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Probability distribution of the eigenvalues of systems governed by the stochastic wave equation

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This paper studies the random eigenvalue problem of systems governed by the one dimensional wave equation. The mass and stiffness properties of the system are taken to vary spatially in a random manner. The probability distribution function of the natural frequencies is characterized in terms of the solution of a first order nonlinear stochastic differential equation. Analytical solutions are obtained based on the method of stochastic averaging. The effect of the mean and autocorrelation of the mass and stiffness processes and also the uncertainty in specifying the boundary conditions are included in the analysis. The theoretical predictions are compared with digital simulations.

INTRODUCTION

Uncertainties are unavoidable in the specification of mass and stiffness of engineering structures. Traditionally in random vibration studies these uncertainties are overlooked. If these uncertainties are to be taken into account in response analysis one faces many intricate questions. For instance, the system natural frequencies and mode shapes will be stochastic in nature and in a dynamical analysis information on the joint probability distributions of these quantities would be required. This problem is associated with the determination of the eigenvalues of random matrices and with stochastic boundary value problems. Very few past studies are available on such problems. A review on the related literature is available in the works of Boyce,[1] Scheidt and Purkert[2] and Ibrahim.[3] A variety of methods based on transfer matrix approach, perturbation analysis variational formulation etc., have been employed in the determination of statistics of natural frequencies and mode shapes. By and large the available studies on random eigenvalues aim at estimating the first two moments and often end up establishing bounds on them.

A fundamentally different approach to random eigenvalue problems has been outlined by Iyengar and Athreya[4] who have studied a second order stochastic boundary value problem with reference to the distribution of the eigenvalues. They have converted the problem to that of studying the zeros of a random initial value problem. A significant feature of this approach is that it enables the application of Markovian methods in the analysis of random eigenvalues. Recently the present authors[5] have studied this approach further and have shown that, although an exact solution is rarely possible, for specific types of stochastic variations the associated initial value problem can be readily handled using approximate techniques. This paper is a continuation of the above study and it considers the case of the stochastic string equation in which both the mass and stiffness are assumed to be random processes. The physical motivation for considering this kind of randomness arises out of modelling the uncertain variations of the material and geometric properties of the system. Thus, for example, in the case of soil layers the mass and stiffness could be stochastic in nature on account of the inhomogeneities in the soil medium. The aim of the present study is to obtain an approximation to the probability distribution function (PDF) of the associated random eigenvalues. First, a first order stochastic nonlinear differential equation, which leads to the PDF of nth eigenvalue, is derived. This closely follows the earlier work of Iyengar and Athreya[4] but is generalized to incorporate different types of boundary conditions and possible uncertainties in their specification. Analytical solutions of this first order equation are obtained based on the method of stochastic averaging. This requires the assumption that there exists a clear cut separation between the characteristic lengths of the system and the stochastic coefficients. The

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solution obtained takes into account the mean, autocorrelation and cross correlation of the mass and stiffness processes. The scope of the approximate solutions is further examined through digital simulations.

INITIAL VALUE APPROACH TO A RANDOM EIGENVALUE PROBLEM

The stochastic wave equation is encountered in the vibration analysis of strings, bars, soil layers and shafts which have randomly varying material properties. In analysing the homogeneous equation for free vibration characteristics, after the space and time variables are separated, one needs to solve a second order stochastic boundary value problem. In general this problem can be stated as finding the nontrivial solutions of the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{[1+\delta g(x)]\frac{\mathrm{d}y}{\mathrm{d}x}\right\}+\lambda^2[1+\epsilon f(x)]y=0 \tag{1}$$

together with the boundary conditions

$$\alpha_1 y(0)+\alpha_2\frac{\mathrm{d}y}{\mathrm{d}x}(0)=0; \quad y(1)=0 \tag{2}$$

Nontrivial solutions are possible for certain values of λ, which are the eigenvalues of the system. In the above equation $g(x)$ and $f(x)$ are taken to be jointly stationary random fields which are bounded in a mean square sense and the parameters ϵ and δ are such that $(1+\epsilon f)>0$ and $(1+\delta g)>0$. For the instance when the governing equation is that of a string, these quantities respectively correspond to the random stiffness and the mass fields. The dependent variable y would in this case correspond to the displacement field of the string. The independent variable, x, has been suitably normalized so that it takes values in the interval $(0,1)$. By assigning different values to α_1 and α_2 one can simulate different fixity conditions. Thus, for example, for a string fixed at both ends, $\alpha_1=1$ and $\alpha_2=0$ and similarly, for a bar fixed at one end and free at the other, $\alpha_1=0$ and $\alpha_2=1$. However, in the ensuing analysis α_1 and α_2 can, in general, be random variables with a specified joint distribution. The eigenvalue parameter λ in this case clearly is a random variable. The aim here is to obtain the PDF of λ_n, the nth eigenvalue. This is achieved by first converting the above boundary value problem into an associated initial value problem. For this purpose, we consider the solution $y^*(x,\lambda)$ of eqn (1), such that

$$\alpha_1 y^*(0)+\alpha_2\frac{\mathrm{d}y^*}{\mathrm{d}x}(0)=0; \quad \frac{\mathrm{d}y^*}{\mathrm{d}x}(0)=1 \tag{3}$$

