Once the true estimate is obtained as the solution of Eq. (33), the error covariance matrix $P_{k+1|k+1}$ associated with the new state may be estimated as

$$P_{k+1|k+1} = [I - K_{k+1}H_{k+1}]P_{k+1|k}[I - K_{k+1}H_{k+1}]^T + K_{k+1}R_{k+1}K_{k+1}^T, \quad (34)$$

where the quantities (without bar) are computed by substituting in the corresponding quantities (with bar) the true value of the estimates obtained from Eq. (33).

4.2. The MTrL-based EKF

The second class of the EKF, referred to as the MTrL-based EKF, may be viewed as an extension and further generalization of the LTL-based EKF. The proposed filter is based on the new and more flexible MTrL method [12] instead of the earlier version of the MTL method [36] wherein the only way to treat the nonlinear terms was to convert them into conditional forcing functions. With the latest version of the MTrL applied to nonlinear structural dynamics, the nonlinear damping and stiffness terms in the original system appear respectively as conditionally determinable damping and stiffness coefficients in the linearized system as well. The fundamental solution matrix (FSM) or the state transition matrix of the linearized system may be obtained through Magnus' expansion [28], which is derivable based on the assumption that the linearized system matrix is a sufficiently smooth Lie element.

For the purpose of implementation, let the subset of the time-axis over $(0, T]$ be divided into M closed–open sub-intervals as

$$(0, T] = \{I_1, I_2, I_3, \dots, I_M\} \\ = \{(0 = T_0, T_1], (T_1, T_2], \dots, (T_{M-1}, T_M = T)\}. \quad (35)$$

Now consider the sub-interval I_1 and let it be ordered into p smaller intervals as

$$(0 = t_0, t_1], (t_1, t_2], \dots, (t_{p-1}, t_p = T_1]$$

with the time-step $h_i = t_{i+1} - t_i$, where $i \in \mathbb{Z}^+$; $i \leq p$. Similarly each succeeding interval is subdivided into p sub-intervals and so ordered that

$$(0, T] = \{(t_0, t_1, t_2, \dots, t_p]; (t_p, t_{p+1}, t_{p+2}, \dots, t_{2p}); \dots; \\ \times (t_{(M-1)p}, t_{(M-1)p+1}, t_{(M-1)p+2}, \dots, t_{Mp})\}, \quad (36)$$

where $t_{jp} = T_j$; $j = 1, 2, 3, \dots, M$.

Let $S_i = \{(i-1)p, (i-1)p+1, (i-1)p+2, \dots, ip\} \subset \mathbb{Z}^+$. The idea is to derive a linearized system for each sub-interval I_i , $1 \leq i \leq M$. Moreover, following the concept of transversal linearization, it is intended that the state estimate computed through an application of the Kalman filter to the linearized equations and the ‘best’ estimates remain identical at all points of discretization in that interval, viz. at $t_{(i-1)p}, t_{(i-1)p+1}, t_{(i-1)p+2}, \dots, t_{ip} \in I_i$.

For the problem of parameter estimation at hand based on Eqs. (5) and (7), it is naturally sought to derive the linearized system such that it is also $(2n+l)$ -dimensional and is obtainable from the given nonlinear system with simple and least alterations. Firstly, the part of time-axis, over which the estimates are sought, needs to be discretized and ordered as mentioned above. The integer M denotes the number of times the MTrL-based linearization procedure has to be applied to get the estimates over the entire time interval of interest $(0, T]$. Each application of the MTrL procedure produces the estimates of $(2n+l)p$ discretized (scalar) state variables at p grid points over each sub-interval I_i . Thus it is a multi-step procedure for the estimation problem at p points over each linearization step. The estimate at the right end of the interval I_i will provide an initial estimate for the linearized system over I_{i+1} .

For further elaboration of the methodology, Eqs. (5) and (6) are considered again and linearized over the interval I_i using the unknown estimates at $p+1$ points as follows:

$$\dot{\check{Z}} = B(\check{Z}, t)\check{Z} + F(t) + G(t)\xi(t), \quad (37)$$

$$\bar{Y} = H(\check{Z}, t)\check{Z} + \eta(t) \quad (38)$$

provided that the vector functions g and \tilde{h} are decomposable as

$$g(Z, t) = B(Z, t)Z, \quad (39)$$

$$\tilde{h}(Z, t) = H(Z, t)Z \quad (40)$$

with $B(Z, t)$ and $H(Z, t)$ bounded over I_i . In the present work \check{Z} is obtained through an interpolation over the known initial vector estimate $\{\hat{Z}_{(i-1)p|(i-1)p}\}$ and unknown vectors $\{\hat{Z}_{k|k}|k \in S_i \text{ and } k \neq (i-1)p\}$ at t_k : $k \in S_i$. This expansion leads to an approximation of $B(Z, t)$ and $H(Z, t)$ over I_i and hence a conditionally linearized set of equations over the i -th time interval. For instance, if one chooses these interpolating functions to be Lagrangian

polynomials of degree p , \widehat{Z} is approximated in a vector space V_p of polynomials spanned by the Lagrange polynomials such that the dimension of this approximating space is p . The set of interpolating Lagrange polynomials $\{P_k|k = 0, 1, 2, \dots, p\}$ over I_i are given by

$$P_k(t) = \prod_{j=(i-1)p}^{ip} \frac{t - t_j}{t_k - t_j} \quad (41)$$

and one may write

$$\check{Z} = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}. \quad (42)$$

One is not restricted in choosing only Lagrangian polynomials to arrive at the vector function \check{Z} . For instance, one may use distributed approximating functionals [47] or wavelet-based [42] interpolating functions to obtain the function. However, irrespective of the specific form of interpolation, the MTrL-based linearized system corresponding to the nonlinear system (5) and (6) takes the form given by Eqs. (37) and (38). It must however be recorded that the use of such expansions is not quite consistent with the nature of strong solutions of a stochastic differential equation under white noise. It is well-known that the solution of an SDE is a semi-martingale and admits an expansion through a stochastic Taylor expansion (STE) (see, for instance, [21,30]). For instance, an STE over a time-step Δt has terms with fractional powers of Δt and such an expansion is therefore not spanned by the monomials $1, t, t^2$ and so on. However, we are presently interested in obtaining a reasonably accurate and numerically stable state estimate, which is itself non-stochastic. Indeed, as substantiated with extensive numerical simulations, the errors in the computed estimates owing to the relaxation of the mathematical rigor are not substantial as long as the additive noise intensity is not too high. The rest of the procedure is quite similar to the LTL-based EKF with the exception that the state transition matrix for the linearized system with time dependent coefficients has to be computed in a different manner (see Appendix A) in the form of a series expansion following Magnus [28]. It should be noted that the STM $\Phi(t)$ for the linearized system over the interval I_i is obtained from the originally nonlinear system of equations through conditional linearization using the state estimates at discrete time-points i.e. $\widehat{Z}_{(i-1)p}$ (known) and $\{\widehat{Z}_k|k \in S_i; k \neq (i-1)p\}$ (unknown) through Eq. (A.5) and thus the STM is a function of these unknown estimates at the grid points. Following a similar line of transversality argument as in the LTL-based EKF, it is evident that for a given set of observations, the ‘best’ estimate manifold and the estimate manifold through the application of the Kalman filter to the linearized system may be made instantaneously identical at the p grid points over I_i through enforcement of appropriate constraint equations (algebraic). The whole procedure can be summarized as follows.

Recast, for notational convenience, Eqs. (37) and (38) as

$$\dot{\check{Z}} = \Psi(t)\check{Z} + F(t) + G(t)\xi(t), \quad (43)$$

$$\bar{Y} = \Psi_{\bar{H}}\check{Z} + \eta(t), \quad (44)$$

where $\Psi(t) = B(\check{Z}, t)$ and $\Psi_{\bar{H}} = H(\check{Z}, t)$. Also let

$$\widehat{Z}_I = \{\widehat{Z}_{(i-1)p+1|(i-1)p+1}, \widehat{Z}_{(i-1)p+2|(i-1)p+2}, \dots, \widehat{Z}_{ip|ip}\} \quad (45)$$

denote the set of unknown estimates over I_i . Given initial estimates of state $\widehat{Z}_{0|0}$ and the error covariance matrix $P_{0|0}$, use a loop over $i = 1, 2, 3, \dots, M$, with $(t \in (0, T])$, to implement the following steps:

1. Start with filter estimates $\widehat{Z}_{(i-1)p|(i-1)p}$ and its error covariance matrix $P_{(i-1)p|(i-1)p}$.
2. Evaluate the predicted states $\{\widehat{Z}_{j|(i-1)p}|j \in S_i; j \neq (i-1)p\}$ as solution of the following nonlinear differential equation at $\{t_j|j \in S_i; k \neq (i-1)p\}$ subject to the initial condition $\widehat{Z}_{(i-1)p|(i-1)p}$ at $t = t_{(i-1)p}$

$$\dot{\widehat{Z}}_{t|t_{(i-1)p}} = g(\widehat{Z}_{t|t_{(i-1)p}}, t) + F(t), \quad (46)$$

which is the process Eq. (5) without noise. This is feasible as the prediction stage basically gives the expected values of the estimate at the forward time-points and hence the additive zero-mean white noise (vector) term is left out. The simultaneous prediction at p points may be achieved using the *class II* MTrL [12]. The class II MTrL avoids the computationally cumbersome Magnus’ expansion by treating the nonlinear functions in the process equation as a conditional forcing function and thereafter solves for the discretized state variables ensuring a transversal intersection of the original and the conditionally linearized flows at the grid points through appropriate constraints.

3. Discretize the conditionally linearized equation as in Eq. (25) and derive p sets of STMs and process noise covariance matrices. The STMs are given by

$$\Phi_j := \Phi(t_{(i-1)p}, t_j, \widehat{Z}_{(i-1)p|(i-1)p}, \widehat{Z}_I) = \exp(\Omega(t)), \quad (47)$$

where $\{j \in S_i; j \neq (i-1)p\}$ and are obtainable by substituting $\Psi(t)$ for $A(t)$ in Eq. (A.4), the limits of integration being $t_{(i-1)p}$ to t_j in the j th case. Denoting the error covariance matrix as $\bar{Q}_j := \bar{Q}_j(\widehat{Z}_{(i-1)p|(i-1)p}, \widehat{Z}_I)$ may be obtained in an identical manner like Eq. (28) by using the STMs as obtained above.

