Nonlinear systems under random differential support motions : response analysis and development of critical excitation models

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Abstract

The response of single degree/multiple degrees of freedom structural systems with nonlinear multiple supports, which are excited by differential, jointly stationary, Gaussian random support motions is considered. The study consists of two parts. In the first part, an approximate response analysis, based on the method of equivalent linearization in conjunction with dynamic stiffness matrix formalisms, is presented. The acceptability of the approximate solutions is examined with the help of Monte Carlo simulation results. The next part of the study addresses the issue of highest response of the system under a set of partially defined inputs. Specifically, it is assumed that the knowledge on auto-power spectral density functions of the support motions is available while the corresponding cross power spectral density functions are considered to be unknowns. These unknown functions are determined such that the steady state variance of a chosen response variable is maximized. The relevance of the study to problems in earthquake engineering is discussed.

Key words : Multi-support excitations; critical excitations; random vibrations.

1.0 INTRODUCTION

The present study deals with the seismic response analysis of structural systems which are supported at multiple points on nonlinear springs. The support motions are taken to be a vector of jointly stationary Gaussian random processes. Typical examples for such type of structural systems arise in the earthquake response analysis of piping structures housed in industrial power plants. These structures are multiply supported and, in the event of an earthquake, the supports suffer differential random motions. Nonlinearity in the supporting system might arise due to the inelastic behavior of the primary structure or due to special supporting devices, such as snubbers and gaps, which are activated during earthquakes.

The literature on random vibration analysis of nonlinear structural systems is extensive (see for example, Roberts and Spanos 1990, Socha and Soong 1992, Ibrahim 1996, Manohar 1995 and Schueller 1997). However, the studies on nonlinear system response to multi-component and multi-support motions are rather limited. The available literature on this topic mostly deal with response analysis of multi-supported secondary piping systems under earthquake excitations; see the review papers by Chen and Soong (1988), Lin (1991) and Villaverde (1997) for overviews on seismic response of secondary systems. Besides, most of these studies treat excitations as being deterministic. Shah and Hartman (1981) employed direct and modal space time integration methods to analyze the response of a typical nuclear power plant piping system. Accurate representation of effect of rigid body motions required a large number of modes to be included in the analysis. Based on results of shake table tests, Suzuki and Sone (1989) have proposed a response reduction factor to compute nonlinear response from the estimates of linear response. The nonlinearities arising from gaps and friction were considered. The reduction factors were reported as a function of input maximum acceleration with the maximum reduction of about 30 % for acceleration of about 0.3 g where g is the accelerations due to gravity. Igusa and Sinha (1991) presented a procedure for linear secondary systems with nonlinear supports subjected to random excitation. Equivalent linearization method was used to linearize the nonlinear behavior of hysteresis loops. This method separates the nonlinear analysis of supports from the linear analysis of secondary systems. They conclude that the mode shapes of equivalent linear structure are the indicators of the effect of stress distribution in the secondary structure. Park (1992) studied seismic response of three dimensional piping systems. An approach was presented to evaluate the stochastic response statistics using plastic hinge method and random vibration theories. An orthotropic bi-axial hysteretic model was used to describe the plastic behavior under stochastic dynamic loads. A response spectrum approach was used in the application of equivalent linearization. Messmer (1993) evaluated the influence of snubbers and gap elements on the response of piping systems under seismic excitations. Based on experimental and theoretical results, the study concluded that the modeling of structures by equivalent linear damping, based on total energy, cannot predict accurately deformation history of piping system. In the context of long span land based structures, studies addressing structural nonlinearities and spatially varying ground motion models appear to be very limited; see the study by Gillies and and Shepherd (1981) on multi-component inputs to inelastic bridge structures and the study by Monti *et al.,* (1996) on multi-span inelastic bridges subjected to non-synchronous input support motions.

The concept of critical random earthquake ground motions is also of relevance to the present study. This concept was introduced by Drenick (1970) and since then it has been studied by several authors. Thus, critical earthquake load models in the form of time histories, response spectra and power spectral density (psd) functions have been developed in the existing literature; the relevant literature has been briefly reviewed by Manohar and Sarkar (1995). We mention here briefly the papers dealing with critical excitations for nonlinear systems. Iyengar (1972) considered inputs having known total energy and obtained critical excitations for a class of nonlinear systems in terms of impulse response function of corresponding linear system. He also treated the input total energy as a random variable and obtained the worst possible distribution of the critical response. Drenick and Park (1975) pointed out that the procedure used in the above analysis enforced additional constraints on the input involving the system response. By linearizing the given nonlinear equation around the critical excitation-response pair, Drenick (1977) obtained critical excitations in terms of impulse response of linearized equations. Philippacopoulos and Wang (1982) presented an approach for generation of critical inelastic response spectra. They applied it to a simple single degree of freedom (sdof) nonlinear system with stiffness nonlinearity. They obtained the approximate solution to the problem by equivalent linearization. Westermo (1985) determined the critical responses of linear, elastoplastic and hysteretic sdof systems. For linear systems, the critical excitations were shown to be harmonic and were derivable from the frequency response functions of the systems. The critical excitations for elastoplastic systems, however, were not harmonic. Pirasteh *et al.,* (1988) presented a methodology for constructing the most critical accelerogram from among a broad class of candidate accelerograms for a given site and structure. The critical accelerograms were obtained based on the nonlinear response of structure.

