Methods of nonlinear random vibration analysis

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Abstract. The various techniques available for the analysis of nonlinear systems subjected to random excitations are briefly introduced and an overview of the progress which has been made in this area of research is presented. The discussion is mainly focused on the basis, scope and limitations of the solution techniques and not on specific applications.

Keywords. Nonlinear systems; random excitations; nonlinear vibration; vibration analysis.

1. Introduction

Random vibration methods are extensively used in earthquake, wind, transportation and offshore structural engineering applications. Here, the uncertainties is specifying the forces acting on the structure are quantified using sophisticated load models based on the theory of probability and stochastic processes. Consequently, the response analysis of structures is also carried out in a probabilistic framework which eventually leads to the assessment of the safety of the structure. In order to maintain a consistent level of sophistication in modelling, the vibrating structure also needs to be modelled with care. This concern leads to questions on modelling nonlinear behaviour of the structural system and also on modelling uncertainties in specifying the structural parameters themselves. The questions of structural nonlinearity are particularly important while addressing the problem of failures and safety assessments, especially, since the nonlinear response is, at times, radically different from the one obtained using a simplified linear model. These questions offer considerable challenge to the analyst and are currently being actively pursued in vibration engineering research as evidenced by a continuous stream of publications in leading international journals. The present paper aims at presenting an overview of the research work in this field highlighting the developments which have taken place over the last decade. The emphasis is, therefore, to focus on the various techniques and methodologies, which have admittedly come to stay as powerful tools in dealing with problems encountered frequently in the area of nonlinear random vibrations.

The sources of nonlinearities in vibration problems can be categorized into four groups:

- Geometric nonlinearities arising out of large deformations;
- nonlinear elastic and dissipation properties of the structural material;

- topological causes as in the case of vibroimpact systems such as rocking blocks and systems with stoppers;
- fluid-structure interactions leading to nonlinear couplings.

Table 1 lists a few examples which have been studied in the literature in the context of nonlinear random vibration of engineering structures. In these problems closed form solutions are rarely possible. Moreover, there exists no single general analytical procedure which leads to acceptable solution under all circumstances. The influential factors in formulating the solution procedures are

- System degrees of freedom, nature of nonlinearity (that is, nonlinearity in mass, stiffness or damping and symmetry/asymmetry of nonlinearity), predominance of the nonlinearity in affecting the system behaviour, including quasiperiodicity and bifurcations, in the absence of random excitations,
- Stationarity/nonstationarity of excitation, probability distribution and strength of excitation,
- Excitation bandwidth in relation to system bandwidth,
- Mechanism of excitation, that is, external or parametric,
- Response variables of interest.

Accordingly, several approximate solution procedures have been developed which lead to acceptable solutions in specific problem domains. Mostly, the approximations are based on the Markovian nature of the response or on the proximity of the response probability density function (pdf) to Gaussian distributions. Many of the methods are ingenious extensions of deterministic nonlinear analysis procedures to stochastic problems. A discussion on the following methods is presented in the sequel: (i) Markov vector approach, (ii) Perturbation, (iii) Equivalent linearization, (iv) Equivalent non-linearization, (v) Closure, (vi) Stochastic averaging, (vii) Stochastic series solution and (viii) Digital simulations.

2. Markov vector approach

When the inputs arise from Gaussian white noise processes the response will be a diffusion process and the associated transitional pdf (tpdf) will satisfy the well-known Kolmogorov equations. The governing equations of motion in these situations can be cast in the form of equations of the Itô type as follows:

$$\mathrm{d}\mathbf{X}(t) = f(\mathbf{X}(t),t)\mathrm{d}t + G(\mathbf{X}(t),t)\mathrm{d}\mathbf{B}(t), \tag{1}$$

under the initial conditions

$$\mathbf{X}(t_0) = \mathbf{Y}, \tag{2}$$

where, $\mathbf{X}(t) = n \times 1$ response vector, $f(\mathbf{X}(t),t) = n \times n$ matrix, $G(\mathbf{X}(t),t) = n \times m$ matrix, $\mathbf{B}(t) = m \times 1$ vector of the Brownian motion processes having the properties

$$\mathbf{E}[\Delta_j(t)] = \mathbf{E}[B_j(t+\Delta t) - B_j(t)] = 0, \tag{3}$$

$$\mathbf{E}[\Delta_i(t)\Delta_j(t)] = 2D_{ij}\Delta t, \tag{4}$$

and $\mathbf{Y} = n \times 1$ vector of initial conditions independent of $\mathbf{B}(t)$. Here $\mathbf{E}[\cdot]$ represents the mathematical expectation operator. The above representation of equations of motion

Table 1. Example of nonlinear engineering vibrating systems.

Nonlinear system	Model	Equation of motion	Reference
Frictional seismic base isolation system	Coloumb's oscillator	$\ddot{x} + \mu g \operatorname{sgn}(\dot{x}) + \omega^2 x = -\ddot{\mu}_g(t)$	Ahmadi & Su (1987)
Plate undergoing large amplitude vibrations	Duffing's oscillator	$\ddot{x} + 2\eta\omega\dot{x} + \omega^2 x + \alpha x^3 = f(t)$	Lin (1967)
Hysteretic structure	Bouc's oscillator	$m\ddot{x} + c\dot{x} + \alpha kx + (1-\beta)kz = f(t)$ $\dot{z} = (1/\eta)[A\dot{x} - v(\beta\lvert\dot{x}\rvert\lvert z\rvert^{n-1}z - \gamma\dot{x}\lvert z\rvert^{n})]$	Wen (1989) Iyengar & Dash (1978)
Vibro-impact system	Rectangular rocking block	$I\ddot{\theta} + WR\sin(\alpha\operatorname{sgn}\theta - \theta)[1 + f(t)]$ $+ WR\cos(\alpha\operatorname{sgn}\theta - \theta)g(t) = 0$ $\dot{\theta}(t_+^*) = c\dot{\theta}(t_-^*);\ t^*: \theta(t^*) = 0.$	Spanos & Koh (1986) Iyengar & Manohar (1991)
Across wind oscillation of chimneys	Van der Pol's oscillator	$m\ddot{y} + c\dot{y} + ky = \alpha_1\dot{y} + \alpha_2\dot{y}^3 + f(t)$	Vickery & Basu (1983) Manohar & Iyengar (1991a, b)
Ship rolling in random seas	—	$\ddot{\phi} + \varepsilon^2 F(\dot{\phi}) + \omega^2\phi[1 + k\varepsilon^2\phi^2 + \varepsilon f(t)] = \varepsilon f(t)$	Roberts (1982)
Offshore structure under wave loads	Modified Morrison's model	$M\ddot{\mathbf{X}} + C\dot{\mathbf{X}} + K\mathbf{X} = \mathbf{P}(t)$ $P_i(t) = \frac{1}{2}C_D\rho A_i[u_i(t) - \dot{x}_i(t)]\lvert u_i(t)$ $- \dot{x}_i(t)\rvert + C_M\rho V_i\dot{u}_i(t) - \bar{C}_M\rho V_i\ddot{x}_i(t)$	Taylor & Rajagopalan (1982)

is fairly general in the sense that it allows for: (1) Multi-degree linear/nonlinear discrete systems, (2) external and parametric excitations, (3) nonstationary excitations, (4) nonwhite excitations, in which case, additional filters to model inputs as filtered white noise processes are to be appended to the system equations with a consequent increase in the size of the problem and (5) random initial conditions. The Kolmogorov equations satisfied by the response tpdf, $p(\mathbf{x}, t|\mathbf{y}, t_0)$ are