4. Estimate the predicted error covariance matrices $\bar{P}_{j|(i-1)p}$, where $\{j \in S_i; j \neq (i-1)p\}$, by

$$\bar{P}_j := \bar{P}_{j|(i-1)p}(\widehat{Z}_{(i-1)p|(i-1)p}, \widehat{Z}_I) = \Phi_j P_{(i-1)p|(i-1)p} \Phi_j^T + \bar{Q}_j. \quad (48)$$

5. The Kalman gains at $\{t_j|j \in S_i; j \neq (i-1)p\}$ are given by

$$\bar{K}_j := \bar{K}_j(\widehat{Z}_{(i-1)p|(i-1)p}, \widehat{Z}_I) = \bar{P}_j \Psi_{\bar{H}}^T \bar{P}_j^{-1} \times [\Psi_{\bar{H}} P_j \Psi_{\bar{H}}^T + R_j]^{-1}. \quad (49)$$

6. Obtain new estimates of states $\{\widehat{Z}_{j|j} | j \in S_i; j \neq (i-1)p\}$ via:

$$\widehat{Z}_{j|j} := \widehat{Z}_{j|j}(\widehat{Z}_{(i-1)p|(i-1)p}, \widehat{Z}_I) = \widehat{Z}_{j|(i-1)p} + \overline{K}_j[\overline{Y}_j - \tilde{h}(\widehat{Z}_{j|(i-1)p}, t_j)]. \tag{50}$$

Now the conditionally linearized estimates and the optimal estimates would become instantaneously identical at $\{t_j | j \in S_i; j \neq (i-1)p\}$ if and only if zeros of the Eq. (31) can be found such that

$$\widehat{Z}_{j|j} = \widehat{Z}_{j|j} \quad \text{s.t. } \{j \in S_i; j \neq (i-1)p\}. \tag{51}$$

In other words, the above constraint condition is automatically satisfied if p sets of $(2n+p)$ dimensional real and physically admissible vector root of the following $(2n+l)p$ dimensional transcendental equation can be found:

$$\widehat{Z}_{j|j} - \widehat{Z}_{j|(i-1)p} + \overline{K}_j[\overline{Y}_j - \tilde{h}(\widehat{Z}_{j|(i-1)p}, t_j)] = 0 \tag{52}$$

s.t. $\{j \in S_i; j \neq (i-1)p\}$.

In order to avoid taking derivatives of the vector functions at all stages, we have presently used fixed-point iteration in preference to the Newton–Raphson method.

7. Once the true estimate is obtained as the solution of Eq. (52), the associated error covariance matrix $P_{ip|ip}$ associated with the new state at the last point of discretization in I_i can be estimated via

$$P_{ip|ip} = [I - K_{ip}\Psi_H]P_{ip|ip}[I - K_{ip}\Psi_H]^T + K_{ip}R_{ip}K_{ip}^T, \tag{53}$$

where the quantities (without bar) are computed by substituting in the corresponding quantities (with bar) the true value of the estimate obtained from Eq. (53).

5. Numerical examples

Numerical illustrations on single and multi-DOF systems are provided in this section. The synthetic measurement data have been generated on a computer by the integrating the process SDE using the stochastic Heun’s method. With these measurements and the associated system models, the problem of parameter identification has been taken up by the three kinds of filters, viz. the conventional EKF, the LTL-based EKF and the MTrL-based EKF. While the first problem involving a hardening Duffing oscillator is extensively studied for a number of sub-cases, a hysteretic as well as an MDOF oscillator have been incorporated to demonstrate the universality of the proposed methods and its extensibility to problems of higher dimensionality. Henceforth, unless mentioned otherwise, all noises refer to zero-mean Gaussian white-noises. The desired standard deviations have been incorporated through appropriate enveloping factors. Wherever used the value of the weighing factor for the WGI scheme has been fixed at 100.

5.1. Example-1 – a hardening, single-well Duffing oscillator

Consider a hardening, single-well Duffing oscillator subjected to sinusoidal driving force corrupted by a white-Gaussian noise:

$$m\ddot{x} + c\dot{x} + kx + \alpha x^3 = F \cos(\lambda t) + \sigma_p \zeta(t). \tag{54}$$

The problem at hand consists of estimating the parameters $\{c, k, \alpha\}$ from a set of observations related directly to the states. Accordingly, these are declared as the additional states corresponding to an augmented system. The mass m has been set to 1 kg and hence does not appear in the expressions to follow. For the present problem, unless mentioned otherwise, the measured component (Y) will be the displacement (x) of the oscillator. Writing Eq. (54) in a state space form and including the measurement model, the mathematical model for the estimation problem is provided by the equations:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -z_4 x_1 - z_3 z_2 - z_5 z_1^3 + F \cos(\lambda t) + \sigma_p \zeta_p(t), \\ \dot{z}_3 &= 0 + \sigma_c \zeta_c(t), \\ \dot{z}_4 &= 0 + \sigma_k \zeta_k(t), \\ \dot{z}_5 &= 0 + \sigma_\alpha \zeta_\alpha(t) \end{aligned} \tag{55}$$

and

$$Y = z_1 + \sigma_Y \zeta_Y(t). \tag{56}$$

Here $z_1 = x$ and $z_2 = \dot{x}$ are the displacement and the velocity states, respectively, z_3, z_4, z_5 are the states in the extended state space corresponding to $\{c, k, \alpha\}$. σ_p is the enveloping factor for the original process noise $\zeta_p(t)$ whereas $\sigma_c, \sigma_k, \sigma_\alpha$ are the enveloping factors for the artificial evolution of the respective parameters. σ_Y is the enveloping factor for the measurement noise. As has been mentioned earlier, Eq. (56) is just a representative form and in practice the measurement can be represented as in Eq. (7). The specific forms of the conditionally linearized equations used for different cases will be indicated at appropriate places. However, the solution procedure by the conventional EKF is well known and will not be elaborated presently.

For the LTL-based EKF, over the time-interval $(t_{i-1}, t_i]$, the conditionally-linearized system has the following state space representation:

$$\begin{aligned} \begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \\ \dot{\bar{z}}_3 \\ \dot{\bar{z}}_4 \\ \dot{\bar{z}}_5 \end{Bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -z_2^* & -z_1^* & -\{z_1^*\}^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \end{Bmatrix} \\ &+ \begin{Bmatrix} 0 \\ F \cos(\lambda t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \sigma_p \zeta_p(t) \\ \sigma_c \zeta_c(t) \\ \sigma_k \zeta_k(t) \\ \sigma_\alpha \zeta_\alpha(t) \end{Bmatrix}, \end{aligned} \tag{57}$$

where z_1^* and z_2^* are the estimates of the states corresponding to still-unknown displacement and velocity, respectively at the i -th grid point. The measurement equation at the i -th time-point (linear in this case) following Eq. (7) is presently written as

$$\bar{Y}_i = [1 \ 0 \ 0 \ 0 \ 0] \begin{Bmatrix} \bar{z}_1^i \\ \bar{z}_2^i \\ \bar{z}_3^i \\ \bar{z}_4^i \\ \bar{z}_5^i \end{Bmatrix} + \sigma_Y \zeta_Y^i. \quad (58)$$

For the MTrL-based Kalman filter, the linearized system involves a known value of the estimates at the initial point and unknown estimates at multiple grid points of the time-interval corresponding to a single linearization step. For an MTrL system of order p , the set of unknown estimates comprises of the vector estimates at p distinct grid points. In the present case, for the time-interval $I_i = (t_{(i-1)p}, t_{(i-1)p+1}, \dots, t_{ip}]$, the linearized process equation may be written as

$$\begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \\ \dot{\bar{z}}_3 \\ \dot{\bar{z}}_4 \\ \dot{\bar{z}}_5 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\Psi_{z_2}(t) & -\Psi_{z_1}(t) & -\{\Psi_{z_1}(t)\}^3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \cos(\lambda t) \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \sigma_p \zeta(t) \\ \sigma_c \zeta(t) \\ \sigma_k \zeta(t) \\ \sigma_\alpha \zeta(t) \end{Bmatrix}, \quad (59)$$

where

$$\Psi_{z_1}(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}^1, \quad (60)$$

$$\Psi_{z_2}(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}^2, \quad (61)$$

where $P_k(t) = \prod_{j=(i-1)p}^{ip} \frac{t-t_j}{t_k-t_j}$ is the set of Lagrange interpolating polynomials and $\widehat{Z}_{k|k}^1$ and $\widehat{Z}_{k|k}^2$ denotes the first and second components of the augmented state vector. The set of measurement equations at the discrete grid points over I_i is presently represented as

$$\bar{Y}_j = [1 \ 0 \ 0 \ 0 \ 0] \begin{Bmatrix} \bar{z}_1^j \\ \bar{z}_2^j \\ \bar{z}_3^j \\ \bar{z}_4^j \\ \bar{z}_5^j \end{Bmatrix} + \sigma_Y \zeta_Y^j \quad \text{such that } \{j \in S; j \neq (i-1)p\}. \quad (62)$$

5.1.1. Results

Subcase-1: The simulation parameters are taken to be $m = 1$ kg, $k = 7$ N/m, $c = 0.9$ N s/m and $\alpha = 3$ N/m³. The test signal is sinusoidal with an amplitude $F = 4$ N and frequency $\lambda = 3.5$ rad/s. While the enveloping factor for the process noise σ_p has been taken to be 0.001 N, that for the evolution of the additional states has been taken as 0.00002 (units of $\sigma_c, \sigma_k, \sigma_\alpha$ are N s/m, N/m and N/m³, respectively). The standard deviation of the observation noise is about 5% of the root mean square value of the response, i.e., the approximate solution of the process equation via the stochastic Heun’s method ($\sigma_Y = 0.030$ m). The order of MTrL (p) used in this example is 2. Note that, owing to the fact that the strong solution of the process SDE is a semi-martingale [31] and therefore does not strictly admit a polynomial-type expansion, the order p has presently nothing to do with the formal order of accuracy of the MTrL-based EKF.

Figs. 1–6 show the results for a time-step of 0.01 s. For the MTrL-based filter the time-step over which linearization takes place is even higher, i.e., 0.02 s. The results show that the estimated parameter values are close to their reference values for all the three filters.