Recently Sarkar and Manohar (1996, 1998) have extended the concept of random seismic critical excitations to cover the cases of multi-component and multi-support excitations. These studies dealt with linear systems subjected to multi-component/multi-support earthquake support motions. These support motions were modeled as a vector of jointly Gaussian random processes with limited knowledge on the power spectral density matrix. The following models were considered

1. The auto-psd functions are known and the amplitude and phase of the cross-psd functions are unknown. In this case the highest response was shown to be produced by fully coherent motions having system dependent phase spectra.

2. The auto-psd functions and the phase spectra associated with the cross-psd functions are known and the coherence spectra are not known. In this case the highest response was shown to be produced by excitations with system dependent coherence functions which represented neither fully coherent motions nor fully incoherent motions.

3. The variance and zero crossing rates of individual components of the excitations are known and also the phase spectra associated with cross psd-functions are known. The psd functions and the coherence functions are not known. This case was analyzed using a sequential linear programming approach.

In the present paper, we extend our earlier studies to determine the critical support motion models for vibrating systems which are multiply supported on nonlinear springs. We begin by considering the case of a sdof system which is doubly supported on cubic nonlinear springs and which is acted upon by a pair of jointly stationary Gaussian random support motions. An approximate analysis based on method of equivalent linearization is developed. This study is further generalized to the analysis of a Euler-Bernoulli beam, which is again taken to be doubly supported on nonlinear springs and which is subjected jointly stationary Gaussian random support motions. The response analysis in frequency domain is carried out by employing method of equivalent linearization in conjunction with response representation in terms of dynamic stiffness coefficients of the beam. Subsequently, for these two systems, a class of critical psd matrix models for the support motions are derived. Also discussed are the issues related to the validation of equivalent linearization using Monte Carlo simulations and questions on feasibility of applying the equivalent linearization method to these problems.

2.0 SINGLE DEGREE OF FREEDOM SYSTEM

2.1 Equation of motion

Consider a doubly supported sdof system with hardening spring characteristics and linear damping subjected to differential support motions as shown in figure 1. The governing equation of motion for this system can be written as

$$m\ddot{y}+\frac{c}{2}[(\dot{y}-\dot{u})+(\dot{y}-\dot{v})]+\frac{k}{2}[(y-u)+(y-v)]+\frac{\alpha}{2}[(y-u)^3+(y-v)^3]=0. \quad (1)$$

Here, y is the absolute displacement response of the mass m, $u(t)$ and $v(t)$ are the support motions, c is the damping coefficient, k is the linear spring rate and α is the nonlinear spring rate. A dot here denotes derivative with respect to time t. The support accelerations are taken to be a pair of jointly stationary, Gaussian random processes with zero mean, auto-psd functions $S_{uu}(\omega)$ and $S_{vv}(\omega)$ and cross psd function $S_{uv}(\omega)=|S_{uv}(\omega)|\exp[-\mathrm{i}\phi_{uv}(\omega)]$.

2.2 Equivalent linearization

To construct an approximate solution to the above equation, we replace the given system by an equivalent system

$$m\ddot{y}+\frac{c}{2}[(\dot{y}-\dot{u})+(\dot{y}-\dot{v})]+\frac{\beta}{2}[(y-u)+(y-v)]=0. \quad (2)$$

Here β is the equivalent parameter to be determined. There are several criteria that one can adopt to determine β (Roberts and Spanos 1990, Socha and Soong 1991). An approach based on equivalence of average total steady state potential energy in the spring is adopted here to find β. Accordingly, we find β such that

$$< E_p >=< E_p^* > \tag{3}$$

where $< \cdot >$ is the mathematical expectation operator, $< E_p >$ is the mean steady state potential energy for the nonlinear system and $< E_p^* >$ is the mean steady state potential energy for the equivalent linear system. These quantities are given respectively by

$$< E_p >= \frac{k}{4}\{< (y-u)^2 + (y-v)^2 >\} + \frac{\alpha}{8}\{< (y-u)^4 + (y-v)^4 >\} \tag{4}$$

and

$$< E_p^* >= \frac{\beta}{4}\{< (y-u)^2 + (y-v)^2 >\}. \tag{5}$$

Using these expressions in equation 3, the equivalent linear spring rate is found to be

$$\beta = \frac{k}{2} + \frac{\alpha}{2}\frac{< (y-u)^4 > + < (y-v)^4 >}{< (y-u)^2 > + < (y-v)^2 >}. \tag{6}$$