- the Chapman–Kolmogorov–Smoluckowski (CKS) integral equation

$$p(\mathbf{x}, t|\mathbf{y}, t_0) = \int_{-\infty}^{\infty} p(\mathbf{x}, t|\mathbf{z}, \tau) p(\mathbf{z}, \tau|\mathbf{y}, t_0) \mathrm{d}\mathbf{z}, \tag{5}$$

- the forward equation or the Fokker–Planck–Kolmogorov (FPK) equation

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial t} = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}[f_j(\mathbf{x}, t) p(\mathbf{x}, t|\mathbf{y}, t_0)] + \sum_{ij=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}[(GDG^{\mathrm{T}})_{ij} p(\mathbf{x}, t|\mathbf{y}, t_0)], \tag{6}$$

- the backward equation

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial t_0} = -\sum_{j=1}^{n} f_j(\mathbf{y}, t) \frac{\partial p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial y_j} - \sum_{ij=1}^{n} [GDG^{\mathrm{T}}]_{ij} \frac{\partial^2 p(\mathbf{x}, t|\mathbf{y}, t_0)}{\partial y_i \partial y_j}. \tag{7}$$

In these equations, the superscript T denotes the matrix transpose operation. The first of these equations represents the consistency condition for the response process to be Markov. Equation (7) is the adjoint of (6) and these two equations can be derived using (5) together with the equation of motion given by (1). It is of interest to note that the forward equation and backward equations are also satisfied by several other response probability functions of interest. Thus, for instance, the probability, $Q(t|\mathbf{y}, t_0)$, that first passage across a specified safe domain will not occur in the time interval $t_0 - t$ for trajectories in the phase plane starting at $\mathbf{y}$ at $t = t_0$, can be shown to satisfy the backward Kolmogorov equation. The formulation of these equations leads to the exact response characterization of a limited class of problems and helps in formulating strategies for approximate analysis for a wider class of problems. The details of the derivation of these equations along with a discussion on the initial conditions, boundary conditions, well posedness, eigenvalues and eigenfunctions and the existence, uniqueness and stability of stationary solutions can be found in the works of Bharucha Reid (1960), Stratonovich (1963), Caughey (1963a, 1971), Feller (1966), Fuller (1969) and Roberts (1986a). A comprehensive treatment of the FPK equation and its application in physical sciences is available in the monographs by Risken (1989) and Horsthemke & Lefever (1984).

2.1 *Exact solutions*

The complete solution of the FPK equation is obtainable for all externally forced linear oscillators (Lin 1967) and for a class of first order nonlinear systems (Caughey & Dienes 1961; Stratonovich 1963; Atkinson & Caughey 1968; Atkinson 1973). In these solutions either the Fourier and Laplace transform-techniques or the method of eigenfunction expansion is used. Methods based on group theory have also been developed (Bluman, 1971). The stationary solution, when it exists, can be found for all first order systems and for a limited set of higher order systems. A general class of systems for which exact stationary solution under external white noise excitation is possible is discussed by Caughey & Ma (1982a, 1982b). This set includes single degree of freedom (sdof) systems with nonlinear stiffness and a class of sdof and multi-degree of freedom (mdof) systems with nonlinear damping and stiffness. Dimentberg (1982) has obtained stationary pdf for a specific sdof system in which both parametric and external white noise excitations are present. This solution has been obtained through an inverse procedure (Dimentberg 1988a). Here an approximate solution is first obtained based on the method of stochastic averaging. This solution is substituted into the governing FPK equation for the original system. This equation would be exactly satisfied provided the system's parameters are related in a certain special way. Thus, this subset of parameters defines a class of systems for which the governing reduced FPK equation is solvable. The concept of detailed balance developed earlier by physicists (Gardiner 1983; Risken 1989), has been used by Yong & Lin (1987) and Langley (1988a) to obtain exact stationary solution for a class of sdof and mdof nonlinear systems under white noise excitations. In this method, the components of response vector **Y** are classified as either even or odd depending upon their behaviour under a time reversal of t to $-t$. The even variables do not change their sign whereas the odd variables undergo a change of sign. These are denoted as

$$\tilde{x}_i = \varepsilon_i x_i, \quad \text{no summation on } i, \tag{8}$$

where $\varepsilon_i = 1$ for even variables and $\varepsilon_i = -1$ for odd variables. The state of detailed balance is defined as

$$p(\mathbf{x}, t|\mathbf{y}, t_0) = p(\tilde{\mathbf{y}}, t|\tilde{\mathbf{x}}, t_0), \quad t > t_0. \tag{9}$$

In the steady state, in terms of the drift coefficients A_i and diffusion coefficient B_{ij} this condition is given by

$$A_i(\mathbf{x})p(\mathbf{x}) + \varepsilon_i A_i(\tilde{\mathbf{x}})p(\mathbf{x}) - \frac{\partial}{\partial x_j}[B_{ij}(\mathbf{x})p(\mathbf{x})] = 0, \tag{10}$$

$$B_{ij}(\mathbf{x}) - \varepsilon_i \varepsilon_j B_{ij}(\tilde{\mathbf{x}}) = 0. \tag{11}$$

Here summation on repeated index is implied. When these conditions are satisfied, the stationary solution expressed as

$$p(\mathbf{x}) = C\exp[-U(\mathbf{x})], \tag{12}$$

can be obtained by solving the equation for $U(\mathbf{x})$, the generalized potential. The class of problems than can be solved using this method, is shown to include the problems considered by Caughey & Ma (1982a, b) and Dimentberg (1982). Furthermore, Lin & Cai (1988) and Cai & Lin (1988a) have shown that the exact stationary solution as in (12) can still be obtained even when one of the conditions for detailed balance, namely (11) is not satisfied. This class of systems has been termed as belonging to the class of generalized potential. More general class of exactly solvable FPK equations have been discussed by Zhu *et al* (1990) and To & Li (1991).

2.2 *Approximate methods*

For analysing problems possessing no exact solutions one has to take recourse to approximate methods. An iterative procedure based on the parametrix method for studying existence and uniqueness of solutions of partial differential equations (Friedman 1964) has been used by a few authors (Caughey 1971; Mayfield 1973). It has been shown that both transient and steady state solutions can be obtained using this method. Also, the method is useful in improving approximate results obtained by other techniques. However, the method has not been used widely in random vibration studies (Roberts 1981). Payne (1968) used a combination of perturbation and eigenfunction expansion techniques to analyse nearly linear first order systems under white noise inputs. He has derived perturbation expansion for eigenvalues and eigenfunctions of the FPK equation up to $O(\varepsilon^2)$ and obtained the corresponding expression for response autocorrelation function. Here ε may be denoted as a smallness parameter associated with the nonlinearity. Iwan & Spanos (1978) have employed similar techniques to obtain nonstationary response envelope distribution of a linear sdof system. Johnson & Scott (1979, 1980) considered the first order system studied by Payne (1968) and extended his analysis to compute expansion terms up to $O(\varepsilon^7)$. Furthermore, they have also applied this method to second-order systems.