Clearly eqns (1) and (3) no longer constitute an eigenvalue problem and hence in these equations λ is not an eigenvalue. For a given value of λ, let $Z_n(\lambda)$ denote the nth zero of the solution $y^*(x,\lambda)$ of the initial value problem given by eqns (1) and (3). It is known from Sturm–Liouville theory[6] that $Z_n(\lambda)$, $n=1,2,\ldots,$ are nonincreasing in λ. The function $y^*(x,\lambda)$ can be a solution of eqns (1) and (2) only for a specific choice of λ for which the condition $y^*(1,\lambda)=0$ is satisfied. These λs are in fact the eigenvalues of eqns (1) and (2). In other words the solution λ_n of the equation

$$Z_n(\lambda)=1 \tag{4}$$

will be the nth eigenvalue of eqns (1) and (2). The study of $Z_n(\lambda)$ can be simplified by introducing the Pruffer substitutions[6] given by

$$y^*(x)=r(x)\sin\phi(x)$$

$$(1+\delta g)\frac{\mathrm{d}y^*}{\mathrm{d}x}=r(x)\lambda\cos\phi(x) \tag{5}$$

In the transformed (r,ϕ) coordinate system, eqns (1) and (3) can be shown to be equivalent to

$$\frac{\mathrm{d}\phi}{\mathrm{d}x}=\lambda\{[\cos^2\phi/(1+\delta g)]+(1+\epsilon f)\sin^2\phi\} \tag{6}$$

$$\frac{\mathrm{d}r}{\mathrm{d}x}=0{\cdot}5\lambda r\sin 2\phi[1-(1+\epsilon f)(1+\delta g)]/(1+\delta g) \tag{7}$$

$$\phi(0)=\tan^{-1}\{-\alpha_2\lambda/[\alpha_1+\alpha_1\delta g(0)]\} \tag{8}$$

$$r(0)=\{(\alpha_2/\alpha_1)^2+([1+\delta g(0)]^2/\lambda^2)\}^{1/2} \tag{9}$$

From the above equations we can observe that $\phi(x,\lambda)$ is nondecreasing in x and also that the nth zero of $y^*(x,\lambda)$ is the root of the equation

$$\phi(x,\lambda)=n\pi \tag{10}$$

Thus $Z_n(\lambda)$ is a random variable satisfying

$$\phi[Z_n(\lambda),\lambda]=n\pi \tag{11}$$

Thus the study of $Z_n(\lambda)$ reduces to the study of $\phi(x,\lambda)$. Since $\phi(x,\lambda)$ is nondecreasing in x, it follows

$$\begin{aligned}P[Z_n(\lambda)\le x]&=P[\phi(Z_n,\lambda)\le\phi(x,\lambda)]\\&=P[n\pi\le\phi(x,\lambda)]\end{aligned} \tag{12}$$

Further, $Z_n(\lambda)$ is nonincreasing in λ and hence

$$\begin{aligned}P[\lambda_n\le\lambda]&=P[Z_n(\lambda)\le 1]\\&=P[n\pi\le\phi(1,\lambda)]\end{aligned} \tag{13}$$

Thus, in order to find the PDF of λ_n, it is necessary to solve eqns (6) and (8) for every value of λ and obtain the probability density of the process ϕ at $x=1$. Although an exact solution of this problem is unlikely, the problem can, however, be handled by approximate methods for analysing stochastic initial value problems. It should be noted that in particular cases where α_1 and α_2 are random variables the probability measure in eqn (13) must be regarded as conditional on α_1 and α_2. To obtain the unconditional PDF of λ_n, further integration on the distributions of α_1 and α_2 would be required.