To determine $< (y-u)^4 >$ and $< (y-v)^4 >$ we follow the argument that the response of the equivalent linear system is Gaussian and this leads to the simplification

$$\beta = \frac{k}{2} + \frac{3\alpha}{2}\frac{\sigma_1^4 + \sigma_2^4}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

where $\sigma_1^2 =< (y-u)^2 >$ and $\sigma_2^2 =< (y-v)^2 >$. Clearly σ_1 and σ_2 represent, respectively, the standard deviation of relative displacements of the mass with respect to the left and right supports and they form important descriptors of the system response. Putting $w_1 = y - u$ and $w_2 = y - v$ in equation 2, one gets

$$\ddot{w}_1 + 2\eta\omega_{eq}\dot{w}_1 + \omega_{eq}^2 w_1 = -\ddot{u} - 2\eta\omega_{eq}\frac{(\dot{u}-\dot{v})}{2} - \omega_{eq}^2\frac{(u-v)}{2} \tag{8}$$

$$\ddot{w}_2 + 2\eta\omega_{eq}\dot{w}_1 + \omega_{eq}^2 w_2 = -\ddot{v} - 2\eta\omega_{eq}\frac{(\dot{v}-\dot{u})}{2} - \omega_{eq}^2\frac{(v-u)}{2} \tag{9}$$

where $\omega_{eq} = \sqrt{\beta/m}$ is the natural frequency of the equivalent linear system (equation 2) and η is the damping ratio. Based on the standard random vibration theory, the steady state variance of w_1 can be written as

$$\sigma_1^2 =< w_1^2 >= \int_0^\infty |H_d(\omega)|^2 S_1(\omega) \mathrm{d}\omega \tag{10}$$

where $H_d(\omega) = [(\omega^2 - \omega_{eq}^2) + \mathrm{i}2\eta\omega_{eq}\omega]^{-1}$. Furthermore, $S_1(\omega)$ is the psd function of the process $[-\ddot{u} - \eta\omega_{eq}(\dot{u} - \dot{v}) - \frac{1}{2}\omega_{eq}^2(u - v)]$. This function, in turn, can be shown to be given by

$$S_1(\omega) = S_{uu}(\omega)\{1+\eta^2(\frac{\omega_{eq}}{\omega})^2+\frac{1}{4}(\frac{\omega_{eq}}{\omega})^4-\frac{1}{2}(\frac{\omega_{eq}}{\omega})^2\}+S_{vv}(\omega)\{\eta^2(\frac{\omega_{eq}}{\omega})^2+\frac{1}{4}(\frac{\omega_{eq}}{\omega})^4-\frac{1}{2}(\frac{\omega_{eq}}{\omega})^2\}$$

$$+|S_{uv}(\omega)|\{2\cos[\phi_{uv}(\omega)][-\eta^2(\frac{\omega_{eq}}{\omega})^2 - \frac{1}{4}(\frac{\omega_{eq}}{\omega})^4 + \frac{1}{2}(\frac{\omega_{eq}}{\omega})^2]2\sin[\phi_{uv}(\omega)]\eta\frac{\omega_{eq}}{\omega}\}. \tag{11}$$

A similar procedure can now be used to find σ_2. This quantity can be shown to be given by

$$\sigma_2^2 =< w_2^2 >= \int_0^\infty |H_d(\omega)|^2 S_2(\omega) \mathrm{d}\omega \tag{12}$$

where $S_2(\omega)$ is the psd function of the process $[-\ddot{v} - \eta\omega_{eq}(\dot{v} - \dot{u}) - \frac{1}{2}\omega_{eq}^2(v - u)]$. $S_2(\omega)$ can be obtained by swapping u with v in the expression for $S_1(\omega)$ given in eq(11). It may be noted that equation 7, together with equations 10 and 12 for σ_1 and σ_2, result in a nonlinear transcendental equation for the equivalent parameter β.