Atkinson (1973) has used an adjoint variational method to find the eigenvalues of the FPK operator. He has generated trial functions that are orthogonal to known stationary solution and has determined response power spectral density (psd) for the case of a Duffing oscillator, a bang-bang system and a system with nonlinear damping. Toland *et al* (1972) proposed a random walk analogy based on a discrete approximation to continuous Markov process, and obtained a recurrence relation for response probabilities. This technique is equivalent to using finite difference approximation on the FPK equation and is time consuming especially when the domain of integration is large. The eigenfunction expansion method is applicable when the time and space variables in the FPK equation can be separated which is generally possible when the drift and diffusion terms are time invariant. In a more general context, the methods of weighted residual have been employed by several authors. Thus, Bhandari & Sherrer (1968) have used the Galerkin technique to find the stationary response of sdof and a two degree of freedom system with polynomial nonlinearities. Wen (1975, 1976) has extended this analysis to nonstationary response analysis and studied the response of the Duffing oscillator and a hysteretic system. The use of the method in determining psd response and the first excursion failure probabilities of response has also been indicated. A similar technique has been used by Solomos & Spanos (1984) for obtaining the amplitude response of a linear sdof system under evolutionary random excitation. Another variant of the weighted residual technique, namely, the method of moments, has been used by Fujita & Hattori (1980) for the analysis of sdof systems with collisions under modulated white noise input. Langley (1985) has applied the finite element method to solve the two dimensional FPK equation associated with the stationary response of a Duffing oscillator and a ship rolling problem. The domain in the phase space to be covered by finite elements was estimated using an equivalent linearization solution. An improper selection of the extent of the domain is shown to result in negative values for response pdf. Bergman & Spencer (1992) have studied the transient solutions of the FPK equations of several second order nonlinear systems using finite element method.

Other discretization procedures based on path integral formalisms (Wehner & Wolfer 1983; Kapitanaik 1985, 1986; Naess & Johnsen 1993) and cell mapping techniques (Sun & Hsu 1990) have also been developed. These methods are related to the iterative technique discussed by Crandall *et al* (1966) and are based on an assumption that the tpdf over short time steps is Gaussian. Sun & Hsu have used a Gaussian closure approximation to evaluate the transitional probabilities, while, Naess and Johnsen have employed cubic **B**-splines to represent the tpdf. It is to be noted that although the analysis makes the short-time Gaussian approximation for the tpdf, the global non-Gaussian nature of the unconditional pdf is, nevertheless, captured by the analysis.

2.3 *Generalizations and moment equations*

Generalization of the FPK equation to non-diffusive Markov processes has been discussed by Pawula (1967). Here the inputs are modelled as white noise arising out of non-Gaussian processes. The response in such cases will still have the Markovian property but the equation of motion of the transitional pdf will have infinite number of terms. Tylikowski & Marowski (1986) have considered the response of a Duffing oscillator to Poissonian impulse excitation and have shown that the transitional pdf satisfies an integro-partial differential equation. The use of truncated generalized FPK equations in computing the lower order moments has been demonstrated by Risken (1989).

When response moments are of interest, the governing equations for the moments can be derived based on the FPK equation (Soong 1973). Thus, the moments of the function $h[\mathbf{X}(t), t]$ of the solution $\mathbf{X}(t)$ of (1) can be shown to be governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{E}[h(\mathbf{X},t)] = \sum_{j=1}^{n} \mathbf{E}\left[f_j \frac{\partial h}{\partial x_j}\right] + \sum_{ij=1}^{n} \mathbf{E}\left[(GDG^T)_{ij} \frac{\partial^2 h}{\partial x_i \partial x_j}\right] + \mathbf{E}\left[\frac{\partial h}{\partial t}\right]. \quad (13)$$

By setting $h[\mathbf{X}(t), t] = X_1^{k_1} X_2^{k_2} X_3^{k_3} \cdots X_n^{k_n}$ and choosing different values for k_i, one can derive equations for the most commonly used moments. These equations can be readily solved for the case of deterministic nonautonomous linear systems under white noise inputs. This forms the basis for obtaining approximate transient solutions using linearization procedures. In the case of linear oscillators with parametric white noise excitations, the response is non-Gaussian and the associated FPK equation is not solvable. However, the exact response moments can be obtained by solving the associated moment equations. In nonlinear problems these equations form an infinite hierarchy and an exact solution of moment equations also is not possible. Based on the principle of maximum entropy, Sobczyk & Trebicki (1990, 1992) and Chang (1991) have developed approximate stationary solutions of nonlinear systems under parametric and external noise excitations. This consists of employing pdf with undetermined coefficients which, in turn, are found by maximizing the entropy subject to the constraints of the pdf normalization and the moment equations which are obtained through the governing FPK equations. Roy & Spanos (1991) have utilized a perturbation solution scheme for the moment equations and have shown that the scheme overcomes the problem of infinite hierarchy. They have proposed Pade type transformations of the series which enables the analysis to be applicable even for strongly nonlinear systems. Furthermore, the same authors (Roy & Spanos 1993) have studied the response power spectral density of nonlinear systems by utilizing formal solutions of the FPK equation as discussed by Risken (1989) in conjunction with a power series expansion in terms of Pade approximants for the response spectra. The method requires the knowledge of the stationary response of the FPK equation.

As has been briefly indicated earlier, the Markov property of response has also been used in the study of first passage probabilities. Here, either the forward or the backward Kolmogorov equation is solved in conjunction with appropriate boundary conditions imposed along the critical barriers. Alternatively, starting from the backward Kolmogorov equation, one can also derive equations for moments of the first passage time, which, in principle, can be solved recursively. Thus, denoting by $\tilde{T}(\mathbf{y})$, the time required by the response trajectory of (1) initiated at the point $\mathbf{x} = \mathbf{y}$ in the phase space at time $t = t_0$ to cross a specified safe domain for the first time, the moments $M_k = \mathbf{E}[\tilde{T}^k]$, $k = 1, 2, \cdots N$, can be shown to be governed by the equation

$$-\sum_{j=1}^{n} f_j(\mathbf{y}, t)\frac{\partial M_k}{\partial y_j} - \sum_{ij=1}^{n} (GDG^{\mathrm{T}})_{ij}\frac{\partial^2 M_k}{\partial y_i \partial y_j} + kM_{k-1} = 0 \quad (k = 0, 1, 2, \cdots), \tag{14}$$

with the condition $M_0 = 1$. These equations are referred to as the generalized Pontriagin–Vitt (GPV) equations in the literature. Although no exact analytical solution exists for finding M_k, several approximations are available and they have been reviewed by Roberts (1986a). These methods include method of weighted residuals (Spanos 1983), random walk models (Toland & Yang 1971; Roberts 1978), finite difference method (Roberts 1986b), finite element method (Spencer & Bergman1985) and cell mapping techniques (Sun & Hsu 1988).