Subcase-2: Consider a forced Duffing oscillator in the chaotic regime [35]. The unknown parameter values have been set to $k = 39.4784$ N/m, $c = 1.5708$ N s/m and $\alpha = 39.4784$ N/m³. The test signal is sinusoidal with an amplitude $F = 1658.094$ N and frequency $\lambda = 2\pi$ rad/s. The standard deviation of the observation noise is about 5% of the root mean square value of the response, i.e., the approximate solution of the process equation via the stochastic Heun’s method ($\sigma_Y = 0.025$ m). All other quantities are the same as in subcase-1. The time-step used in this problem is 0.01 s. Figs. 7–9 show that among the three filters, only the LTL- and MTrL-based filter estimates are close to their reference values while estimate from the other filter is not so. However, for all the three filters, the estimate values are close to the reference values when the

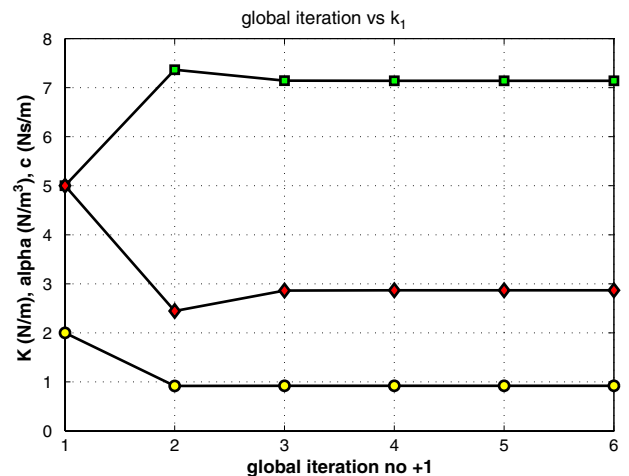


Fig. 1. Estimated parameter values in different global iterations for the Duffing oscillator of example-1 via EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

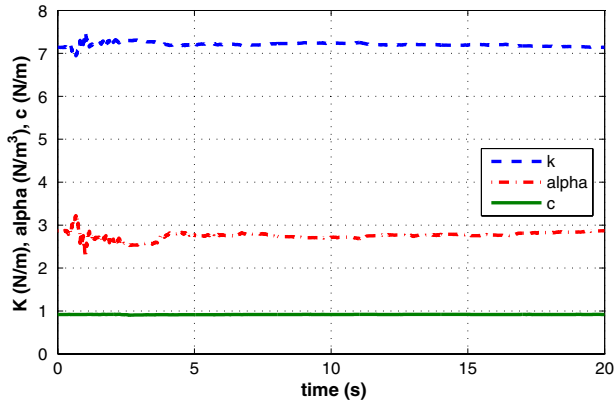


Fig. 2. Time evolution of estimated parameters in the last global iteration for the Duffing oscillator of example-1 via EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

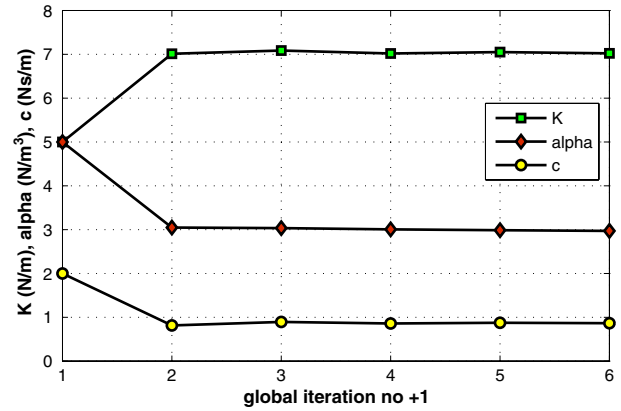


Fig. 5. Estimated parameter values in different global iterations for the Duffing oscillator of example-1 problem via MTrL-based EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

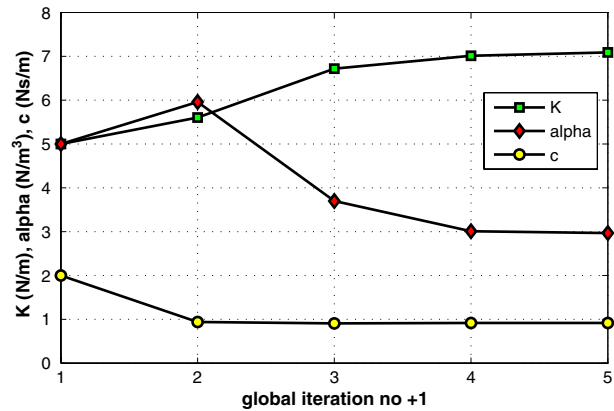


Fig. 3. Estimated parameter values in different global iterations for the Duffing oscillator of example-1 problem via LTL-based EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

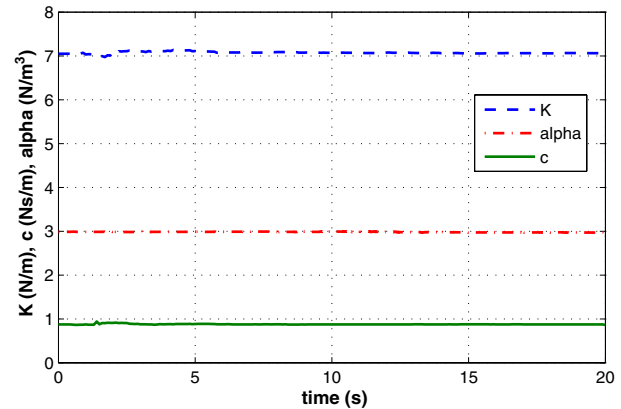


Fig. 6. Time evolution of estimated parameters in the last global iteration for the Duffing oscillator of example-1 via MTrL-based EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

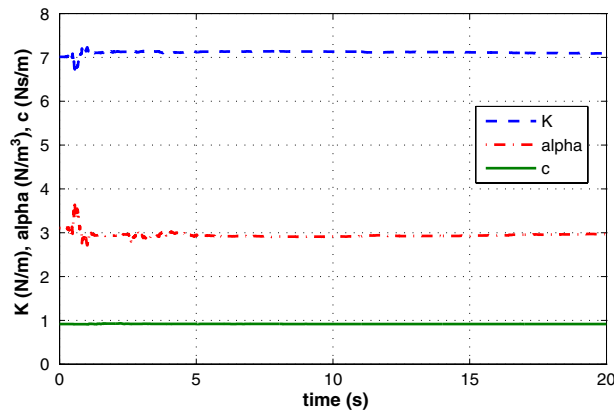


Fig. 4. Time evolution of estimated parameters in the last global iteration for the Duffing oscillator of example-1 via LTL-based EKF for subcase-1 ($h = 0.01$ s), reference values of parameters: $c = 0.9$ N s/m; $k = 7$ N/m; $\alpha = 3$ N/m³.

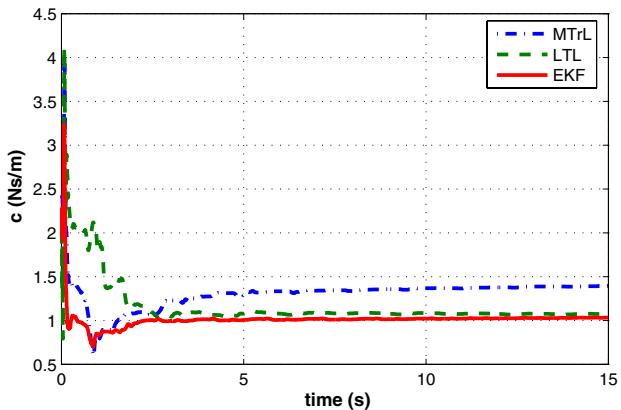


Fig. 7. Time evolution of damping parameter c for the Duffing oscillator of example-1 for subcase-2 ($h = 0.01$ s), reference value of $c = 1.5708$ N s/m.

time-step used is lower e.g., $h = 0.005$ s, 0.002 s or 0.001 s. Figs. 10–12 show the effect of time-step on the estimated

value of the stiffness parameter (k). From Fig. 13 it could be observed that for smaller values of the time-step, the

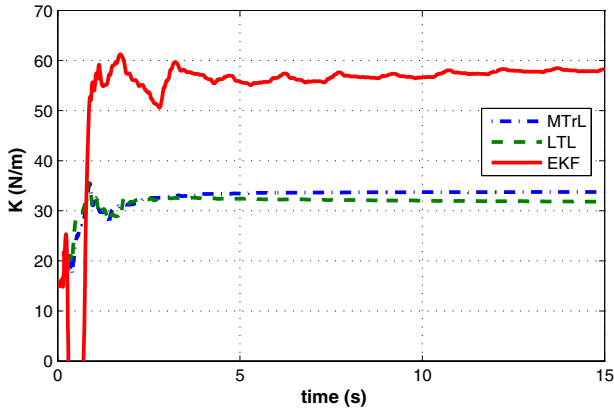


Fig. 8. Time evolution of stiffness parameter k for the Duffing oscillator of example-1 for subcase-2 ($h = 0.01$ s), reference value of $k = 39.4784$ N s/m.

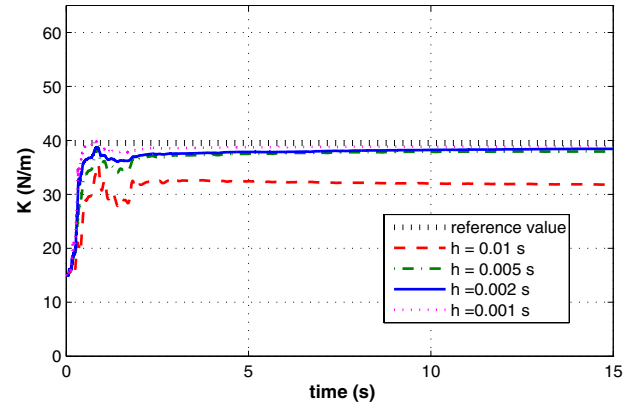


Fig. 11. Time evolution of estimated parameter k for the Duffing oscillator of example-1 via LTL-based EKF for subcase-2 for various time-steps, reference value: 39.4784 N/m.

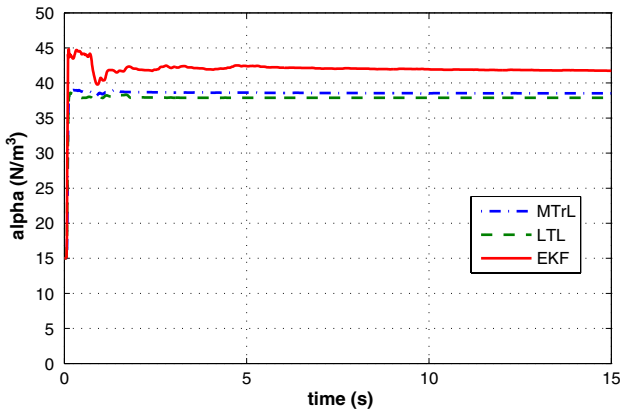


Fig. 9. Time evolution of the hardening parameter α for the Duffing oscillator of example-1 for subcase-2 ($h = 0.01$ s), reference value of $\alpha = 39.4784$ N/m³.

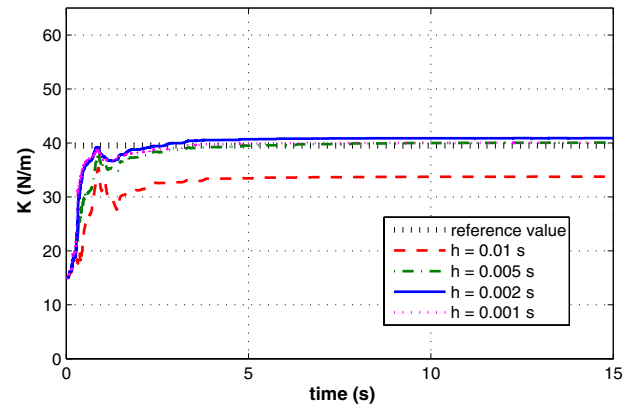


Fig. 12. Time evolution of estimated parameter k for the Duffing oscillator of example-1 via MTrL-based EKF for subcase-2 for various time-steps, reference value: 39.4784 N/m.