For further analysis, it is convenient to rewrite the processes $g(x)$ and $f(x)$ as follows

$$1/[1+\delta g(x)] = g_0 + \sigma\bar{g}(x)$$
$$= g_0[1+\nu\bar{g}(x)]$$
$$g_0 = \langle 1/[1+\delta g(x)]\rangle$$
$$\nu = \sigma/g_0$$
$$\sigma^2 = \langle 1/[1+\delta g(x)]^2\rangle - \langle 1/[1+\delta g(x)]\rangle^2$$
$$\langle\bar{g}\rangle = 0; \quad \langle\bar{g}^2(x)\rangle = 1$$
$$f(x) = m_f + \bar{f}(x); \quad \langle\bar{f}\rangle = 0 \tag{14}$$

Here $\langle\cdot\rangle$ denotes the expectation operator. The process $\bar{g}(x)$ can be interpreted as the randomness in the flexibility of the system. Furthermore, the phase process can be redefined as

$$\phi(x,\lambda) = \lambda x + \Theta(x,\lambda) \tag{15}$$

With these substitutions, eqns (6) and (8) are equivalent to

$$\frac{d\Theta}{dx} = \lambda[(g_0-1)\cos^2(\Theta+\lambda x) + \epsilon m_f \sin^2(\Theta+\lambda x)$$
$$+ g_0\bar{g}\cos^2(\Theta+\lambda x) + \epsilon\bar{f}\sin^2(\Theta+\lambda x)]$$
$$\Theta(o) = \tan^{-1}\{-\alpha_2\lambda/[\alpha_1+\alpha_1\delta g(o)]\} \tag{16}$$

The PDF of λ_n is now given by

$$P[\lambda_n \le \lambda] = 1 - \int_o^{(n\pi-\lambda)} p(\Theta,\lambda)\,d\Theta \tag{17}$$

Here $p(\Theta,\lambda)$ is the probability density of the process $\Theta(x,\lambda)$ at $x=1$.

STOCHASTIC AVERAGING

Equation (16) is a nonlinear differential equation with stochastic coefficients. The exact solution of this equation, even with simple models for $\bar{f}(x)$ and $\bar{g}(x)$ is difficult to obtain. For instance, if $\bar{f}$ and $\bar{g}$ are obtained as filtered white noise processes, one can derive the Fokker–Planck equation for the transitional probability density function of the response vector $(\Theta,\bar{f},\bar{g})$. However, the exact solution of this equation is not possible. Alternatively, one can obtain approximate solutions of eqn (16) which are acceptable over limited ranges of system parameters. Thus, under the assumption that the quantities (g_0-1), ϵ and ν are much less than unity and also that the correlation lengths of processes $\bar{g}(x)$ and $\bar{f}(x)$ are much less than the relaxation length of the process $\Theta(x)$, one can employ the method of stochastic averaging to further analyse eqn (16). In the case of random vibration problems this method is a powerful tool for analysing problems in which the correlation time of the excitation is small compared to the relaxation time of the system.[7] It is interesting to examine the scope of the method when applied to stochastic differential equations in spatial variables such as eqn (16). It is first noted that for small values of ϵ, ν and (g_0-1), $\Theta(x)$ will be a slowly varying random process. The trigonometric terms present in the right side of eqn (16) give rise to rapid oscillations of small magnitudes in Θ. Thus this equation may be approximately replaced by an averaged equation in which these rapid oscillations are suitably eliminated. First, the terms not containing $\bar{f}(x)$ and $\bar{g}(x)$ are replaced by their spatial averages over $(0, 2\pi/\lambda)$ to get

$$\frac{d\Theta}{dx} = 0{\cdot}5(g_0-1)\lambda + 0{\cdot}5\epsilon\lambda m_f + \nu g_0\bar{g}\cos^2(\Theta+\lambda x)$$
$$+ \epsilon\lambda\bar{f}\sin^2(\Theta+\lambda x) \tag{18}$$

It is important to note that in the above equation $\bar{f}(x)$ and $\bar{g}(x)$ are not delta-correlated and hence $\Theta(x)$ will be correlated with them. This implies that $\Theta(x)$ is not a Markov process. In the next stage of averaging, based on the assumption that the band widths of $\bar{f}$ and $\bar{g}$ are much greater than that of Θ, the fluctuating components in the above equation are replaced by equivalent delta-correlated processes. This leads to a Markovian approximation for $\Theta(x)$. The simplified equation for further analysis is found to be