2.3 Numerical results and Validation

A sdof system with m=1000 Kg, k= 1.58E+05 N/m, α = 6.1E+08 N/m^3 and $\eta = 0.05$ is considered. We take the support accelerations $\ddot{u}(t)$ and $\ddot{v}(t)$ to possess the psd functions of the form

$$S_{uu}(\omega) = S_{vv}(\omega) = \{\phi_0 \frac{\omega_g^4 + (2\eta_g\omega_g\omega)^2}{(\omega_g^2 - \omega^2)^2 + (2\eta_g\omega_g\omega)^2}\}\{\frac{[\frac{\omega}{\omega_f}]^4}{\{(1 - \frac{\omega^2}{\omega_f^2})^2 + 4\eta_f^2\frac{\omega^2}{\omega_f^2}\}}\}. \tag{13}$$

It may be observed that this psd function is written as a product of two functions: the first part corresponds to the well known Kanai-Tajimi psd model for earthquake ground motions and the second part represents a filter which serves to suppress the singularity at $\omega = 0$ in the psd functions for displacement which the Kanai-Tajimi spectrum possesses (Clough and Penzien 1993). The cross psd function $S_{uv}(\omega)$ is taken to be of the form

$$S_{uv}(\omega) = |S_{uv}(\omega)| \exp[-\mathrm{i}\omega\tau] \tag{14}$$

In the numerical work, it is assumed that ω_g=8 π rad/s, ϕ_0=1.0E-03 $\mathrm{m}^2/\mathrm{s}^3$, η_g=0.6, ω_f=2.5 rad/s and η_f=0.8. These parameters correspond to a support motion with root mean square acceleration of $0.028g$ with an average peak ground acceleration of about $0.085g$. Two sets of calculations were done: one for the case of the support motions being stochastically independent ($|S_{uv}(\omega)| = 0$) and the other for the case of the two motions being fully coherent ($|S_{uv}(\omega)| = \sqrt{\{S_{uu}(\omega)S_{vv}(\omega)\}}$). An iterative procedure was used to evaluate the equivalent parameter β. An accuracy of 10^{-4} was sought and this was realized in about six iteration cycles. The possible of presence of multi-valued solutions for β was examined but, however, for the parameter ranges studied, it was found that the governing equation for β lead to single valued solutions. Figures 2-4 show the results of a parametric study in which the steady state standard deviation σ_1 of the relative displacement $w_1(t) = y(t) - u(t)$ is computed by varying the time lag τ (figure 2), nonlinearity parameter α (figure 3) and the ground frequency ω_g (figure 4). These figures also show results for the case of input critical excitation models and these will be explained later in section 4.3. The simulation results shown in these figures were obtained by simulating an ensemble of 250 samples of the input processes $\ddot{u}(t)$ and $\ddot{v}(t)$. The governing equation of motion (equation 1) was integrated numerically for about 50 cycles using a fourth order Runge-Kutta algorithm with as step size of 1/160 s. The response data in the last 30 cycles were temporally averaged for each time history. This temporally averaged result was again used to obtain the ensemble average over the 250 samples. An immediate observation that can be made from figures 2-4 is that the theoretical and simulation results show reasonably good mutual agreement.

The results of equivalent linearization are valid only when the nonlinear system response has broad features of a Gaussian random process. Thus, for instance, the method is likely to succeed only if response amplitude has unimodal probability density function. It has been established in the literature (see, for example, Iyengar 1988, Roberts 1991, Langley 1988), that for nonlinear systems which exhibit multiple modes in the probability density function of the response amplitude, the equivalent linearization does not give valid estimates of nonlinear response. Figure 5 show plots of probability distribution function (PDF) of the response amplitude defined as

$$A(t) = \sqrt{(y-u)^2 + [\frac{\dot{y}-\dot{u}}{\lambda_0^+}]^2} \tag{15}$$

where λ_0^+ is the rate of zero crossing of $y-u$ with positive slope. In these plots the dominant frequency of input is varied and these frequencies are marked on the plots. Two possible estimates of λ_0^+ given by $\lambda_0^+ = \omega_0$ and $\lambda_0^+ = \omega_g$ were considered. The results on probability distribution of $A(t)$ for these two choices of λ_0^+ were observed to be broadly similar. The lack of any abrupt changes in slopes of PDF indicate that the response density functions are essentially unimodal. This would indicate that the equivalent linearization method can be expected to be applicable to analyze the problem on hand.