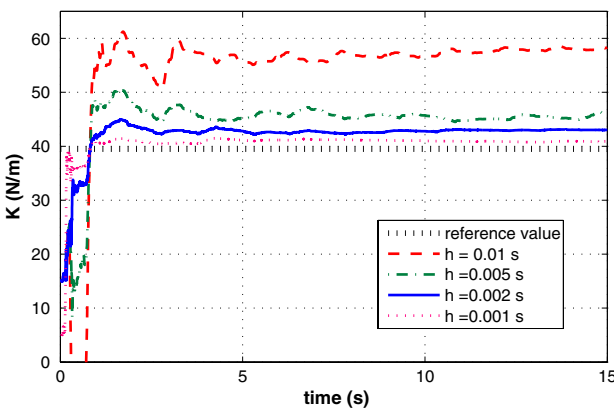


Fig. 10. Time evolution of estimated parameter k for the Duffing oscillator of example-1 via EKF for subcase-2 for various time-steps, reference value: 39.4784 N/m.

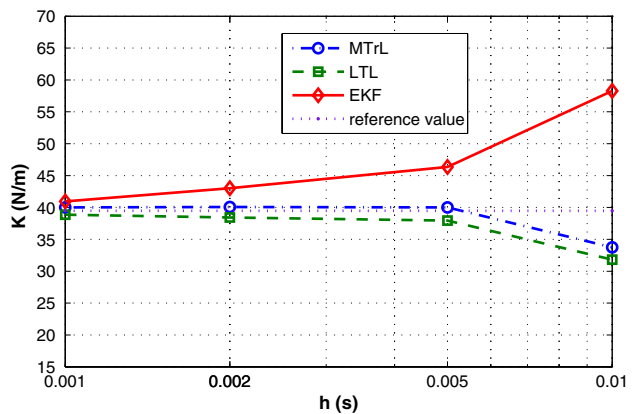


Fig. 13. Converged estimate of k for different time-steps for the Duffing oscillator of example-1 via EKF, LTL-based EKF and MTrL-based EKF for subcase-2.

estimates of k as per the three methods lie close to the reference value. As the step-size is increased from 0.001 s to 0.005 s, the results of the LTL- and MTrL-based filters remain insensitive to the changes in h , while the EKF esti-

mates move away from the reference value. Beyond $h = 0.005$ s, the EKF results deteriorate rapidly, while the LTL- and MTrL-based filters perform relatively better. Fig. 14 shows the time-evolution of the variance associated

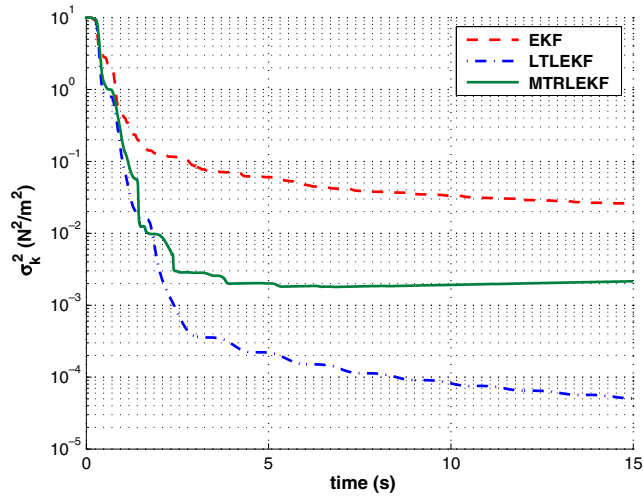


Fig. 14. Time evolution of the variance of the stiffness parameter k for the Duffing oscillator of example-1 via EKF, LTL-based EKF and MTrL-based EKF for $h = 0.01$ s in subcase-2.

with the estimates of the stiffness parameter k for $h = 0.01$ s. As has been mentioned earlier, the actual variability of the identified parameter is not captured accurately within the framework of extended Kalman filtering. Though a further investigation into the real nature of the uncertainty associated with the identified parameters is pending, it can be observed that of the three filtering schemes studied, the results from MTrL-based EKF show faster numerical convergence as the measurements are assimilated sequentially. Quantitatively, however, the predicted variances from the three methods show notably large differences. This clearly points towards the need for further research into the convergence properties of the algorithms.

Subcase-3: Although most of the problems considered in the present work do have essentially time-invariant parameters, a very limited study for a time-varying parameter

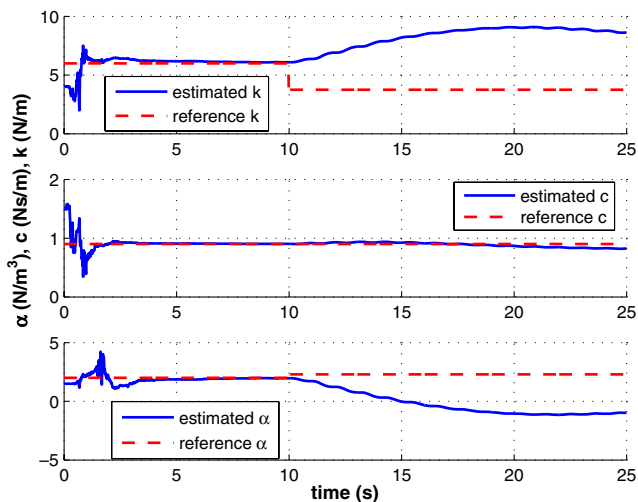


Fig. 15. Time evolution of estimated parameters, c ; k ; α , for the Duffing oscillator of example-1 via EKF for subcase-3 ($h = 0.01$ s).

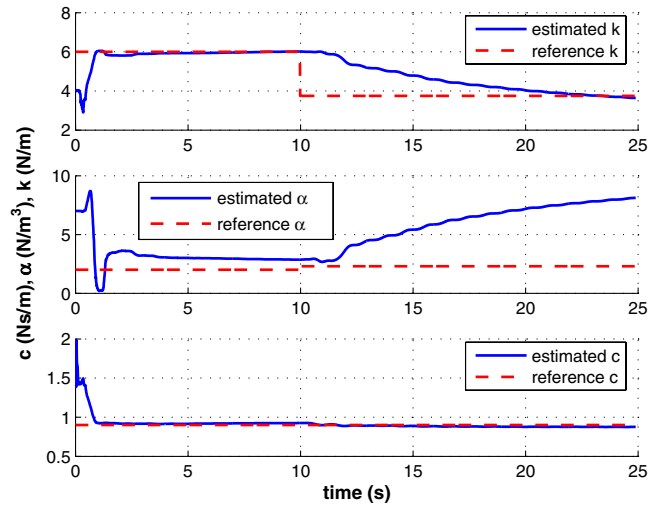


Fig. 16. Time evolution of estimated parameters, c ; k ; α , for the Duffing oscillator of example-1 via LTL-based EKF for subcase-3 ($h = 0.01$ s).

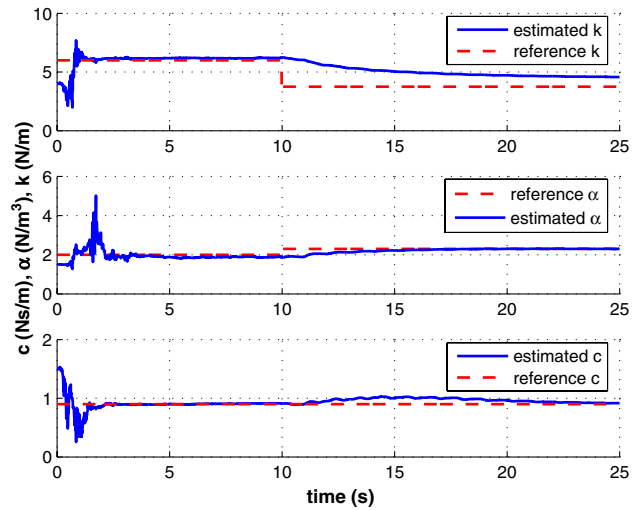


Fig. 17. Time evolution of estimated parameters, c ; k ; α , for the Duffing oscillator of example-1 via MTrL-based EKF for subcase-3 ($h = 0.01$ s).

estimation will now be taken up. Here the unknown parameters k and α are chosen such that there is a sudden change in the parameter values after some time in the observation period. Thus, during the first 10 s of simulation, the values of k and α are set to 6 N/m and 2 N/m³, respectively. At $t = 10$ s, a discontinuous change (jump) of these two parameters are assumed to happen and the new values of k and α are set to 3.75 N/m and 2.3 N/m³, respectively and the latter values remain unchanged till the end of the observation period (i.e., 25 s). The value of all other simulation quantities are taken to be same as in *subcase-1*. The time-step used is 0.01 s. Figs. 15–17 report time-evolutions of the parameters. Only the MTrL-based EKF is successful in tracking the temporal variation of parameters. We emphasize that no adaptive tracking techniques [48] have been incorporated in any of the three filters.

5.2. One-and-a-half DOF oscillator with nonlinear hysteretic damping

In the current example, we consider a one-and-a-half DOF oscillator with the Bouc–Wen hysteretic model. Hysteresis is a mechanical phenomenon which is useful to model structures under earthquake loads. Hence it has been studied widely: for instance the recent work by Ghahem and Ferro [11] which considers the identification of a structure with hysteretic behavior modeled by the Bouc–Wen hysteretic model using the ensemble Kalman filter. The governing equation of motion is given by

$$m\ddot{x} + 2c\dot{x} + k[\alpha x + (1 - \alpha)r] = F \cos(\lambda t) + \sigma_p \xi_p(t), \tag{63}$$

$$\dot{r} = A\dot{x} - \beta\dot{x}|r|^n - \gamma|\dot{x}|r|^{n-1} + \sigma_H \xi_H(t). \tag{64}$$

Here m is the mass of the system (set to unity in the present case), α is a scalar parameter and r is the restoring force. The present problem consists of estimating the hysteretic parameters A, β, γ, n is assumed to be known in the present case. The measured quantities are the displacement and velocity histories of the oscillator. Again declaring these unknown parameters as additional states of the system, the extended state space equations corresponding to the process and the measurement models may be written as

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -\alpha k z_1 - 2c z_2 - (1 - \alpha)k z_3 + F \cos(\lambda t) + \sigma_p \xi(t), \\ \dot{z}_3 &= z_4 z_2 - z_6 z_2 |z_3|^n - z_5 |z_2 |z_3|^{n-1} + \sigma_H \xi_H(t), \\ \dot{z}_4 &= 0 + \sigma_A \xi_A(t), \\ \dot{z}_5 &= 0 + \sigma_\gamma \xi_\gamma(t), \\ \dot{z}_6 &= 0 + \sigma_\beta \xi_\beta(t) \end{aligned} \tag{65}$$

and

$$Y = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} + \begin{Bmatrix} \sigma_{Y_1} \xi_{Y_1}(t) \\ \sigma_{Y_2} \xi_{Y_2}(t) \end{Bmatrix}, \tag{66}$$

where $z_1 = x$ and $z_2 = \dot{x}$ are the displacement and the velocity states, respectively, z_4, z_5, z_6 are the states in the extended state space corresponding to $\{A, \gamma, \beta\}$. σ_p and σ_H are the enveloping factors for the original process noises $\xi_p(t)$ and $\xi_H(t)$, respectively whereas $\sigma_A, \sigma_\gamma, \sigma_\beta$ are the enveloping factors for the artificial evolution of the respective parameters. For the LTL-based EKF, over the time-interval $(t_{i-1}, t_i]$, the conditionally linearized system has the following state space representation:

$$\begin{aligned} q \begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \\ \dot{\bar{z}}_3 \\ \dot{\bar{z}}_4 \\ \dot{\bar{z}}_5 \\ \dot{\bar{z}}_6 \end{Bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha k & -2c & (\alpha - 1)k & 0 & 0 & 0 \\ 0 & 0 & 0 & z_2^* & -|z_2^* |z_3^*|^{n-1} & -z_2^* |z_3^*|^n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\times \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \\ \bar{z}_6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \cos(\lambda t) \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \sigma_p \xi_p(t) \\ \sigma_H \xi_H(t) \\ \sigma_A \xi_A(t) \\ \sigma_\gamma \xi_\gamma(t) \\ \sigma_\beta \xi_\beta(t) \end{Bmatrix}. \end{aligned}$$