$$\frac{d\Theta}{dx} = A + BW(x) \tag{19}$$

where

$$\langle W(x_1)W(x_2)\rangle = \delta(x_1-x_2) \tag{20}$$
$$A = 0{\cdot}5\lambda(g_0-1) + 0{\cdot}5\epsilon\lambda m_f + 0{\cdot}25\epsilon^2\lambda^2\Gamma_1$$
$$+ 0{\cdot}25g_0^2\nu^2\lambda^2\Gamma_2 - 0{\cdot}25\epsilon g_0\nu\lambda^2\Gamma_3 \tag{21}$$
$$B^2 = 0{\cdot}125\epsilon^2\lambda^2[2S_{\bar{f}}(o) + S_{\bar{f}}(2\lambda)]$$
$$+ 0{\cdot}125g_0^2\nu^2\lambda^2[2S_{\bar{g}}(o) + S_{\bar{g}}(2\lambda)]$$
$$+ 0{\cdot}125\epsilon\lambda^2 g_0\nu\Gamma_4 \tag{22}$$
$$\Gamma_1 = \int_{-\infty}^{0} \langle\bar{f}(x)\bar{f}(x+\tau)\rangle \sin 2\lambda\tau\,d\tau \tag{23}$$
$$\Gamma_2 = \int_{-\infty}^{0} \langle\bar{g}(x)\bar{g}(x+\tau)\rangle \sin 2\lambda\tau\,d\tau \tag{24}$$
$$\Gamma_3 = \int_{-\infty}^{0} \langle\bar{f}(x)\bar{g}(x+\tau) - \bar{g}(x)\bar{f}(x+\tau)\rangle \sin 2\lambda\tau\,d\tau \tag{25}$$
$$\Gamma_4 = \int_{-\infty}^{0} (2-\cos 2\lambda\tau)$$
$$\times [\langle\bar{g}(x)\bar{f}(x+\tau) + \bar{g}(x+\tau)\bar{f}(x)\rangle]\,d\tau \tag{26}$$

Here $S_{\bar{f}}$ and $S_{\bar{g}}$ are respectively the power spectral density function of the processes $\bar{f}$ and $\bar{g}$. It may be noted that the parameters Γ_3 and Γ_4 are in terms of the cross correlation functions of the processes $\bar{g}$ and $\bar{f}$. It follows from eqn (19) that $\Theta(1,\lambda)$ is a gaussian random

variable with mean $A + \Theta(o)$ and variance B^2. Accordingly, the PDF of λ_n as per eqn (17) is given by

$$P[\lambda_n \leq \lambda] = 1 - \text{erf}\{[n\pi - \lambda - A - \Theta(o)]/B\} \quad (27)$$

where erf$(\cdot)$ denotes the error function.

NUMERICAL RESULTS

Numerical results were obtained to illustrate the above theoretical solution. It was assumed that the parameters α_1 and α_2 and hence the boundary conditions of eqn (2) were deterministic. Four specific examples were considered which highlight different features of the theoretical solution.

Example 1

Here $g_0 = 1$, $\nu = 0$, $\alpha_1 = 1$, $\alpha_2 = 0$ and $\bar{f}(x)$ is taken to be a stationary random process with power spectral density function given by

$$S_{\bar{f}}(\lambda) = 2D/(\alpha^2 + \lambda^2) \quad (28)$$

This corresponds to a case of a fixed–fixed string with randomly distributed mass. The PDF of the first two eigenvalues obtained using eqn (27) for $m_f = 1$, $D = 0{\cdot}2$, $\alpha = 4{\cdot}0$ and various values of ϵ are shown in Figs 1 and 2. The accuracy of these solutions has been further examined through the use of digital simulation results, obtained by assuming a solution of the form

$$y(x) = \sum_{j=1}^{10} a_j \phi_j(x) \quad (29)$$

for eqns (1) and (2). In the present case $y(o) = y(1) = 0$ and hence $\phi_j(x)$ are taken to be

$$\phi_j(x) = \sin j\pi x \quad (30)$$

In the simulation work samples of $\bar{f}(x)$ with power spectral density given by eqn (28) are to be generated. This requires knowledge of the PDF of the process $\bar{f}(x)$. In situations where this information is not readily available it is expedient to assume that $\bar{f}(x)$ is gaussian distributed. Accordingly samples of $\bar{f}(x)$ can be generated as the stationary solution of the equation

$$\frac{d\bar{f}}{dx} + \alpha\bar{f} = \tilde{W}(x) \quad (31)$$

Here $\tilde{W}(x)$ is a gaussian white noise process with $\langle\tilde{W}(x_1)\tilde{W}(x_2)\rangle = 2D\delta(x_1 - x_2)$. In the numerical work the above equation is integrated with a step size = 0·005. The first 200 numbers of the solution are ignored to ensure that the transients are dissipated. For every sample of $\bar{f}(x)$, the series solution of eqn (29) via Galerkin's approximation leads to sample realization of eigenvalues and eigenvectors. One hundred such samples are obtained for the first two eigenvalues to construct the PDF which are shown in Figs 1 and 2.