Digital simulation results provide a quantitative means to examine the accuracy of the approximate solutions such as those predicted by equivalent linearization. The acceptability of an approximate solution can also be examined in a qualitative manner by examining the stability of solutions, see for example, the papers by Iyengar(1988), Manohar and Iyengar (1990), Roberts (1991) and Koliopulos and Langley (1993). According to this criterion, an approximate solution $y_0(t)$ is deemed acceptable if a small perturbation $z(t)$ imposed on $y_0(t)$ is stochastically stable. Thus, in the present study, if the approximate solution given by the equivalent linearization, is perturbed, that is, if we take $y(t) = y_0(t) + z(t)$, the equation governing the perturbation $z(t)$ can be deduced from equation 1 as

$$\ddot{z}+2\eta\omega_0\dot{z}+\omega_0^2 z+\frac{\alpha}{2m}\{2z^3+3z^2[(y_0-u)+(y_0-v)]+3z[(y_0-u)^2+(y_0-v)^2]\} = 0. \tag{16}$$

Here $\omega_0^2 = k/m$. For the equivalent linear system to be acceptable, it is required that the solution of the above equation is stochastically stable. The studies conducted, for instance, by Iyengar (1988), Manohar and Iyengar (1990) and Koliopulos and Langley (1993) adopt approximate analytical approach to ascertain the stochastic stability. A detailed examination of this issue, however, has not been attempted in this study. Instead, limited numerical experimentation has been carried out to examine the time evolution of the perturbation $z(t)$. This consists of obtaining samples of $z(t)$ and examining their behavior. Thus, to implement this, we need to numerically integrate equations 2 and 16 simultaneously, for different samples of $u(t)$ and $v(t)$. Notice that equation 16 has a zero right hand side and, consequently, to obtain non-trivial solutions, it is essential to employ non-zero initial conditions on $z(t)$ and $\dot{z}(t)$. In our study we have taken $z(0)$ and $\dot{z}(0)$ to be a pair of random variables with zero mean and a small standard deviation of about 1 % of the theoretical standard deviation of the steady state $y_0(t)$ and $\dot{y}_0(t)$ respectively. Figure 6 shows a few samples of amplitude of $z(t)$ and the ensemble mean and standard deviation of perturbation amplitude across a sample of 250 members is shown in figure 7. For the parameter ranges considered, the simulation results showed that the samples and the first two moments of the amplitude of perturbation go to zero for large times. This, again, lends credence to the use of equivalent linearization in analyzing the problem on hand.

3.0 DOUBLY SUPPORTED EULER-BERNOULLI BEAM

3.1 Equation of motion

An Euler-Bernoulli beam supported on two discrete springs and which is subjected to support motions $u(t)$ and $v(t)$ is shown in figure 8. The springs are taken to be of the Duffing type, that is, they possess a cubic force-deflection characteristics. The support motions $u(t)$ and $v(t)$ are taken to constitute a vector of zero mean, stationary, Gaussian random processes with psd matrix given by

$$S(\omega) = \begin{bmatrix} S_{uu}(\omega) & S_{uv}(\omega) \\ S_{vu}(\omega) & S_{vv}(\omega) \end{bmatrix}. \tag{17}$$

The cross psd function can be represented as

$$S_{uv}(\omega) = |S_{uv}(\omega)| \exp[-\mathrm{i}\phi_{uv}(\omega)]. \tag{18}$$

The governing equations of motion for this system is given by

$$EI\, Y^{''''}(x,t) + m\ddot{Y}(x,t) + c\dot{Y}(x,t) = 0 \tag{19}$$

with boundary conditions

$$EI\ Y^{'''}(0,t) = -k_1[Y(0,t) - u] - \alpha[Y(0,t) - u]^3$$
$$EI\ Y^{'''}(L,t) = k_2[Y(L,t) - v] + \alpha[Y(L,t) - v]^3$$
$$Y^{''}(0,t) = 0$$
$$Y^{''}(L,t) = 0 \tag{20}$$

and initial conditions

$$Y(x,0) = 0$$
$$\dot{Y}(x,0) = 0. \tag{21}$$

Here, EI= flexural rigidity, m=mass per unit length, c=viscous damping coefficient, k_1, k_2= linear spring rates and α_1, α_2= nonlinear spring rates. A prime in the above equations denotes differentiation with respect to the spatial variable x and a dot, as before, represents derivative with respect to time t. It may be noted that the above governing equation of motion constitute a nonlinear partial differential equation with randomly time varying boundary conditions. The field equation here is linear but the nonlinearity enters through the boundary conditions. Exact solution to this type of problems is currently not possible and one has to take recourse to approximate analysis procedures or to digital simulation strategies. As in the previous section, we apply the method of equivalent linearization to obtain approximate solutions and validate them by using Monte Carlo simulations.