In the above equations, z_1^*, z_2^* and z_3^* are the estimates of states corresponding to displacement, velocity and restoring force, respectively at the i -th grid point and they are still-unknown. The measurement equation (linear in this case), following Eq. (7), at the i -th time-point may be written as

$$\begin{Bmatrix} \bar{Y}_i^1 \\ \bar{Y}_i^2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1^i \\ \bar{z}_2^i \\ \vdots \\ \bar{z}_6^i \end{Bmatrix} + \begin{Bmatrix} \sigma_{Y_1} \xi_{Y_1}^i \\ \sigma_{Y_2} \xi_{Y_2}^i \end{Bmatrix}. \tag{67}$$

For the MTrL-based Kalman filter, the linearized system involves the known values of the estimates at the initial time-point of the time-interval for one linearization step and unknown estimates at multiple time-points over the same interval. For an MTrL system of order p (not the formal order of accuracy of the filter), the set of unknown estimates comprises the vector estimates at p distinct grid points. In the present case, for the time-interval $I_i = (t_{(i-1)p}, t_{(i-1)p+1}, \dots, t_{ip}]$, the linearized process equation can be written as

$$\begin{aligned} &\begin{Bmatrix} \dot{\bar{z}}_1 \\ \dot{\bar{z}}_2 \\ \dot{\bar{z}}_3 \\ \dot{\bar{z}}_4 \\ \dot{\bar{z}}_5 \\ \dot{\bar{z}}_6 \end{Bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\alpha k & -2c & (\alpha - 1)k & 0 & 0 & 0 \\ 0 & 0 & 0 & -\Psi_{z_2}(t) & -|\Psi_{z_2}(t)|\Psi_{z_3}(t)|\Psi_{z_2}(t)|^{n-1} & -\Psi_{z_2}(t)|\Psi_{z_3}(t)|^n \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\times \begin{Bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \\ \bar{z}_5 \\ \bar{z}_6 \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \cos(\lambda t) \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0 \\ \sigma_p \xi_p(t) \\ \sigma_H \xi_H(t) \\ \sigma_A \xi_A(t) \\ \sigma_\gamma \xi_\gamma(t) \\ \sigma_\beta \xi_\beta(t) \end{Bmatrix}, \end{aligned} \tag{68}$$

where

$$\Psi_{z_1}(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}^1 \tag{69}$$

and

$$\Psi_{z_2}(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}^2 \tag{70}$$

and

$$\Psi_{z_3}(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \widehat{Z}_{k|k}^3, \tag{71}$$

where $P_k(t) = \prod_{j=(i-1)p}^{ip} \frac{t-t_j}{t_k-t_j}$ is the set of Lagrange interpolating polynomials and $\widehat{Z}_{k|k}^1, \widehat{Z}_{k|k}^2, \widehat{Z}_{k|k}^3$ are respectively the

first, second and third components of the extended state vector. The set of measurement equations at the discrete time-points over I_i can be represented as

$$\begin{Bmatrix} \bar{Y}_j^1 \\ \bar{Y}_j^2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1^j \\ \bar{z}_2^j \\ \vdots \\ \bar{z}_6^j \end{Bmatrix} + \begin{Bmatrix} \sigma_{Y_1} \zeta_{Y_1}^j \\ \sigma_{Y_2} \zeta_{Y_2}^j \end{Bmatrix} \text{ such that } \{j \in S_i; j \neq (i-1)p\}. \tag{72}$$

5.2.1. Results

The simulation parameters have been taken to be $m = 1 \text{ kg}$, $k = 7 \text{ N/m}$, $c = 0.02 \text{ N s/m}$, $\alpha = 0.25$ and $n = 2$. The test signal is sinusoidal with an amplitude $F = 4 \text{ N}$ and frequency $\lambda = 3 \text{ rad/s}$. σ_p is taken as 0.02 N and σ_H is also taken as 0.02 m/s , while the enveloping factors for the evolution of the additional states are each equal to 0.00002 (σ_A is non-dimensional while units of both σ_γ , σ_β are m^{-2}). The values of the parameters to be estimated have been taken to be $A = 1$, $\beta = 0.5 \text{ m}^{-2}$ and $\gamma = 0.5 \text{ m}^{-2}$. The standard deviations of the observation noises is about 3% of the root mean square value of the responses, i.e., the approximate solution of the process equation via the stochastic Heun’s method ($\sigma_{Y_1} = 0.008 \text{ m}$ and $\sigma_{Y_2} = 0.025 \text{ m/s}$). The order of MTrL (p) used in this example is 2.

All the three filtering algorithms have been used to solve for the unknown parameters with a time-step of 0.01 s . The converged estimates for all the three filters are close to their reference values. Figs. 18–23 show the results of the parameter estimation problem by the three filters.

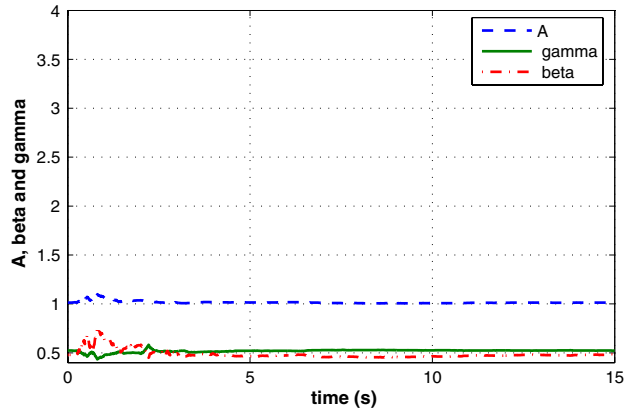


Fig. 19. Time evolution of estimated parameters in the last global iteration for the hysteretic oscillator via EKF of example-2, $h = 0.01 \text{ s}$, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

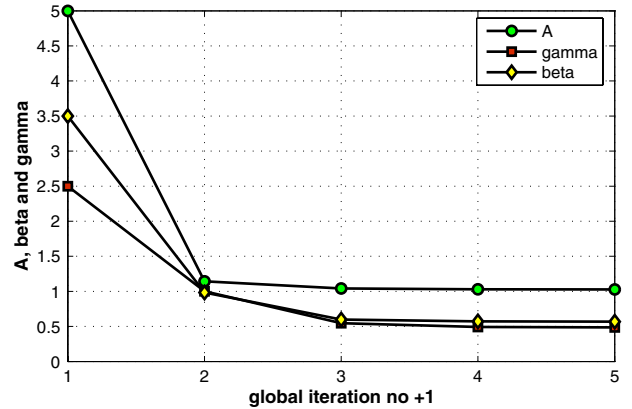


Fig. 20. Estimated parameter values in different global iterations for the hysteretic oscillator via LTL-based EKF of example-2, $h = 0.01 \text{ s}$, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

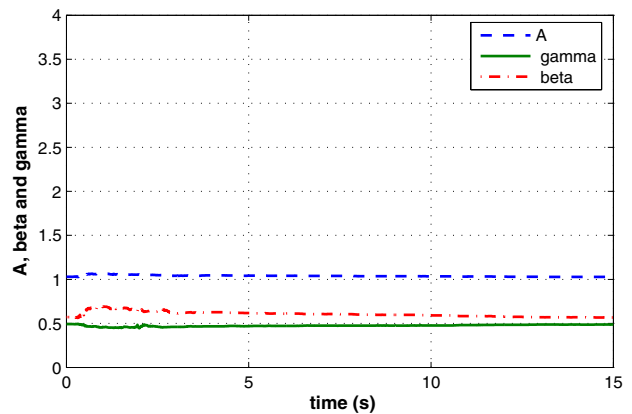


Fig. 21. Time evolution of estimated parameters in the last global iteration for the hysteretic oscillator via LTL-based EKF of example-2, $h = 0.01 \text{ s}$, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

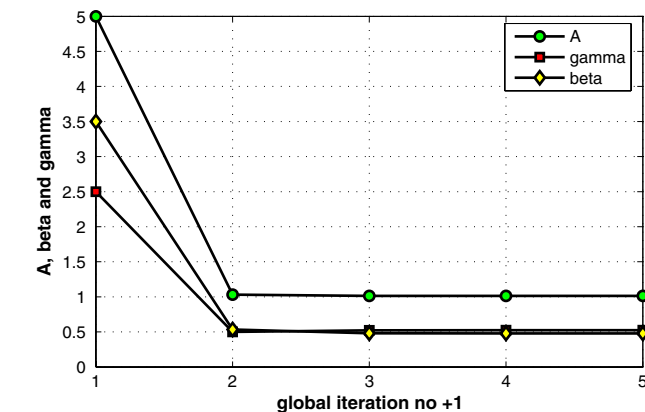


Fig. 18. Estimated parameter values in different global iterations for the hysteretic oscillator via EKF of example-2, $h = 0.01 \text{ s}$, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

5.3. Planar three-story linear shear building

Consider an idealized planar three DOF shear building system with known masses (m_1, m_2, m_3). The inter-story

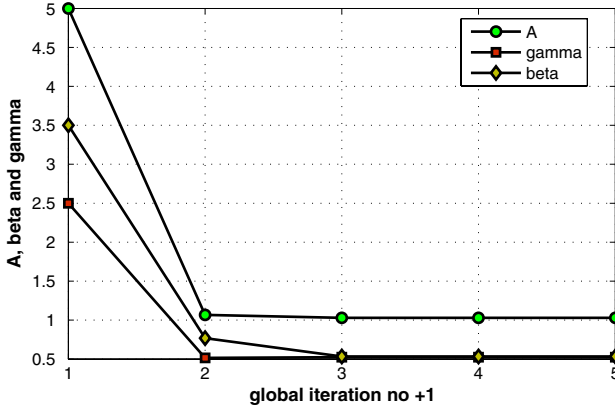


Fig. 22. Estimated parameter values in different global iterations for the hysteretic MTrL-based EKF of example-2, $h = 0.01$ s, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