Example 2

Here the system parameters are as in the previous example but the process $f(x)$ is taken as

$$f(x) = F^2(x) \quad (32)$$

where $F(x)$ is a zero mean gaussian stationary random process with autocorrelation

$$\langle F(x_1)F(x_2)\rangle = \beta^2 \exp(-\alpha|x_1 - x_2|) \quad (33)$$

The process $f(x)$ can be written in the form

$$f(x) = m_f + \bar{f}(x)$$

$$m_f = \beta^2$$

$$\bar{f}(x) = F^2(x) - \beta^2$$

$$S_{\bar{f}}(\lambda) = 2\sigma^4/(4\alpha^2 + \lambda^2) \quad (34)$$

It may be noted that the form of the above power spectral density function is similar to that considered in the previous example. The process $\bar{f}(x)$ in the present

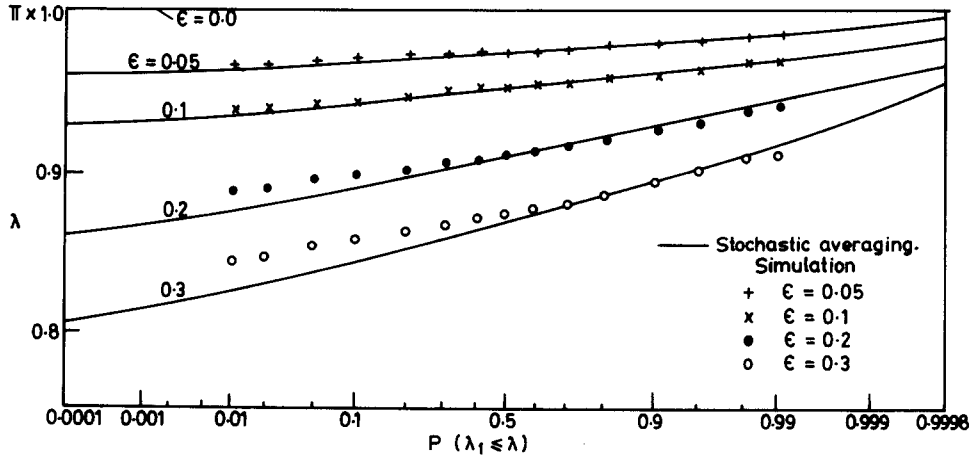

Fig. 1. Probability distribution of the first eigenvalue. Example 1; $m_f = 1{\cdot}0$, $\alpha = 4{\cdot}0$, $D = 0{\cdot}2$.

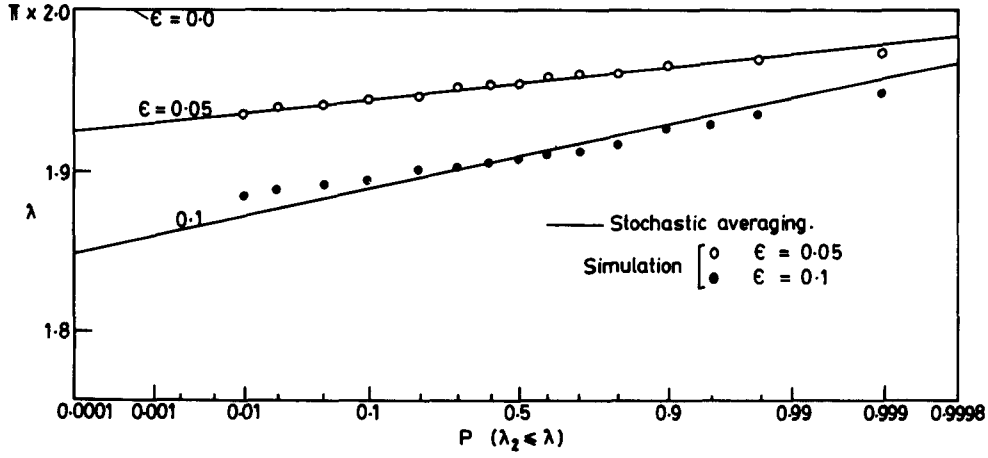

Fig. 2. Probability distribution of the second eigenvalue. Example 1; $m_f = 1{\cdot}0$, $\alpha = 4{\cdot}0$, $D = 0{\cdot}2$.

example is, however, nongaussian. The PDF of the first two eigenvalues for $\beta^2 = 0{\cdot}75$, $\alpha = 3{\cdot}0$ and for different values of ϵ are shown in Figs 3 and 4. These figures show two sets of simulation results obtained using eqns (29) and (30). In the first set (I set) gaussian distributed samples of $F(x)$ are generated using a procedure similar to the generation of $\bar{f}(x)$ in Example 1 and the samples of $f(x)$ are then obtained using the relation $f(x) = F^2(x)$. In the second set (II set) samples of $\bar{f}(x)$ of eqn (34) are generated under the assumption that $\bar{f}(x)$ is a gaussian random process. It is important to note that the process $f(x)$ in both these sets has the same mean and autocovariance but the corresponding probability density functions are different.