2.4 *Summary*

The FPK equation approach is the only source of exact solutions in nonlinear random vibration problems. It also forms a powerful tool for approximate analysis. The application of the method is, however, limited to Markovian responses. The solution procedures are not tractable when dealing with large number of variables or with nonstationary inputs. Although the method is applicable when the input is a filtered white noise, this class of problems has not received much attention in the literature.

3. Perturbation method

This is a straightforward extension of the technique used in deterministic problems. The method is applicable when the equations of motion contain a small parameter characterizing the nonlinearity in the system. The solution is expanded in a power series in small parameter which leads to a set of linear differential equations which can further be solved sequentially. The method is applicable to both sdof and mdof systems under additive or multiplicative, stationary or nonstationary stochastic inputs. It was first used by Crandall (1963) to evaluate response moments of sdof and mdof systems with nonlinear stiffness under stationary Gaussian excitations. Shimogo (1963a, b) considered symmetric and asymmetric nonlinear systems under stationary random inputs and computed the response psd using an iterative technique, which, in essence, is identical to the perturbation method. Crandall *et al* (1964) and Khabbaz (1965) applied the method to systems with nonlinear damping and evaluated the response psd. They also indicated the possibility of the even ordered response moments and the psd function becoming negative for large values of the nonlinearity parameter. To a first order of approximation, the psd function obtained using this method and the equivalent linearization technique have been shown to be identical (Crandall 1964). Manning (1975) has estimated the response of the Duffing oscillator to stationary excitation and has shown that by evaluating the response to second order of the nonlinearity parameter, it is possible to display the effects of nonlinear resonances in the response psd. The response of the Duffing oscillator to nonstationary random inputs has been studied by Soni & Surendran (1975). The application of the method to systems with random parametric excitation has been discussed by Soong (1973).

Perturbation method is well suited for polynomial nonlinearities and is useful in computing response moments and sometimes the psd function. Determination of the response pdf using this method is, however, not possible because of the non-Gaussian nature of the higher order corrections. Furthermore, obtaining second or higher order corrections involves cumbersome calculations and is not practicable. It can be noted that the method is asymptotic in nature and the accuracy markedly worsens with the increase in the value of the nonlinearity parameter.

4. Equivalent linearization

This method is by and large the most popular approach in nonlinear random vibration problems. It is extension of the well known harmonic linearization technique to stochastic problems and is applicable to both sdof and mdof systems under stationary or nonstationary inputs. The method consists of optimally approximating the nonlinearities in the given system by linear models so that the resulting equivalent system is amenable for solution. For evaluating the parameters in the equivalent system, an additional assumption that the response is Gaussian is generally made. This method was developed in 1950's in the context of random vibration problems (see Caughey 1963b for earlier references) and nonlinear stochastic controls (Booton 1954). The subsequent developments of the method in vibration problems may be found in the works of Spanos (1981a), Roberts & Spanos (1990) and Socha & Soong (1991) and those in the field of control in the work of Sinitsyn (1974).

Caughey (1963b) applied this technique to find stationary response of nonlinear sdof systems and a class of mdof systems under stationary inputs. The method was generalized to a wider class of mdof systems by Foster (1968), Iwan & Yang (1972) and Atalik (1974). When nonstationary response is of interest the equivalent parameters will be functions of time and accordingly the equivalent linear system will be time varying. For the case of Markovian responses Iwan & Mason (1980) and Wen (1980) have obtained the nonstationary response of mdof systems by solving numerically the moment equations derived from the governing FPK equation. Spanos (1980b) modified the method to deal with mdof systems having asymmetric nonlinearities in which case the response has a nonzero mean. When the inputs are nonwhite the solution of time varying equivalent linear system is generally difficult. Ahmadi (1980b) and Sakata & Kimura (1980) have suggested different schemes to deal with such problems. These schemes have also been extended to analyze the nonstationary response of asymmetric sdof and mdof systems to nonwhite inputs (Kimura & Sakata 1981, 1987). In the study of continuous nonlinear systems, linearization can be done after discretizing the equation of motion or at the level of partial differential equation itself. The latter class of problems have been studied by Iwan & Krousgrill (1983) and Iwan & Whirley (1993).

4.1 *Extensions and improvements*

The method is versatile, easy to implement and computationally efficient. It is applicable when the nonlinearities and the excitations are such that the response is unimodal and nearly Gaussian. This does not necessarily imply that the nonlinearity should be small. Crandall (1973) has demonstrated that for a first order system with cubic nonlinearities, the mean square response using this technique is fairly accurate even when nonlinearity parameter is of the order of 100. The method invariably leads to Gaussian response pdf and fails to display nonlinear resonances in the response psd. Furthermore, the method is not applicable to systems with parametric excitations.

Several modifications to the method have been proposed to overcome some of the above limitations. Crandall (1973) has obtained a nonlinear integral equation for the improved estimate for the stationary response autocorrelations. Generalization of the method for the case of nonlinear systems with both parametric and external white noise excitations has been suggested by Bruckner & Lin (1987a). Here the original Itô equation for the nonlinear system is optimally replaced by a linearized Itô equation so that the first and the second moments computed from the equivalent system has minimum mean square error. For evaluating equivalent parameters use is made of the higher order moment equations and no appeal to the Gaussianness of the response nor for invoking any other closure approximation is necessary. In the methods discussed above the nonlinear system of a given order is invariably replaced by an equivalent system of the same order. Iyengar (1988a) has explored the possibility of replacing a given nonlinear system by an equivalent linear system of a higher order. In his study the nonlinear terms of the given equation are substituted by new dependent variables. Additional equations which govern these new variables and which are nonlinear in nature are obtained by suitably differentiating the given nonlinear equation. The resulting higher order system of nonlinear equations is further analyzed using the usual linearization scheme. For the case of the Duffing oscillator under white noise excitation, the results obtained using this method are shown to be better than the usual linearization solution and in particular, the response psd obtained using this method is shown to display the effects of secondary resonance in the form of an additional peak at about three times the primary resonance frequency. Extensions of equivalent linearization procedures to allow for non-Gaussian nature of the response have also been proposed. Thus, Manohar & Iyengar (1990) considered the broad band excitation of the Van der Pol oscillator and replaced the nonlinear oscillator by a linear system excited by a non-Gaussian input. The non-Gaussian excitation allowed for the limit cycle oscillations of the system in the absence of the external noise. This enabled the correct prediction of bimodal pdf of the response displacement and velocity which the traditional linearization fails to predict (Zhu & Yu 1987). Pradlwarter *et al* (1988), Pradlwarter (1989) and Schueller *et al* (1991) have considered stochastic response of inelastic systems and have proposed a non-Gaussian linearization scheme which is based on the theoretical results shown by Kozin (1987) that linear systems exist which lead, at least for white noise excitations, exactly to the first statistical moments of the response of the respective true nonlinear systems. The method involves nonlinear transformation between nonlinear response and the linear response which has to be chosen based on physical considerations. Iyengar (1992) considered Duffing's oscillator under narrow band excitation and developed an equivalent linear system whose stiffness parameter is random in nature. This parameter is shown to be a function of response envelope which is approximated as a random variable. While the response conditioned on the stiffness is Gaussian, the unconditioned response becomes non-Gaussian in nature.