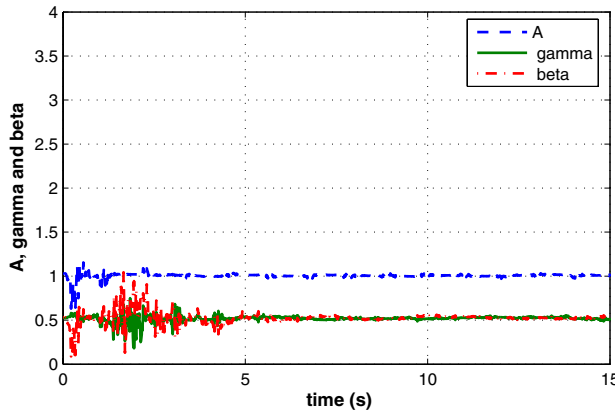


Fig. 23. Time evolution of estimated parameters in the last global iteration for the hysteretic oscillator via MTrL-based EKF of example-2, $h = 0.01$ s, reference values of parameters: $A = 1$; $\beta = 0.5$; $\gamma = 0.5$.

stiffnesses (k_1, k_2, k_3) are assumed to be unknown along with the inter-story viscous damping coefficients c_1, c_2, c_3 [46,33]. The linear equation of motion of this system subject to floor level excitations $F(t)$ is given by

$$M\ddot{x} + C\dot{x} + Kx = F(t) + \Xi_p(t). \quad (73)$$

Here M, C, K are the mass, damping and stiffness matrices, respectively. $X_1 := x = \{x_1, x_2, x_3\}^T$ is the displacement vector, $X_2 := \dot{x} = \{\dot{x}_1, \dot{x}_2, \dot{x}_3\}^T$ is the velocity vector, $\Xi_p(t)$ is the process noise vector (3×1), Z denotes the vector corresponding to the augmented state space and its first 6 elements are obtained as $\{z_1 \ z_2 \ \dots \ z_6\}^T = \{X_1^T, X_2^T\}^T$

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 + k_3 \end{bmatrix},$$

$$C = \begin{bmatrix} c_1 & -c_1 & 0 \\ c_1 & c_1 + c_2 & -c_2 \\ 0 & -c_2 & c_2 + c_3 \end{bmatrix}.$$

Declaring $k_1, k_2, k_3, c_1, c_2, c_3$ as additional states $\{z_7, z_8, z_9, z_{10}, z_{11}, z_{12}\}^T$, the extended state space equations for the identification model can be written as

$$\begin{cases} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \\ \dot{z}_5 \\ \dot{z}_6 \\ \{\dot{z}_p\}_{(6 \times 1)} \end{cases} = \begin{bmatrix} & z_4 \\ & z_5 \\ & z_6 \\ -M^{-1}K_Z \{z_1 \ z_2 \ z_3\}^T - M^{-1}\{C_Z\} \{z_4 \ z_5 \ z_6\}^T \\ & [0]_{(6 \times 6)} \end{bmatrix}$$

$$+ \begin{cases} 0 \\ 0 \\ 0 \\ -M^{-1}F \\ \{0\}_{(6 \times 1)} \end{cases} + \begin{cases} 0 \\ 0 \\ 0 \\ \sigma_p \zeta_p(t) \\ \{\sigma_p \zeta_p(t)\}_{(6 \times 1)} \end{cases}. \quad (74)$$

Here K_Z and C_Z are obtained from the stiffness and damping matrices, respectively by replacing the element stiffnesses and dampings by the corresponding additional states $z_p = \{z_7, z_8, z_9, z_{10}, z_{11}, z_{12}\}^T$; and $\{\sigma_p \zeta_p(t)\}_{(6 \times 1)}$ denotes the enveloping factors associated with the artificial evolution of parameters.

For notational convenience we compress Eq. (74) and rewrite it following Eq. (5) as

$$\dot{z} = g(z) + f(t) + \zeta(t).$$

It is evident that in spite of the linear mechanics of the problem, the (augmented) identification model is nonlinear in the states of the system. As before, the linearization procedure for the two proposed filters will be indicated. In the LTL-based EKF, the coefficient matrix of the linearized process equation over the time-interval $(t_{i-1}, t_i]$ following Eq. (20) can be written as

$$B(\widehat{Z}_i, t_i) = \begin{bmatrix} [B_1]_{3 \times 6} & [B_2]_{3 \times 6} \\ [B_3]_{3 \times 6} & [B_4]_{3 \times 6} \\ [B_5]_{6 \times 6} & [B_6]_{6 \times 6} \end{bmatrix},$$

$$\text{where } [B_1] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$[B_3] = \begin{bmatrix} -\alpha \frac{(\sigma_7^*)}{m_1} & \alpha \frac{(\sigma_7^*)}{m_1} & 0 & -\alpha \frac{(\sigma_{10}^*)}{m_1} & \alpha \frac{(\sigma_{10}^*)}{m_1} & 0 \\ \alpha \frac{(\sigma_7^*)}{m_2} & -\alpha \frac{(\sigma_7^* + \sigma_8^*)}{m_2} & \alpha \frac{(\sigma_8^*)}{m_2} & \alpha \frac{(\sigma_{10}^*)}{m_2} & -\alpha \frac{(\sigma_{10}^* + \sigma_{11}^*)}{m_2} & \alpha \frac{(\sigma_{11}^*)}{m_2} \\ 0 & \alpha \frac{(\sigma_8^*)}{m_3} & -\alpha \frac{(\sigma_8^* + \sigma_9^*)}{m_3} & 0 & \alpha \frac{(\sigma_{11}^*)}{m_3} & -\alpha \frac{(\sigma_{11}^* + \sigma_{12}^*)}{m_3} \end{bmatrix}, \quad (75)$$

$$[B_4] = \begin{bmatrix} \beta \frac{(\sigma_5^* - z_4^*)}{m_1} & 0 & 0 & \beta \frac{(\sigma_5^* - z_4^*)}{m_1} & 0 & 0 \\ \beta \frac{(\sigma_3^* - z_2^*)}{m_2} & \beta \frac{(\sigma_3^* - z_2^*)}{m_2} & 0 & \beta \frac{(\sigma_4^* - z_5^*)}{m_2} & \beta \frac{(\sigma_6^* - z_5^*)}{m_2} & 0 \\ 0 & \beta \frac{(\sigma_2^* - z_3^*)}{m_3} & -\beta \frac{z_3^*}{m_3} & 0 & \beta \frac{(\sigma_8^* - z_6^*)}{m_3} & -\beta \frac{z_6^*}{m_3} \end{bmatrix} \quad (76)$$

and B_2 is a (3×6) matrix of zeros while B_5 and B_6 are (6×6) matrices of zeros. Also $z_1^*, z_2^*, \dots, z_{12}^*$ are the

unknown estimates at the i -th grid point and $\alpha, \beta \in [0, 1]$ are real parameter such that $\alpha + \beta = 1$. These parameters, somewhat similar to those in the Newmark method of numerical integration of ODEs, allow a splitting of terms, as indicated above and below, of the system coefficient matrix corresponding to the linearized (augmented) process equation. In this way, the linearized coefficient matrix as well as the FSM may be made to be more populated thereby increasing the accuracy as well as convergence of the modified EKF algorithm. The measured quantities in this case are the floor displacements and the measurement equation at the i -th time instant can be represented as

$$\bar{Y}_i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1^i \\ \bar{z}_2^i \\ \vdots \\ \bar{z}_{12}^i \end{Bmatrix} + \begin{Bmatrix} \sigma_{Y_1} \zeta_i^1 \\ \sigma_{Y_2} \zeta_i^2 \\ \sigma_{Y_3} \zeta_i^3 \end{Bmatrix} \quad (77)$$

For an MTrL-based EKF of order p , the set of unknown estimates comprises the vector estimates at p distinct grid points. In the present case, for the time interval $I_i = (t_{(i-1)p}, t_{(i-1)p+1}, \dots, t_{ip}]$, the coefficient matrix of the linearized process equation following (43) can be written as

$$\Psi(t) = \begin{bmatrix} [\Psi_{B_1}]_{3 \times 6} & [\Psi_{B_2}]_{3 \times 6} \\ [\Psi_{B_3}]_{3 \times 6} & [\Psi_{B_4}]_{3 \times 6} \\ [\Psi_{B_5}]_{6 \times 6} & [\Psi_{B_6}]_{6 \times 6} \end{bmatrix},$$

where $[\Psi_{B_1}] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$,

$$[\Psi_{B_3}] = \begin{bmatrix} -\alpha \frac{(\Psi_{z_7})}{m_1} & \alpha \frac{(\Psi_{z_7})}{m_1} & 0 & -\alpha \frac{(\Psi_{z_{10}})}{m_1} & \alpha \frac{(\Psi_{z_{10}})}{m_1} & 0 \\ \alpha \frac{(\Psi_{z_7})}{m_2} & -\alpha \frac{(\Psi_{z_7} + \Psi_{z_8})}{m_2} & \alpha \frac{(\Psi_{z_8})}{m_2} & \alpha \frac{(\Psi_{z_{10}})}{m_2} & -\alpha \frac{(\Psi_{z_{10}} + \Psi_{z_{11}})}{m_2} & \alpha \frac{(\Psi_{z_{11}})}{m_2} \\ 0 & \alpha \frac{(\Psi_{z_8})}{m_3} & -\alpha \frac{(\Psi_{z_8} + \Psi_{z_9})}{m_3} & 0 & \alpha \frac{(\Psi_{z_{11}})}{m_3} & -\alpha \frac{(\Psi_{z_{11}} + \Psi_{z_{12}})}{m_3} \end{bmatrix}, \quad (78)$$

$$[\Psi_{B_4}] = \begin{bmatrix} \beta \frac{(\Psi_{z_2} - \Psi_{z_1})}{m_1} & 0 & 0 & \beta \frac{(\Psi_{z_5} - \Psi_{z_4})}{m_1} & 0 & 0 \\ \beta \frac{(\Psi_{z_1} - \Psi_{z_2})}{m_2} & \beta \frac{(\Psi_{z_3} - \Psi_{z_2})}{m_2} & 0 & \beta \frac{(\Psi_{z_4} - \Psi_{z_5})}{m_2} & \beta \frac{(\Psi_{z_6} - \Psi_{z_5})}{m_2} & 0 \\ 0 & \beta \frac{(\Psi_{z_2} - \Psi_{z_3})}{m_3} & -\beta \frac{\Psi_{z_3}}{m_3} & 0 & \beta \frac{(\Psi_{z_5} - \Psi_{z_6})}{m_3} & -\beta \frac{\Psi_{z_6}}{m_3} \end{bmatrix}, \quad (79)$$

where Ψ_{B_2} is a (3×6) matrix of zeros while Ψ_{B_5} and Ψ_{B_6} are (6×6) matrices of zeros and $\Psi_z^l(t) = \sum_{k=(i-1)p}^{ip} P_k(t) \hat{Z}_{k|k}^l$ and $l = 1$ to 12 denoting the various states of the augmented system. Also, as before, α and $\beta \in [0, 1]$ are the splitting parameters such that $\alpha + \beta = 1$ and $P_k(t) = \prod_{j=(i-1)p}^{ip} \frac{t-t_j}{t_k-t_j}$ is the set of Lagrange interpolating polynomials. The set of measurement equations at the grid points over I_i can be represented as

$$\bar{Y}_j = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \bar{z}_1^j \\ \bar{z}_2^j \\ \vdots \\ \bar{z}_{12}^j \end{Bmatrix} + \begin{Bmatrix} \sigma_{Y_1} \zeta_j^1 \\ \sigma_{Y_2} \zeta_j^2 \\ \sigma_{Y_3} \zeta_j^3 \end{Bmatrix} \quad \text{such that } \{j \in S_i; j \neq (i-1)p\}. \quad (80)$$