Example 3

Here a string fixed at both ends with random variations in both the mass and the stiffness is considered. The processes $\bar{f}$ and $\bar{g}$ are assumed to be jointly stationary independent processes with power spectra given by

$$S_{\bar{f}}(\lambda) = 2D_{11}/(\alpha^2 + \lambda^2)$$
$$S_{\bar{g}}(\lambda) = 2D_{22}/(\mu^2 + \lambda^2) \tag{35}$$

The theoretical PDF of λ_1 given by eqn (27) for $D_{11} = 0{\cdot}2$, $D_{22} = 1{\cdot}0$, $\alpha = 4$, $\mu = 1$ and for different values of g_0, ϵ and ν are shown in Fig. 5.

Example 4

The system parameters here are identical to those in the previous example except that the boundary conditions are now chosen such that $\alpha_1 = 0$ and $\alpha_2 = 1$. These conditions correspond, for example, to a stochastic shear beam fixed at one end and free at the other. Figure 6 shows the PDF of the first eigenvalue for different values of g_0, ϵ and ν.

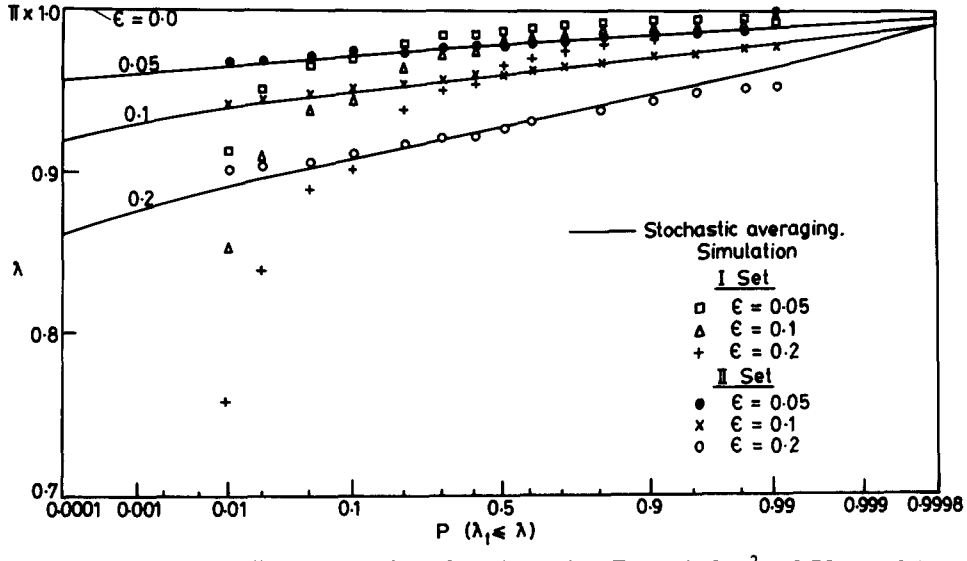

Fig. 3. Probability distribution of the first eigenvalue. Example 2; $\sigma^2 = 0{\cdot}75$, $\alpha = 3{\cdot}0$.

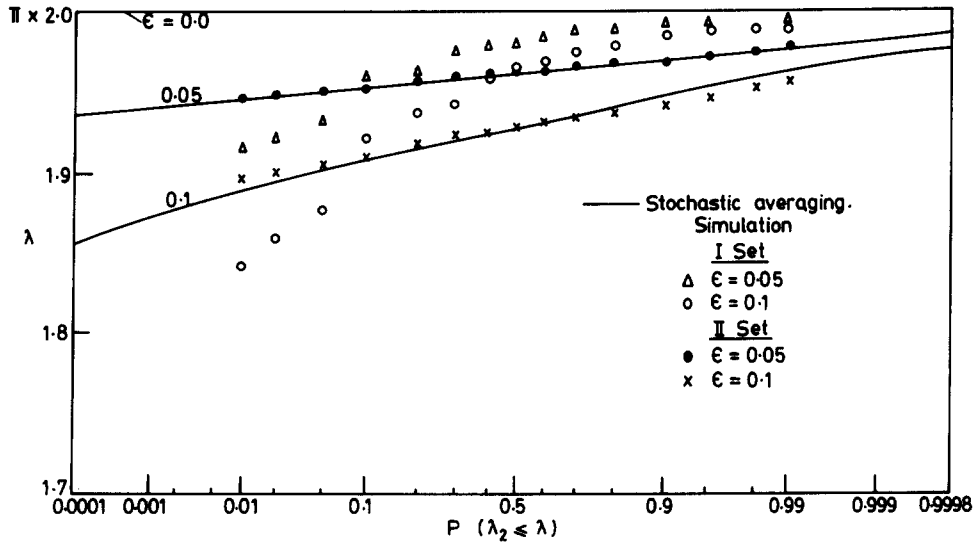

Fig. 4. Probability distribution of the second eigenvalue. Example 2; $\sigma^2 = 0{\cdot}75$, $\alpha = 3{\cdot}0$.