3.2 Equivalent linearization

For the problem on hand (figure 8), to apply the method of equivalent linearization, we replace the nonlinear springs by a pair of equivalent linear springs with spring rates β_1 and β_2. The field equation governing the behavior of the equivalent linear system is identical to equation 19. However the boundary conditions now get linearized and are modified to read

$$\begin{aligned} EI\ Y^{'''}(0,t) &= -\beta_1\ [Y(0,t) - u] \\ EI\ Y^{'''}(L,t) &= \beta_2\ [Y(L,t) - v] \\ Y^{''}(0,t) &= 0 \\ Y^{''}(L,t) &= 0. \end{aligned} \tag{22}$$

To determine the equivalent linear spring rates β_1 and β_2 we demand that the average steady state potential energy stored in the nonlinear system and the linearized system are the same. That is

$$\lim_{t\to\infty} < E^l_{NL}(t) >= \lim_{t\to\infty} < E^l_L(t) >$$
$$\lim_{t\to\infty} < E^r_{NL}(t) >= \lim_{t\to\infty} < E^r_L(t) > \quad (23)$$

where, $E_{NL}(t)$ and $E_L(t)$ are the potential energies stored in the nonlinear and equivalent linear springs respectively. The superscripts l and r denote respectively the left and right springs. These quantities are given by

$$\begin{array}{c} < E^l_{NL} >= \frac{k_1}{2} < (Y(0,t)-u)^2 > + \frac{\alpha_1}{4} < (Y(0,t)-u)^4 > \\ < E^r_{NL} >= \frac{k_2}{2} < (Y(L,t)-v)^2 > + \frac{\alpha_2}{4} < (Y(0,t)-v)^4 > \\ < E^l_L >= \frac{\beta_1}{2} < (Y(0,t)-u)^2 > \\ < E^r_L >= \frac{\beta_2}{2} < (Y(L,t)-v)^2 > . \end{array} \quad (24)$$

In these equations we have omitted to write $\lim_{t\to\infty}$ with an understanding that we are considering the expectations in the steady state. As may be observed, the expression for the expected potential energy in nonlinear springs contain fourth order moments $< (Y(0,t)-u)^4 >$ and $< (Y(L,t)-v)^4 >$. In implementing the equivalent linearization procedure it is assumed $< (Y(0,t)-u)^4 >= 3 < (Y(0,t)-u)^2 >^2$ and $< (Y(L,t)-v)^4 >= 3 < (Y(L,t)-v)^2 >^2$ the justification for this being that the response of the linearized system is going to be Gaussian. Consequently, employing equation 23, the equivalent linear parameters β_1 and β_2 are obtained as

$$\beta_1 = k_1 + \tfrac{3}{2}\alpha\sigma^2_{z_1}$$
$$\beta_2 = k_2 + \tfrac{3}{2}\alpha\sigma^2_{z_2}. \quad (25)$$

Here, $\sigma^2_{z_1}$ and $\sigma^2_{z_2}$ are variance of relative displacements $z_1(t) = [Y(0,t)-u(t)]$ and $z_2(t) = [Y(L,t)-v(t)]$ in the left and right springs respectively; these are given by

$$\sigma^2_{z_1} =< z_1(t)^2 >=< (Y(0,t)-u)^2 >$$
$$\sigma^2_{z_2} =< z_2(t)^2 >=< (Y(L,t)-v)^2 > . \quad (26)$$

It must be noted that the equivalent spring rates β_1 and β_2 are functions of $\sigma_{z_1}^2$ and $\sigma_{z_2}^2$ which in turn depend upon β_1 and β_2. In other words, equation 25 represents a pair of coupled nonlinear equations for β_1 and β_2.

To determine the equivalent parameters β_1 and β_2 we first obtain the psd functions for the processes $z_1(t)$ and $z_2(t)$. This is done by carrying out a stationary response analysis of the equivalent linear system using the dynamic stiffness matrix formulations (Paz 1984). This leads to expressions for the psd functions $S_{z_1z_1}(\omega)$ and $S_{z_2z_2}(\omega)$ given by

$$\begin{aligned} S_{z_1z_1}(\omega) &= H_1(\omega)S_{uu}(\omega) + H_2(\omega)S_{vv}(\omega) + H_5(\omega)|S_{uv}(\omega)| \\ S_{z_2z_2}(\omega) &= H_3(\omega)S_{uu}(\omega) + H_4(\omega)S_{vv}(\omega) + H_6(\omega)|S_{uv}(\omega)|. \end{aligned} \tag{27}$$