5.3.1. Results

The simulation parameters are presently taken to be $k_1 = 5000$ N/m, $k_2 = 5000$ N/m, $k_3 = 5000$ N/m, $c_1 = 200$ N s/m, $c_2 = 200$ N s/m, $c_3 = 250$ N/m. The values of the masses, assumed to be known, are set to $m_1 = 100$ kg, $m_2 = 200$ kg and $m_3 = 300$ kg, respectively. Accordingly, the natural frequencies of the system to be identified are 0.3949 Hz, 0.9798 Hz and 1.5038 Hz. The test signal is in the form of a sample realization of a zero-mean white Gaussian noise vector with a standard deviation of

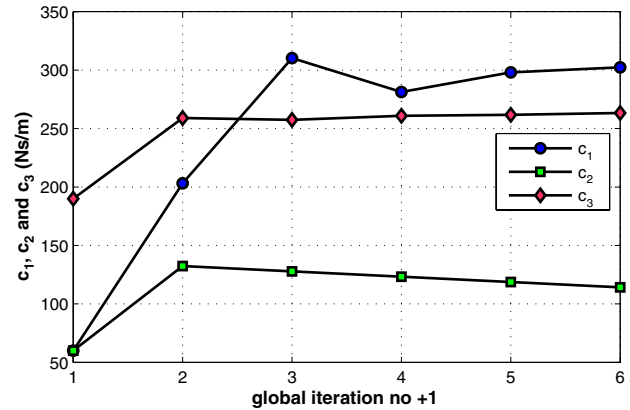


Fig. 24. Estimated damping values in different global iterations via EKF for example-3, $h = 0.1$ s, reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

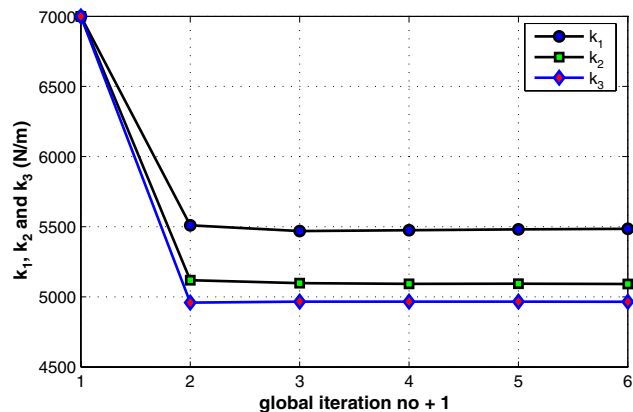


Fig. 25. Estimated stiffness values in different global iterations via EKF for example-3, $h = 0.1$ s, reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

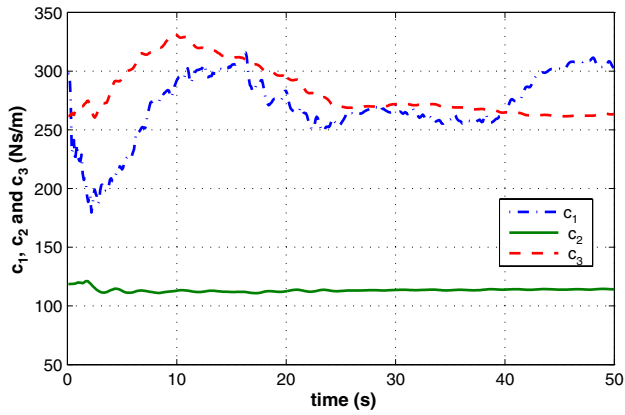


Fig. 26. Time evolution of estimated damping parameters in the last global iteration of example-3 via EKF ($h = 0.1$ s), reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

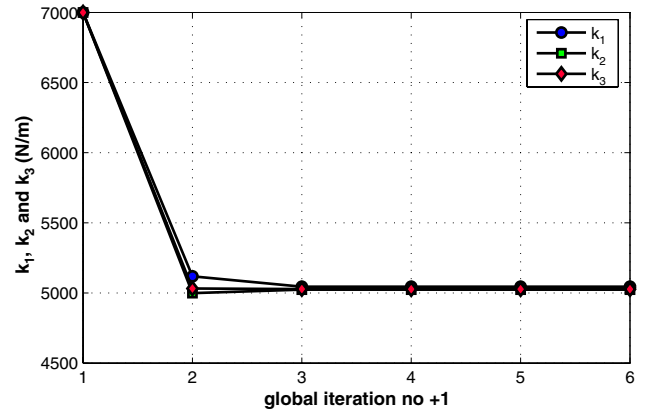


Fig. 29. Estimated stiffness values in different global iterations via LTL-based EKF for example-3, $h = 0.1$ s, reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

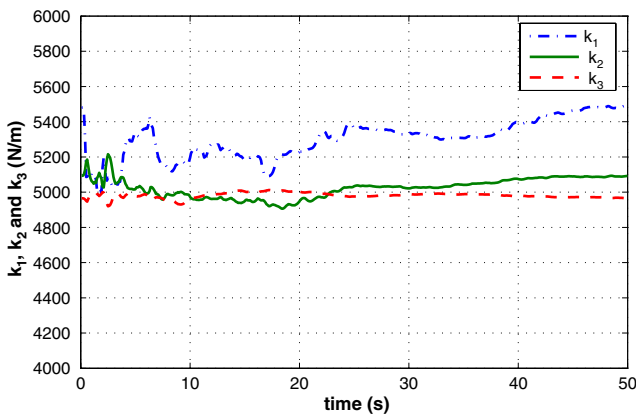


Fig. 27. Time evolution of estimated stiffness parameters in the last global iteration of example-3 via EKF ($h = 0.1$ s), reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

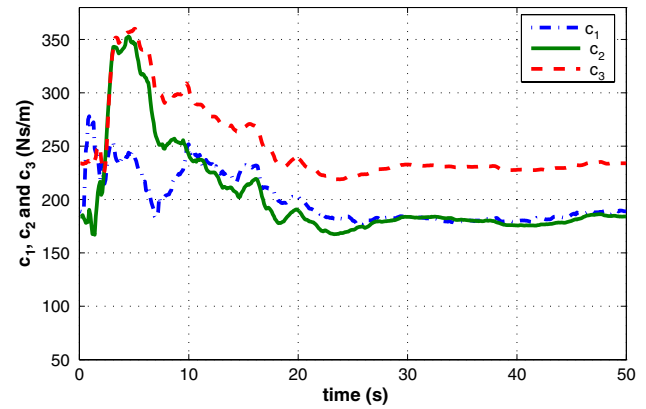


Fig. 30. Time evolution of estimated damping parameters in the last global iteration of example-3 via LTL-based EKF ($h = 0.1$ s), reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

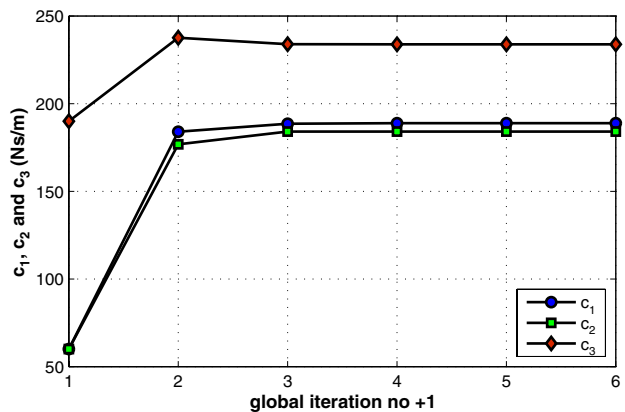


Fig. 28. Estimated damping values in different global iterations via LTL-based EKF for example-3, $h = 0.1$ s, reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

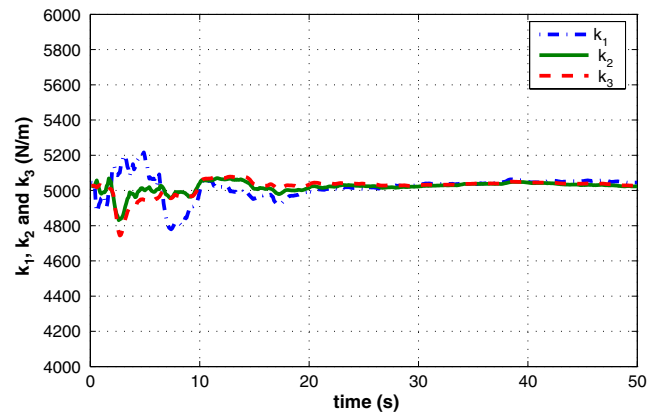


Fig. 31. Time evolution of estimated stiffness parameters in the last global iteration of example-3 via LTL-based EKF ($h = 0.1$ s), reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

5 N. The enveloping factor for each of the process noise components have been taken to be 0.001 N, while the enveloping factors for the evolution of the additional states

are taken as 0.00002 (units corresponding to damping states and stiffness states are N s/m and N/m, respectively). The standard deviation of the observation noise is about

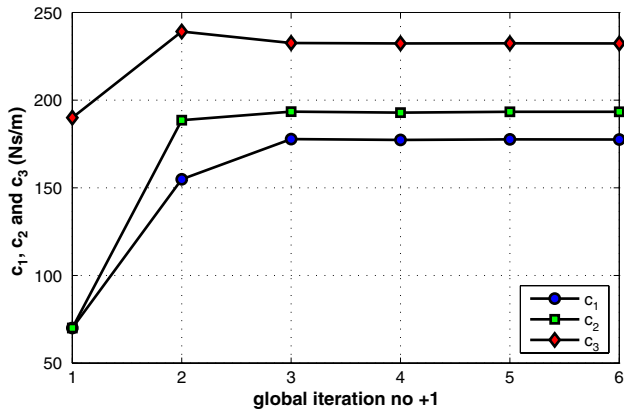


Fig. 32. Estimated damping values in different global iterations via MTrL-based EKF for example-3, $h = 0.1$ s, reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