DISCUSSION

A penalty which one must pay in applying the method of stochastic averaging to eqn (16) is that one gets a gaussian approximation to the process $\Theta(x)$. This would mean that there will be a small nonzero probability of the process $\phi(x)$, given by eqn (15) becoming negative which as per eqn (6) is not admissible. However, in the numerical work, for the parameter ranges considered, this probability is found to be negligibly small. Thus, in Example 1, for $x = 1$, $\alpha = 2{\cdot}0$ and $\epsilon = 0{\cdot}1$ the maximum value of this probability is $4{\cdot}9 \times 10^{-3}$ and for $x = 1$, $\alpha = 4{\cdot}0$ and $\epsilon = 0{\cdot}1$ (Fig. 1) the value is $1{\cdot}19 \times 10^{-7}$.

It is important to note that the applicability of the stochastic averaging method does not impose any restriction on the PDF of the random functions. Since the method cannot take into account the PDF of the random quantities, the information on these distributions is, however, not required to apply the method. On the other hand, this information is essential in a Monte Carlo simulation study. In Example 1, $f(x)$ is assumed to be gaussian distributed. In this case, the mean and the autocovariance completely specify the process. It should be noted that, since the quantity $(1 + \epsilon f)$ represents the mass process, it is required to be strictly positive. Thus, the assumption that $f(x)$ is

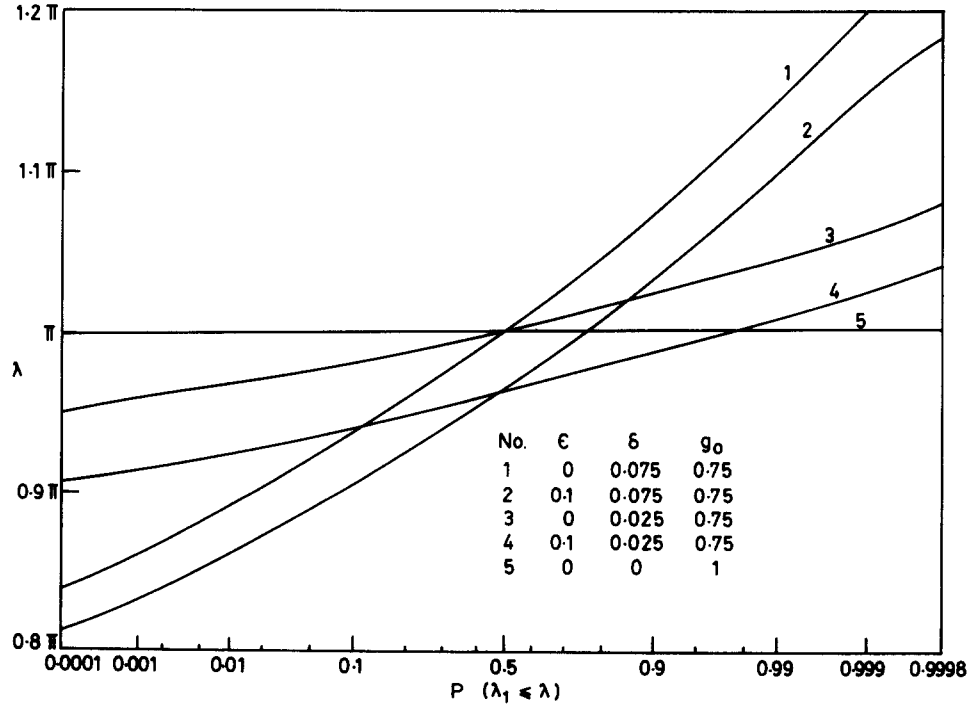

Fig. 5. Probability distribution of the first eigenvalue. Example 3; $D_{11} = 0{\cdot}2$, $D_{22} = 1{\cdot}0$, $\alpha = 4{\cdot}0$, $\mu = 1{\cdot}0$.