Here $H_i(\omega)$ $(i = 1, ..., 6)$ are generalized system frequency response functions which are expressed in terms of the elements of the reduced dynamic stiffness matrix $\mathcal{D}(\omega)$ as follows (see Appendix A for details of $\mathcal{D}(\omega)$):

$$H_1(\omega) = \frac{1}{\omega^4}(1 - 2\mathrm{Re}[\mathcal{D}(1,1)] + |\mathcal{D}(1,1)|^2)$$

$$H_2(\omega) = \frac{1}{\omega^4}|\mathcal{D}(1,2)|^2$$

$$H_3(\omega) = \frac{1}{\omega^4}|\mathcal{D}(3,1)|^2$$

$$H_4(\omega) = \frac{1}{\omega^4}(1 - 2\mathrm{Re}[\mathcal{D}(3,2)] + |\mathcal{D}(3,2)|^2)$$

$$H_5[\omega, \phi_{uv}(\omega)] = g_1(\omega)\cos[\phi_{uv}(\omega)] + g_2(\omega)\sin[\phi_{uv}(\omega)]$$

$$H_6[\omega, \phi_{uv}(\omega)] = g_3(\omega)\cos[\phi_{uv}(\omega)] + g_4(\omega)\sin[\phi_{uv}(\omega)]$$

$$g_1(\omega) = \frac{2}{\omega^4}\Big(\mathrm{Re}[\mathcal{D}(1,1)]\mathrm{Re}[\mathcal{D}(1,2)] + \mathrm{Im}[\mathcal{D}(1,1)]\mathrm{Im}[\mathcal{D}(1,2)]\Big)$$

$$g_2(\omega) = -\frac{2}{\omega^4}\Big(\mathrm{Im}[\mathcal{D}(1,1)]\mathrm{Re}[\mathcal{D}(1,2)] - \mathrm{Re}[\mathcal{D}(1,1)]\mathrm{Im}[\mathcal{D}(1,2)]\Big)$$

$$g_3(\omega) = \frac{2}{\omega^4}\Big(\mathrm{Re}[\mathcal{D}(3,1)]\mathrm{Re}[\mathcal{D}(3,2)] + \mathrm{Im}[\mathcal{D}(3,1)]\mathrm{Im}[\mathcal{D}(3,2)]\Big)$$

$$g_4(\omega) = -\frac{2}{\omega^4}\Big(\mathrm{Im}[\mathcal{D}(3,1)]\mathrm{Re}[\mathcal{D}(3,2)] - \mathrm{Re}[\mathcal{D}(3,1)]\mathrm{Im}[\mathcal{D}(3,2)]\Big). \tag{28}$$

It is clear that the functions $H_2(\omega)$ and $H_4(\omega)$ are non-negative. It must also be noted that the functions $H_1(\omega)$ and $H_3(\omega)$ are also non-negative. This can be demonstrated by considering $u(t) \neq 0$ and $v(t) = 0$. In this case we get $S_{z_1z_1}(\omega) = H_1(\omega)S_{uu}(\omega)$ and $S_{z_2z_2}(\omega) = H_3(\omega)S_{uu}(\omega)$. Let the functions $H_1(\omega)$ and $H_3(\omega)$ be negative for $\omega = \bar{\omega}$. If we take now $S_{uu}(\omega)$ to be a narrow banded with central frequency at $\omega = \bar{\omega}$, it follows that the response psd $S_{z_1z_1}(\omega)$ and $S_{z_2z_2}(\omega)$ are negative at $\omega = \bar{\omega}$. This violates the well known fact that psd function cannot be negative. Thus, the premise that $H_1(\omega)$ and $H_3(\omega)$ can become negative is invalid. It is easy to deduce that the functions H_5 and H_6 on the other hand can take either negative or positive values for different values of ω. It is important to note that these properties of the frequency response functions $H_i(\omega)$ $(i = 1, 2, \cdots, 6)$ are central to the results on response bounds to be discussed in section 4.0. In further work an alternative representation of equation 27 as given below becomes useful:

$$\begin{aligned}
S_{z_1z_1}(\omega) &= H_1(\omega)S_{uu}(\omega) + H_2(\omega)S_{vv}(\omega) + R_1(\omega)|S_{uv}(\omega)| \cos[\alpha_1(\omega) - \phi_{uv}(\omega)] \\
S_{z_2z_2}(\omega) &= H_3(\omega)S_{uu}(\omega) + H_4(\omega)S_{vv}(\omega) + R_2(\omega)|S_{uv}(\omega)| \cos[\alpha_2(\omega) - \phi_{uv}(\omega)]
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
R_1(\omega) &= \sqrt{g_1^2(\omega) + g_2^2(\omega)} \\
R_2(\omega) &= \sqrt{g_3^2(\omega) + g_4^2(\omega)} \\
\alpha_1(\omega) &= \tan^{-1}\{\frac{g_2(\omega)}{g_1(\omega)}\} \\
\alpha_2(\omega) &= \tan^{-1}\{\frac{g_4(\omega)}{g_3(\omega)}\}.
\end{aligned} \tag{30}$$

Once $S_{z_1z_1}(\omega)$ and $S_{z_2z_2}(\omega)$ are determined, the steady state variance of the relative displacements $z_1(t)$ and $z_2(t)$ can be found using the well known relations

$$\begin{aligned}
\sigma_{z_1}^2 &= \int_0^\infty S_{z_1z_1}(\omega)d\omega \\
\sigma_{z_2}^2 &= \int_0^\infty S_{z_2z_2}(\omega)d\omega.
\end{aligned} \tag{31}$$

As has been already noted, equation 25 represents a pair of nonlinear equations for the equivalent linear parameters β_1 and β_2.