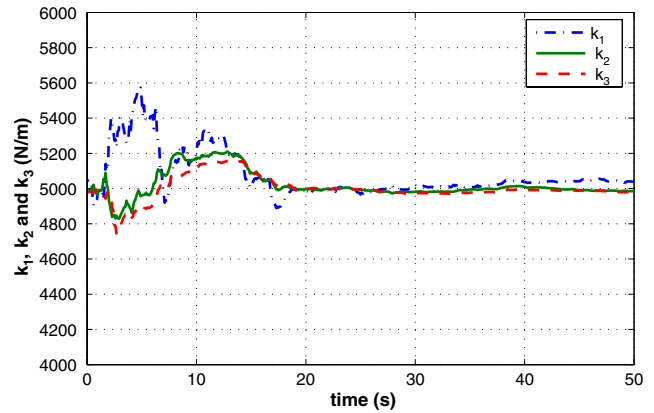


Fig. 35. Time evolution of estimated stiffness parameters in the last global iteration of example-3 via MTrL-based EKF ($h = 0.1$ s), reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

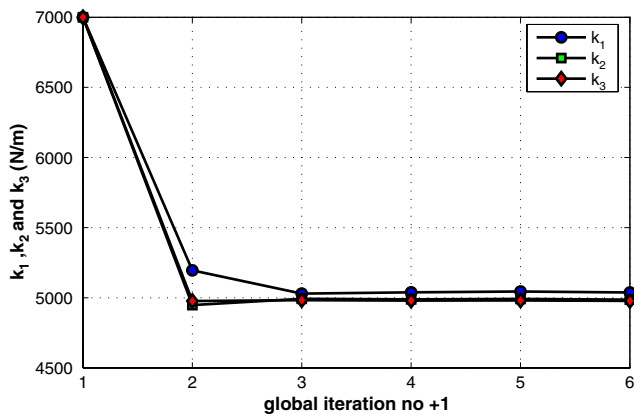


Fig. 33. Estimated stiffness values in different global iterations via MTrL-based EKF for example-3, $h = 0.1$ s, reference values of parameters: $k_1 = 5000$ N/m; $k_2 = 5000$ N/m; $k_3 = 5000$ N/m.

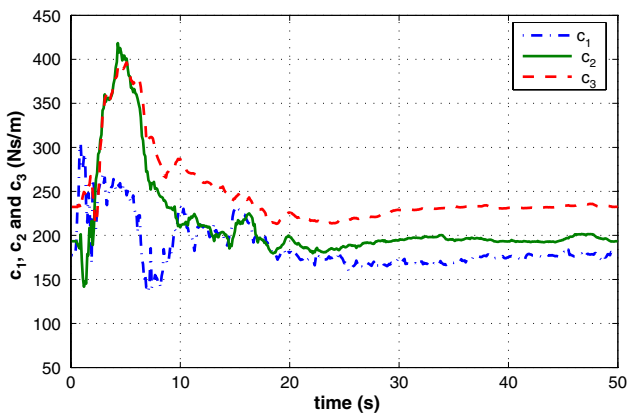


Fig. 34. Time evolution of estimated damping parameters in the last global iteration of example-3 via MTrL-based EKF ($h = 0.1$ s), reference values of parameters: $c_1 = 200$ N s/m; $c_2 = 200$ N s/m; $c_3 = 250$ N s/m.

values of α and β are taken to be 0.5, partly inspired by the commonly used values of these splitting parameters in the Newmark method of numerical integration.

All the three filtering algorithms have been used to solve for the unknown parameters with a time-step of 0.1 s. Though all the filters perform well in the estimation of the stiffness parameters, the damping parameter estimates with the conventional EKF are away from their reference values (as also reported in [44,48]) while the two proposed filters estimate the damping terms with significant accuracy. Figs. 24–35 shows the results of the three filters.

6. Conclusions

Two new forms of EKF are proposed with emphasis on nonlinear system identification of interest in structural engineering. The derivations of these filters are based on the concept of transversal linearization and, in the process, these filters do not need computing derivatives of the nonlinear vector field and they (in particular, the MTrL-based EKF) are found to be less sensitive to the choice of time-step sizes vis-à-vis the conventional EKF. Though a formal convergence analysis is still to be performed, a reasonably large class of numerical examples are solved to demonstrate the versatility and advantages of the proposed filtering algorithms in identification of model parameters. The filters perform satisfactorily in the chaotic regime and in the case of jumps in parameter values. The performance of the conventional EKF is known to be poor in the last case. Discontinuous changes in the temporal variations of parameters assume importance in view of the fact that, for health monitoring of structures, it is important to identify the damage when it occurs. Pending a more thorough investigation into convergence and confidence analysis, the superior performance of the MTrL-based EKF is tentatively attributable to a more accurate computation of the state transition matrix within a Lie algebra framework. However, it must as well be emphasized that the usage of polynomials (such as Lagrange interpolating polynomials)

3% of the root mean square value of the response, i.e., the approximate solution of the process equation via the stochastic Heun’s method ($\sigma_{Y_1}, \sigma_{Y_2}, \sigma_{Y_3} = 0.008$ m). The order of MTrL (p) used in this example is 2. In both the cases the

in achieving the MTrL-based linearization is not entirely consistent with the nature of solutions of the (augmented) process equations, which are presently SDEs under additive noises. Indeed, Taylor expansions of solutions of such SDEs contain fractional exponents of the time-step Δt . Accordingly, the formal order of accuracy of the MTrL-based EKF of so-called ‘order’ p will be less than Δt^p , an order that is readily achievable for deterministic dynamical systems. However, we emphasize that if the effect of imposed white noise is so predominant as to induce a bifurcation and thus to affect the dynamical characteristics of the process equation, then the very purpose of system identification is defeated and we possibly end up estimating a wrong value of the parameter. In other words, despite the noise terms in the process and measurement equations, it is generally important to maintain a correspondence with the deterministic dynamical response (corresponding to zero noise terms). Given that the MTrL is well-suited in accurately capturing the deterministic response, it should be reasonable to ignore the effects (in the sense of expectations) of the stochastic terms on the linearized expansion of the vector field as well in Magnus’ expansion. Note that the LTL- or MTrL-based schemes, as applied to SDEs, are drift-implicit methods. Owing to their superior stability characteristics, drift-implicit methods are widely used in the literature for strong or weak solutions of SDEs [5].

The effect of interpolating polynomials on the MTrL-based EKF needs further study. Particularly for problems with jumps in parameter values, it is anticipated that use of more sophisticated basis functions e.g., B-splines or NURBS should lead to much more accurate tracking of such jumps. Moreover, the proposed filters may be viewed as a first step toward explorations of implicit techniques that offer higher numerical accuracy and stability of estimation procedures. It can be mentioned in this context that techniques to tackle divergence problems, e.g., adaptive two-stage Kalman filters [20], have been proposed. Combining these techniques with the present form of EKF constitutes an interesting element of future study. Applicability of the present approach to very large dimensional problems is closely related to the applicability of the classical form of the EKF to such problems and the identifiability of the inverse problem at hand. Thus, the problem of coupling these methods as well as any related tools (such as the two-stage Kalman filter [1], fading Kalman filter [20] and even inverse approaches based on non-stochastic optimization and regularization) with commercial FE codes for larger problems is of immense practical importance. Studies on such issues have been taken up by the authors and the findings will be reported elsewhere.

Appendix A. An adaptation of the Magnus expansion for the MTrL-based EKF

For the purpose of deriving the state transition matrix of the linearized system, let us consider the following differential equation:

$$\dot{Y} = A(t)Y \quad (\text{A.1})$$

such that $t \geq 0$ and $Y(0) = Y_0 \in G$ where G is a Lie group, $A : \mathbb{R}^+ \rightarrow \mathfrak{g}$ is Lipschitz continuous and \mathfrak{g} is the Lie algebra of G . It is well known that solution of Eq. (A.1) stays on G for all $t \geq 0$. The important geometric structural features of the differential equation (A.1) should ideally be retained under numerical discretization or approximation. This issue is not explored further in this paper. However, it is interesting to note that [17] an appealing feature of a Magnus numerical method is that, whenever the exact solution of Eq. (A.1) evolves in a Lie Group, so does the numerical solution.

Recall that a Lie group is a differentiable manifold equipped with a group structure, which is continuous with respect to underlying topology of the manifold. The Lie algebra \mathfrak{g} is a tangent space of G . The tangent space consists of all $\gamma(t)$, where $\gamma(t) : \mathbb{R} \rightarrow G$ is a smooth curve lying on the manifold and $\gamma(0) = Id_m$, the identity of G . The Lie algebra \mathfrak{g} is a linear space closed under a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. This binary operation is bilinear, antisymmetric and subjected to the Jacobi identity, i.e.

$$\begin{aligned} [a, b] &= -[b, a], \\ [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= 0; \quad \text{where } a, b, c \in \mathfrak{g}. \end{aligned} \quad (\text{A.2})$$

Although the present discussion is based on *matrix Lie groups* only, the restriction to *matrix Lie groups* is not severe, since Ado’s theorem guarantees that any analytical group is locally isomorphic to a *matrix Lie groups* [45]. The binary operation of Lie algebra \mathfrak{g} is called the Lie bracket, which is given by $[a, b] = ab - ba$ where $a, b \in \mathfrak{g}$. The Lie algebra corresponding to matrix Lie groups are well known [32]. Thus, to summarize the internal operations in each space for instance, while we are allowed to proceed by multiplication in G , it is forbidden to multiply elements of the algebra \mathfrak{g} . However, the algebra \mathfrak{g} is a vector space and we can add and commute elements. The essential feature of \mathfrak{g} is that it can be mapped to G through the exponential map. Thus we are allowed to move from the algebra to the group. Given $A \in \mathfrak{g}$ and $t \in \mathbb{R}^+$, $e^{(At)} \in G$ is the flow generated by the infinitesimal generator A (see [32]).

The solution of Eq. (A.1) is well researched starting with the work of Magnus [28]. As shown by Magnus, the solution may be locally represented in the form

$$Y(t) = \exp(\Omega(t))Y_0, \quad (\text{A.3})$$

where $\Omega(t) : \mathbb{R}^+ \rightarrow \mathfrak{g}$ is an infinite sum of elements in \mathfrak{g} , known as the Magnus expansion. Ωt is expressible as a specific expansion in integrals of Lie bracket commutators. Iserles and Norsett [16] have proved the convergence of Magnus’ expansion. For instance, terms containing up to three integrals in Magnus’ expansion (the continuous analogue of Baker Housdorff formula) may be written as

$$\begin{aligned} \Omega(t) = & \int_{t_0}^t A(s) ds + \frac{1}{2} \int_{t_0}^t \left[A(s), \int_{t_0}^s A(s_1) ds_1 \right] ds \\ & + \frac{1}{4} \int_{t_0}^t \left[A(s), \int_{t_0}^s \left[A(s_1), \int_{t_0}^{s_1} A(s_2) ds_2 \right] ds_1 \right] ds \\ & + \frac{1}{12} \int_{t_0}^t \left[\left[A(s), \int_{t_0}^s A(s_1) ds_1 \right], \int_{t_0}^s A(s_1) ds_1 \right] ds + \dots \end{aligned} \quad (\text{A.4})$$

Now following Magnus' formula, the FSM or state transition matrix (STM) from time t_0 to t for Eq. (A.1) is given by

$$\Phi(t) = \exp(\Omega(t)). \quad (\text{A.5})$$

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