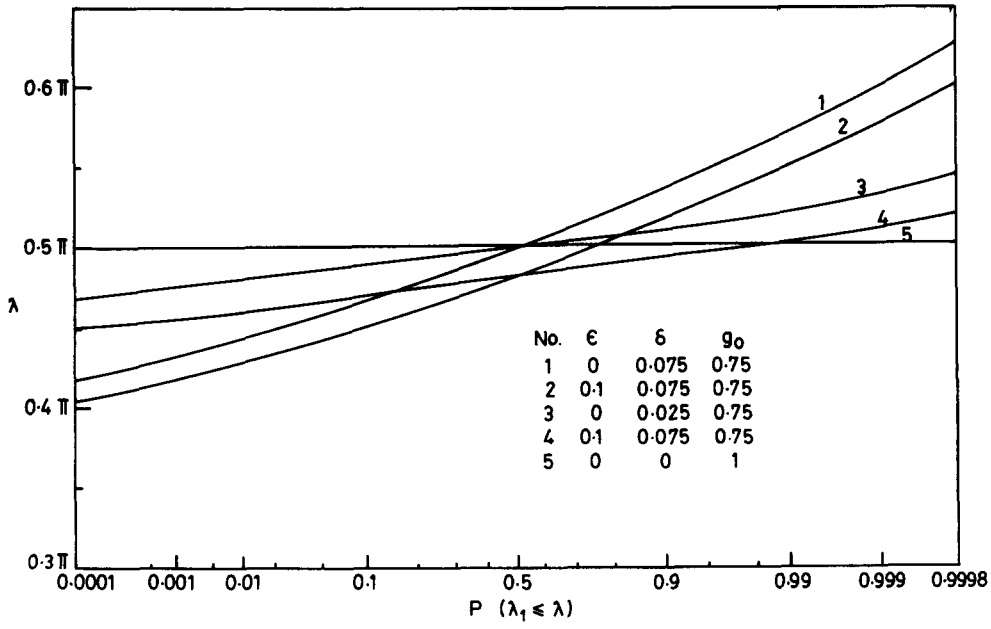

Fig. 6. Probability distribution of the first eigenvalue. Example 4; $D_{11} = 0{\cdot}2$, $D_{22} = 1{\cdot}0$, $\alpha = 4{\cdot}0$, $\mu = 1{\cdot}0$.

gaussian distributed introduces a certain amount of error into the analysis. For small variations this error is, however, expected to be negligibly small. The theoretical solutions shown in Figs 1 and 2 are found to compare well with the digital simulations. In Examples 1 and 2, since $\nu = 0$, the case of $\epsilon = 0$ corresponds to a deterministic system. In this case $P[\lambda_n = n\pi] = 1$. This result is also shown in Figs 1 to 4. When $\epsilon = 0$, for the type of $f(x)$ considered, additional mass would be added to the system which leads to a decrease in the natural frequencies. This is reflected in Figs 1 to 4 where the PDF corresponding to a higher value of ϵ always lie below the functions corresponding to lower ϵ.

The scope of the stochastic averaging solution for different probability distributions of $f(x)$ is examined in Example 2. Here the simulation results corresponding to two different mass processes, having the same mean and autocovariance but different probability distributions, are compared with the theoretical solution. Since both processes have the same mean and covariance they lead to the same theoretical solution. On the other hand, the results obtained using digital simulations, which take into account the additional information on the probability distributions, would be different for the two cases. From Figs 3 and 4 it is observed that the theoretical results do not compare well with the simulation results corresponding to the case where $f(x)$ is nongaussian. On the other hand, this comparison is quite good for the case of $f(x)$ being gaussian distributed. This would indicate the limitation of the stochastic averaging method when the random functions involved are not gaussian distributed.

In Examples 3 and 4 the effect of randomness in both the flexibility and the mass terms on the PDF of the first eigenvalue is investigated. Here, larger values of the parameter ν imply a wider variation in flexibility which in turn results in wider ranges for the system natural frequencies. Thus it can be observed from Fig. 5 that for $\epsilon = 0$, $\delta = 0{\cdot}025$ the range of first eigenvalue is about $0{\cdot}1\pi$ whereas for $\epsilon = 0$ and $\delta = 0{\cdot}075$ it is more than $0{\cdot}3\pi$. For the system considered in Example 4 (Fig. 6) the boundary conditions are $y(1) = 0$ and $\mathrm{d}y/\mathrm{d}x(0) = 0$. It may be noted that the deterministic solution in this case is given by $P[\lambda_n = 0{\cdot}5n\pi] = 1$.

CONCLUSION

The eigenvalues of a second order stochastic boundary value problem can be characterized in terms of zeros of an associated initial value problem. It has been shown that the PDF of eigenvalues is related to the solution of a first order nonlinear stochastic differential equation. This equation is treated analytically using the method of stochastic averaging. The solution is valid when the random variations are small and when the characteristic length of the system is much greater than the correlation length of the random fluctuations. The solution takes into account the mean and the correlations of the random coefficient processes and also the uncertainty in specifying the boundary conditions. The theoretical predictions have been further verified with a limited number of digital simulations.

The extension of the present approach to discrete random multidegree systems and to higher order systems such as stochastic beams is currently being investigated.

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