4.2 *Nonuniqueness of solutions and stochastic stability*

The solution obtained using equivalent linearization is not necessarily unique in more than one sense. Firstly, the answers depend on criterion of equivalence adopted. While the most commonly used criterion requires that the mean square equation difference be minimized, other criteria involving alternative norms of differences or other averaging operators are also admissible (Spanos1981a; Bolotin 1984). Recently Elishakoff & Zhang (1993) have applied several optimization schemes on a randomly driven first order nonlinear system and have shown that, when the differences are averaged with respect to a weight function, which, in turn, is taken to be a nonlinear function of the system potential energy, the estimate of the response variance coincides with the known exact solution. Casciati *et al* (1993) have considered second order nonlinear systems under broad band excitation and established the equivalence by requiring that the upcrossing rate of a specified critical level for the nonlinear and for the equivalent linear oscillator be equal. Obviously the success of this approximation depends upon the accuracy with which the upcrossing statistics are known.

Another source of nonuniqueness, which is, perhaps more subtle, arises within the framework of a specified equivalence criterion. Thus, the linearization technique when applied to nonlinear systems under combined harmonic and noise excitations (Iyengar 1986; Manohar & Iyengar 1991a), narrow band excitations (Richard & Anand 1983; Davies & Nandlall 1986; Davies & Rajan 1986; Jia & Fang 1987; Iyengar 1988b, 1989; Manohar & Iyengar 1991b; Roberts 1991; Kolioupulos & Langley 1993) or for systems with multiple stable equilibrium states under broad band excitations (Langley 1988b; Fan & Ahmadi 1990) leads to multivalued response statistics. It may be noted in this context that Spanos & Iwan (1978) have earlier demonstrated the uniqueness of equivalent linear systems under certain conditions but not of the solutions generated by the equivalent systems.

The occurrence of multivalued response statistics apparently resemble the coexistence of multiple steady states encountered in the deterministic nonlinear oscillation problems. In random vibration context, however, it is important to note that stationary response statistics, when they exist, are necessarily unique. This follows from the fact that the steady solution of the governing FPK equation is always unique (Fuller 1969). The scope of this result includes all the nonlinear dynamical systems which are governed by equations of the form as given in (1) and, as has been noted in § 2, this class is fairly extensive. This fact was not recognized in some of the earlier studies on narrow band excitation of the Duffing oscillators (Davies & Nandlall 1986; Davies & Rajan 1986; Jia & Fang 1987). These authors used a stability analysis of moment equations and concluded that response statistics are multivalued and display the jump behaviour. Iyengar (1986, 1988b, 1989) suggested that the realisability of the multiple solutions must be decided based on the almost sure stochastic stability of the multiple solutions and not on the stability analysis of moment equations. His study showed that in regions of multiple solutions, the linearization solution based on the assumption of Gaussianess of the response is stochastically unstable and, therefore, not valid. Furthermore, simulation studies on response amplitude showed that the probability density function is bimodal in regions where linearization predicts multiple solutions. The studies by Langley (1988b) and Fan & Ahmadi (1990) on broad band excitation of nonlinear systems show that mean square response predicted by the linearization technique may not be unique while the corresponding exact solutions are unique. The system considered in these studies had multiple stable equilibrium states and the random response had multimodal probability density functions. The question of possible relationship between the multiple solutions predicted by linearization and local behaviour of the sample functions has been considered by few authors (Dimentberg 1988b; Roberts 1991; Koliopulos & Langley 1993). It is suggested that the multiple solutions correspond to local behaviours near the modes of the response pdf. The relationship between such local behaviour and global behaviour in an ensemble sense is not obvious and further research is clearly required to resolve this issue.

5. Equivalent nonlinearization

This method is conceptually similar to the method of equivalent linearization and can be viewed as a generalization leading to non-Gaussian estimates for the response. The method has been introduced by Caughey (1986). It consists of replacing the given nonlinear system by an equivalent nonlinear system which belongs to the class of problems which can be solved exactly. This method is related to the class of exactly solvable FPK equations and thus is applicable only to systems under white noise inputs. The criterion of replacement is again the minimization of the mean square error. The method leads to non-Gaussian stationary response pdf and estimates correctly the random response of limit cycle systems in which case, equivalent linearization fails. Cai & Lin (1988b) have developed a similar technique and have applied it to systems in which parametric excitations are also present. Here, the replacement oscillator belongs to the class of systems possessing generalized stationary potential and is selected on the basis that the average energy of dissipation remains unchanged. For a specific system, the solution obtained using this method is shown to be superior to that obtained by stochastic averaging. The application of the method to randomly excited hysteretic structures has also been developed (Cai & Lin 1990). The method has also been studied by Zhu & Yu (1989) who have chosen equivalent nonlinear systems which are energy dependent. They have indicated that the method is asymptotically exact and is equivalent to the method of stochastic averaging of the energy envelope. In a study on random response of limit cycle systems, Manohar (1989) has developed equivalent nonlinearization solutions for the randomly driven Van der Pol oscillator. In one scheme, the Van der Pol oscillator under white noise excitation is replaced by a Van der Pol–Rayleigh oscillator which can be solved exactly. This involves linearization of only a part of the original equation and the equivalent parameters are found based on minimization of mean square error. In an attempt to model the multimodal pdf of the response phase process, Manohar has applied a similar partial linearization procedure to the simplified equations for the response amplitude and phase processes which, in turn, were obtained using a second order stochastic averaging procedure. Furthermore, the same procedure was also used to investigate the effect of noise on frequency entrainment of harmonically driven Van der Pol oscillator (Manohar & Iyengar 1991a) and in the study of rocking of rigid blocks under random base motions (Iyengar & Manohar 1991). The scheme of partial linearization has also been studied recently by

Elishakoff & Cai (1993). To & Li (1991) have presented a systematic equivalent nonlinearization procedure which is again based on the broad class of exactly solvable FPK equations and utilizes calculus of variations to derive the optimal replacement system. In a study on systems with asymmetric nonlinearities, Spanos & Donley (1991) and Li & Kareem (1993) have developed equivalent systems with quadratic nonlinearities. The evaluation of the equivalent parameters is based on an approximate analysis of the equivalent systems using Volterra series representations (see § 8).

6. Closure approximations

In nonlinear random vibration problems the equations for response moments and correlations form an infinite hierarchy and exact solutions are not possible. This is true even for the class of systems for which exact response pdf is obtainable using the FPK equation. The closure problem consists of approximately replacing the infinite hierarchy of equations with a finite set so that estimates for the important lower order moments can be obtained.

6.1 *Closure using assumed probability density function*

The closure approximation can be made either in conjunction with an assumed response pdf or directly on the moment equations. Dashevskii (1967), Assaf & Zirkle (1976) and Crandall (1980, 1985) have employed a series representation for response pdf in terms of the Hermite polynomials. The series is truncated after a finite number of terms. The first term in the series has the form of a Gaussian pdf. The unknown coefficients of the series are related to the response cumulants, central moments or expectations of the Hermite polynomials in response variables. The equations needed to determine these coefficients are generated from the governing equation of motion. This procedure can be viewed as the generalization of equivalent linearization technique wherein the response was assumed to be Gaussian and the parameters in the distribution were determined using moment identities derived from equations of motion. Liu & Davies (1988) have applied Hermite polynomial approximation for the pdf and studied the nonstationary response of nonlinear second-order systems. Furthermore, the same authors (Davies & Liu 1992) have also studied the power spectrum of Duffing's oscillator using a similar procedure. Iyengar (1975) and Iyengar & Dash (1976, 1978) have developed a closure technique in which the response variables and input variables, either as they appear or after a transformation, are assumed to be jointly Gaussian. It is possible in this formulation to take into account non-Gaussian excitations and amplitude limited responses. The method handles nonlinear and stochastically time varying systems in a unified manner (Dash & Iyengar 1982).

6.2 *Closure in terms of moments or cumulants*

In the second approach, one directly deals with moment equations. Here the unknown higher order moments are approximated as functions of lower order moments, thereby truncating the hierarchy of these equations. Thus, Ibrahim (1978), Ibrahim *et al* (1985), Bolotin (1984) and Wu & Lin (1984) have considered different schemes for closing hierarchy of moment equations. Appeal is generally made to the quasinormal approximation which connects are higher order moments to the lower order moments through relations that are strictly valid only for Gaussian variables. This can be expressed in terms of either the direct moments, central moments or cumulants. The quasinormal approximation using cumulants amounts to setting cumulants beyond a given order to zero. For a Gaussian random variable, it may be recalled, all the cumulants beyond order two vanish. Other closure schemes such as discarding the direct or central moments beyond an order or ignoring the correlations among the response variables have also been proposed (Soong 1973). Bellman & Richardson (1968), Wilcox & Bellman (1970) and Sancho (1970) have used a mean square closure technique in which the unknown higher order moment is expressed as an optimal linear combination of lower order moments.

6.3 *Limitations and improvements*

The closure methods are applicable to a wide class of systems and excitations. They have been extensively used in the response and stability analysis of sdof and mdof systems with parametric and nonparametric excitations. However, it has not been possible to justify closure approximations through analytical approaches. Most of the closure schemes are theoretically inconsistent at some level as they violate well-known identities and inequalities of probability theory. Thus, for example, the moment closure scheme of setting direct moments beyond order n to zero violates the inequality $\mathbf{E}[x^{2n}] > \mathbf{E}[x^n]^2$. The cumulant closure violates the theorem due to Marcienkiewicz (Gardiner 1983) which states that the cumulant generating function cannot be a polynomial of order higher than two, that is, either all but the first two cumulants vanish or there are an infinite number of nonvanishing cumulants.

Bellman & Richardson (1968) have derived a condition under which the truncated equations obtained using the mean square closure technique preserves moment properties. No similar results are available for other closure schemes. Although it has been demonstrated with a specific example that the accuracy of closure scheme systematically improves as the order of approximation is increased, examples to counter this are also readily available (Crandall 1985). Instances of the estimated pdf becoming negative have also been encountered (Crandall 1985). Sun & Hsu (1987) have applied second, fourth, and sixth-order cumulant neglect schemes to a specific problem for which exact stationary solution is available and they have delineated regions where the schemes yield acceptable results. In certain parameter regions the fourth- and sixth-order schemes are shown to predict erroneous behaviour including a faulty jump in the response. Fan & Ahmadi (1990) have considered a system with multiple stable equilibrium driven by white noise excitation. They have shown that the stationary response statistics generated by Gaussian closure and non-Gaussian cumulant neglect closure techniques are not unique and are dependent on initial conditions. This contradicts the uniqueness property of the stationary solutions of the FPK equation (see §4·1). Pawleta & Socha (1992) have compared approximate nonstationary solutions obtained using closure approximations with the corresponding exact solutions for the case of parametrically excited linear systems and have shown that near stability boundaries the approximations are not acceptable.

The second-order cumulant neglect and the Gaussian closure techniques are similar to equivalent linearization method and are consistent closure schemes. However, they yield acceptable results only when the response has features of Gaussian variables. For example, when applied to random vibration of self-excited systems (Bolotin 1984) they lead to drastically wrong results (Zhu & Yu 1987; Manohar & Iyengar 1990) (see § 4·1). Recently Grigoriu (1991) has developed a consistent closure procedure which is based on an estimator of the response pdf that consists of superposition of specified kernels weighted by undetermined parameters. These unknown parameters are determined based on the criterion that the moment equations are optimally satisfied up to a specified closure level.

7. Stochastic averaging methods

In these methods the response of lightly damped systems to broad band excitation is approximated by a diffusion process. The coefficients of the associated FPK equation are derived based on an appropriate averaging of the equations of motion. The appeal of these methods lies in the fact that they often reduce the dimensionality of the problem and significantly simplify the solution procedures. On account of this advantage they are also applied to systems wherein the response is already Markov. Different versions of the method are available and are widely used in the problems of response prediction, stability analysis and the first passage and fatigue failure analyses. Extensive surveys of related literature have been published (Ibrahim 1985; Roberts & Spanos 1986; Roberts 1986a; Zhu 1988).

7.1 *Averaging of amplitude and phase*

The method was originally proposed by Stratonovich (1963, 1967) as a generalization of the deterministic averaging method developed earlier by Bogoliubov and Mitropolsky (1961). He considered sdof nonlinear systems under random excitation and showed that when the relaxation time of the oscillator is large compared to the correlation time of the excitation, the response can be approximated by a diffusion Markov process. Subsequently, Khasminiskii (1966) provided a rigorous mathematical proof for Stratonovich's arguments. The necessary requirements for applying the method are satisfied if the system is lightly damped and the excitation power spectrum is slowly varying in the neighbourhood of the system's natural frequency. The response in such a case will be a narrow band process with slowly varying amplitude and phase. The averaging procedure is a combination of temporal and ensemble averaging and it aims at eliminating rapid oscillations from the dominant slowly varying components and also at replacing randomly fluctuating components by equivalent delta correlated processes. This results in a pair of Itô differential equations for amplitude and phase which will have to be analysed using the FPK equation. In many cases the equation for amplitude gets uncoupled from that of the phase thus enabling the determination of the stationary distribution of the amplitude process. In fact this is the main advantage of this method. In order to determine the stationary pdf of displacement and velocity variables, the knowledge of joint pdf of amplitude and phase is essential. But it is in general difficult to obtain this pdf. However under the assumption that amplitude and phase are independent it is still possible to obtain an approximation to pdf of displacement and velocity.

Stratonovich used this method to examine the response of sdof self-excited systems to parametric and nonparametric excitations. Subsequently, the method has been generalized to include mdof systems and nonstationary inputs and widely used in random vibration studies (Roberts & Spanos 1986). The method has also formed the basis for the study of first passage failures (Roberts 1986a) and stability analysis (Ibrahim 1985). For problems wherein the time varying nature of the system such as deterministic excitations or nonstationary inputs needs to be preserved, Lin (1986) has proposed that the temporal averaging in Stratonovich's procedure may be dispensed with. Heuristic arguments for relaxing restrictions on time constants of input and response for problems of stability analysis have also been given.

7.2 *Quasistatic averaging*

A variation of the standard stochastic averaging method, known as the method of quasistatic averaging, has also been developed by Stratonovich (1967). The method is applicable to problems in which the correlation time of the excitation greatly exceeds the relaxation time of the system. This requirement is contrary to the one stipulated for the applicability of the standard stochastic averaging method. The method consists of only temporal averaging. The ensemble averaging with its attendant Markovian approximation is dispensed with. In applying this method, envelope representation is used for both the input and the response processes. During temporal averaging the amplitude and phase angle are approximated as random variables and hence as constants. This finally leads to a nonlinear memoryless transformation relating the input and the output amplitudes and phase angles. Thus the solution of the given random differential equation is converted to a problem in nonlinear transformation of random variables. The method has been used in the study of nonlinear systems under narrow band excitations by several authors (Lennox & Kuak 1976; Sato *et al* 1985; Richard & Anand 1983; Iyengar 1986).

7.3 *Averaging of energy envelope*

The stochastic averaging method is found to give acceptable results for systems with nonlinear damping. In fact when damping is amplitude dependent and the excitation is white noise, the method leads to the known exact solutions (Roberts 1976). However, for a system with nonlinear stiffness, such as Duffing's oscillator, the solution does not display the effects of nonlinearity. In such cases a higher order averaging procedure needs to be used (Stratonovich 1967; Ibrahim 1985). This, however, involves cumbersome calculations. A simpler alternative is to examine whether a one dimensional Markovian approximation can be obtained for the energy envelope of the response. This was originally proposed by Stratonovich (1963) who considered systems under white noise inputs and reduced the two dimensional Markovian vector consisting of a slowly varying energy envelope and a rapidly varying displacement component to a one dimensional Markovian approximation for the energy envelope. This method has further been developed by Roberts (1976, 1978) and generalized to incorporate nonwhite inputs (Roberts 1982), parametric excitations (Zhu 1983) and nonstationary inputs (Red-Horse & Spanos 1992). Here, the averaging is carried out over a period equal to the undamped natural period of the system, which, now depends on the energy of the response. The results obtained using this method also agree with the available exact solutions. Zhu & Lin (1991) and Zhu *et al* (1994) have considered systems with correlated Gaussian excitations and have included the additional contributions to damping and stiffness made by the Wong–Zakai correction terms. These additions are incorporated into the definition of the energy envelope and the consequent new results are shown to be improvements over earlier averaging results. Another version of stochastic averaging is also available (Sunahara *et al* 1977). In this a deterministic averaging is carried out directly on the coefficients of the governing FPK equation. This method has been shown to be equivalent to the averaging of the amplitude or the energy envelope of the response (Zhu 1988).

7.4 *Combination of averaging with other methods*

The method of stochastic averaging has also been used in combination with other methods of random vibration analysis. Thus, Iwan & Spanos (1978) proposed a combination of equivalent linearization and stochastic averaging to analyze systems with nonlinear stiffness. For the case of the Duffing oscillator under white noise input, the method improves the results obtained using averaging of response amplitude but does not lead to the exact solutions. Furthermore, Ariaratnam (1978) has questioned the consistency of approximations made in this analysis. Stratonovich (1967) has used equivalent linearization technique to solve simplified equations obtained using stochastic averaging. Bruckner & Lin (1987b) have adopted a complex form of stochastic averaging which eases the application of non-Gaussian closure technique to the simplified equations and is particularly useful in analyzing nonlinear mdof systems. In the study of nonlinear systems under combined harmonic and random excitations or when a higher order averaging in Cartesian co-ordinates is done, the resulting simplified equations do not get uncoupled, and, in general, are unsolvable within the framework of the Markov process theory. Under such situations Manohar & Iyengar (1990, 1991a) have proposed combining averaging with equivalent nonlinearization technique. This procedure is shown to give satisfactory results for the case of Van der Pol's oscillator under broad band and combined harmonic and white noise excitations.

7.5 *Method of stochastic normal forms*

An alternative way of reducing the dimensionality of the problem using modern bifurcation theories, *viz*, center manifold theory (Guckenheimer & Holmes 1983), has been developed by Sri Namachchivaya & Lin (1991). The method consists of eliminating certain response variables which are asymptotically stable as being unimportant with the essential behaviour of the system restricted to the dynamics of the remaining critical variables. The differences between this method and the traditional averaging arise in carrying out the 'temporal' part of the averaging, while, the ensemble averaging, with the consequent Markovian approximation, remains essentially the same. In fact, the equivalence of this method with a higher order stochastic averaging has been demonstrated (Sri Namachchivaya & Leng 1990). The approach has been employed in the study of the effects of noise on bifurcations in nonlinear systems and for specific cases, the method is shown to be more generally applicable than the stochastic averaging (Sri Namachchivaya 1991; Leng *et al* 1992).

7.6 *System stochasticity problems*

Although, the averaging methods are widely used in vibration problems, the idea of applying them to problems of spatial variability is novel. Thus, the usefulness of the averaging method in the study of stochastic boundary value problems has been investigated by Manohar & Iyengar (1993, 1994) in the context of the determination of the eigensolutions of stochastic wave equations. Here, the given boundary value problem is converted into a set of initial value problems and, these are, in turn, simplified by averaging over spatial domain. The results obtained on the pdf of the eigensolutions using this approximation is found to compare very well with digital simulation results.

7.7 *Summary*

The methods of stochastic averaging enhance the scope of the FPK equation approach in random vibrations. The different versions of this method are mathematically well founded (Zhu 1988). This is in contrast to other approximate techniques discussed earlier. The other merit of these methods is that they lead to non-Gaussian estimates for the response.

8. Stochastic series solutions

A widely used method in deterministic problems is the one based on the representation of the solution in an infinite series. Here an unknown function is expanded in a set of known functions. A few studies based on the extension of this concept to stochastic problems are available in random vibration literature. Thus, Iyengar & Dash (1976) have considered a linear sdof system with both parametric and external random excitations. The parametric excitation is taken to be a nonwhite process. The response is expanded in a power series in the random coefficient process. The unknown coefficients in this series are taken to be deterministic and are determined based on the minimization of the mean square error in an interval of time. Ahmadi and his coworkers (Ahmadi 1980a; Jahedi & Ahmadi 1983; Orabi & Ahmadi 1987a, 1987b, 1988) have used the Weiner–Hermite functions in the study of the Duffing oscillator under stochastic excitation. These functions are a set of statistically orthogonal functions and form a complete random basis for expanding a given random process. See the book by Schetzen (1980) for a systematic account of the Volterra and Wiener theories of nonlinear systems. In the solution both the input and the response processes are expanded in a set of these functions. The orthogonality property further leads to a set of nonlinear coupled integro-differential equations for the unknown kernels in the expansion. These equations have been further solved using an iterative technique. For a Gaussian random process the series consists of only one term and thus the higher order terms in the series are non-Gaussian corrections. Thus the method systematically leads to non-Gaussian estimates for response statistics. However, it is not possible to obtain the expressions for the non-Gaussian pdf with this method. Using a single term in the expansion has been shown to be equivalent to the technique of equivalent linearization (Ahmadi & Orabi 1987). Recently, Ghanem & Spanos (1993) have studied random response of second order nonlinear systems using series expansions for both the excitation and response processes which consist of unknown deterministic functions of time which are weighted by known set of orthogonal random variables.

The excitations are represented exactly using the Karhunen–Loeve expansions. Following the concept of Galerkin expansions, the response is also expanded using the same basis random variables which are used for representing the excitation. This leads to a set of deterministic ordinary differential equations for the unknown functions which can be solved numerically. The method has been applied to the response analysis of Duffing's oscillator subjected to filtered white noise excitation yielding satisfactory results.

9. Digital simulation technique

For problems which are beyond the reach of exact or approximate analysis, the digital simulation technique forms the only means of solution. This technique is also useful in checking the validity of approximate analysis procedures. Here one follows a sample function approach in which the problem is handled largely in a time domain deterministic framework. The application of this method consists of three steps: (i) Simulation of random inputs, (ii) discretization of the stochastic model and generation of response, and (iii) statistical processing of samples of response. Thus, at every stage, the method requires the availability of efficient computers. The basic tool for generating random inputs is the pseudo-random number generator which is a deterministic algorithm that produces a set of numbers which are statistically indistinguishable from uniformly distributed random numbers (Chambers 1967). Scalar and vector random variables of specified distributions can be obtained by suitably transforming the uniformly distributed random numbers (Ripley 1987). Comprehensive reviews on generation of random processes and fields using spectral representations and discrete time series models are available respectively, in the works of Shinozuka & Deodatis (1991) and Spanos & Mignolet (1989). In the response analysis numerical schemes such as Runge–Kutta or predictor–corrector algorithms are used to solve the equations of motion. The method has been widely used in nonlinear response analysis (see, for example, Lutes & Shah 1973; Vaicatis *et al* 1974, Spanos 1980a, 1981b, Zhu *et al* 1993, Manohar & Iyengar 1990) and in first passage problems (Crandall *et al* 1966; Roberts 1976; Pi *et al* 1971; Spanos 1983; Iyengar & Manohar 1991). It has also been extensively used in problems of system stochasticity, parametric excitations and stochastic stability analyses (Shinozuka & Deodatis 1991).

The simulation technique has vast scope and is uniformly applicable to nonlinear and parametric response analysis of sdof and mdof systems. However, in order to obtain reliable estimates of response variables, sufficiently large size of samples should be used in the analysis. This fact makes the method significantly expensive. This is particularly true in response analysis involving estimation of rare events and in the study of large mdof systems. The method is well suited for stationary response analysis where assumption of ergodicity is admissible. Spanos (1981a) has estimated that the cost of simulation studies is typically 100 to 1000 times that of an approximate analysis using equivalent linearization. It has been found that the cost of simulation increases linearily with sample size while the accuracy improves in proportion to the square root of sample size (Spanos & Lutes 1987). Nevertheless, given the strides made over the last few years in computer technology, one can readily forsee the broadening of the scope of digital simulation techniques in engineering stochastic analyses.

10. Summary and conclusions

Various methods for stochastic response analysis have been outlined in the previous sections. Exact solutions are obtainable from the FPK equation approach but are scarce. It is generally necessary to take recourse to one of the several approximate procedures available. Many of the approximations are based on the assumption that the response process is nearly Gaussian distributed and/or is Markovian or nearly Markovian in nature. The approximate techniques based on the FPK equation are applicable to Markovian responses and are largely confined to lower order systems. The perturbation method is applicable to weakly nonlinear systems under weak stochastic inputs. They are useful in getting non-Gaussian estimates for response moments but are cumbersome and relatively inefficient. Equivalent linearization is useful for systems with Gaussian inputs and is applicable over a wide range of nonlinearity. The method leads to Gaussian estimates for the response and hence is suited for problems where the excitation and the system are such that the response is unimodal and nearly Gaussian. When applied to systems with multimodal response probability densities, the method leads to nonunique response statistics and requires careful interpretation. The method is not applicable for evaluating parametric responses. Equivalent nonlinearization methods are based on the class of exactly solvable FPK equations and are limited to systems driven by white noise excitations. The closure schemes are an improvement over equivalent linearization as they can take into account parametric excitations and can lead to non-Gaussian estimates for the response. The methods are however mathematically not well founded. The perturbation, equivalent linearization and closure methods are all fairly general and are applicable to both transient and steady state analysis of sdof and mdof systems. Stochastic averaging methods are applicable to lightly damped systems with parametric and external broad band inputs. They are mathematically well founded and in the case of systems under white noise for which the FPK equations are solvable, the methods lead to the exactly known solutions. These methods reduce the dimensionality of the problem and widen the scope of the FPK equation approach. Their applicability, however, is largely limited to sdof systems. Digital simulation technique is universally applicable and leads to estimates of the response to any desired level of accuracy. The method relies on the availability of a fast computer and is quite expensive, especially, in the study of large scale systems and in the analysis of rare events.

The above methods have been applied in the past to study a variety of nonlinear problems such as structures undergoing large amplitude vibrations, yielding systems, self-excited systems, hysteretic systems, vibroimpact systems and rocking of blocks. The developments of these methods are characterized by two conflicting objectives. Firstly, the methods are expected to be viable when applied to large-scale engineering structures, while, on the other hand, they need to capture correctly, the qualitative behaviour of nonlinear systems. The linearization and closure methods are, perhaps, the only feasible analytical methods which can be used in conjunction with computational structural models for studying large scale mdof systems. One of the major drawbacks of these methods, however, lies in their inability to capture correctly the interactions between equilibrium states of the unforced system and external random excitations. It has been noted by Andronov and others as far back as in 1933 (Bolotin 1967; Kozin 1969) that, for systems under white noise excitations, the most probable response states correspond to the stable equilibrium states of the unforced system (also see Kapitanaik 1986). Similar behaviour can be expected in the study of the random response of systems exhibiting complicated bifurcation patterns such as those associated with jumps, limit cycles, nonlinear resonances and entrainments. It is important to recognize that corresponding to multiple stable solutions of the unforced system, the stochastic response probability density functions can be multimodal. Evidently, the linearization and closure techniques are ill equipped to model these nonlinear features satisfactorily. On the other hand, the averaging and FPK equation based approaches are mathematically well founded and perform well when applied to simple systems displaying the above mentioned complicated response patterns, but are, however, of limited use in analysing large scale structures. Thus, methods to overcome these limitations still need to be developed.

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