3.3 Numerical Results

The formulations presented in the previous sections is illustrated by considering the response of system shown in figure 8 with the area of cross section $A_0 = 3.71E - 03$ m^2, moment of inertia I=2.1E+04 m^4, mass density ρ=2700 kg/m^3, Young's modulus E=2.1E+05 N/m^2, damping constant C=63.0 N-s/m^2, linear spring stiffness k_1=k_2=1.767E+05 N/m and length L= 8 m. The nonlinear spring rate α_1 and α_2 are taken to be in the range of 2.0E+06 to 4.0E+08. The first few natural frequencies of this system, with $\alpha_1 = \alpha_2 = 0$, that, is with nonlinearity, are found to be 21.1, 57.8, 97.7, 173.7 and 310.0 rad/s. The auto-psd functions $S_{uu}(\omega)$ and $S_{vv}(\omega)$ for the ground accelerations $\ddot{u}(t)$ and $\ddot{v}(t)$ are again taken to be given by

$$S_{uu}(\omega) = S_o\{\frac{(1+4\eta_{g1}^2(\frac{\omega}{\omega_{g1}})^2)}{(1-(\frac{\omega}{\omega_{g1}})^2)^2+(4\eta_{g1}^2(\frac{\omega}{\omega_{g1}})^2)}\}\{\frac{(\frac{\omega}{\omega_f})^4}{(1-(\frac{\omega}{\omega_f})^2)^2+(4\eta_f^2(\frac{\omega}{\omega_f})^2}\} \quad (32)$$

$$S_{vv}(\omega) = S_o\{\frac{(1+4\eta_{g2}^2(\frac{\omega}{\omega_{g2}})^2)}{(1-(\frac{\omega}{\omega_{g2}})^2)^2)+(4\eta_{g2}^2(\frac{\omega}{\omega_{g2}})^2)}\}\{\frac{(\frac{\omega}{\omega_f})^4}{(1-(\frac{\omega}{\omega_f})^2)^2+(4\eta_f^2(\frac{\omega}{\omega_f})^2}\}. \quad (33)$$

In the numerical work the parameters S$_o$, ω_{g1}, ω_{g2}, η_{g1} and η_{g2} are taken to be 0.01, 20 rad/s, 20 rad/s, 0.6 and 0.4 respectively. The above parameters represent a root mean square acceleration level of input as 0.081g and the corresponding zero period acceleration of 0.241g for a peak factor of 3.0 where g is acceleration due to gravity. The filter function parameters ω_f and η_f are taken to be 5 rad/s and 0.52 respectively. Detailed numerical studies have been conducted to examine the nature and validity of the solution based on equivalent linearization. An iterative procedure was employed to determine the equivalent parameters β_1 and β_2. The iterative cycles involved in computing β_1 and β_2 were seen to converge to an accuracy of about 1.0E-06 within about 6 to 8 number of cycles. For the parameter ranges considered, β_1 and β_2 were observed to be single valued. Figures 9-11 show the steady state variance of the force in the left spring as a function of linear spring rate $k = k_1 = k_2$ (figure 9), nonlinear spring rate $\alpha = \alpha_1 = \alpha_2$ (figure 10) and excitation frequency parameter $\omega_g = \omega_{g1} = \omega_{g2}$ (figure 11). Results from both the analytical and Monte Carlo simulations are shown in these figures. In the simulation studies, the beam was discretized into 5 number of elements using the traditional finite element method. The discretized equations were integrated using a step size of 1/2800 s. The integration was carried out for a length of about 78 s using a fourth order Runge-Kutta algorithm. The first 4 s of this sample was discarded to allow for the dissipation of the transients. Variance of the response was estimated by employing temporal averaging of the shortened sample. Twenty estimates of sample variance were obtained and these were subsequently averaged to arrive at the final estimate of the response variance. Figure 12 shows the sample probability density functions of the force in the left spring. These density functions are estimated using 10000 number of points. Results from 20 samples are superposed in this figure. In these plots the response mean has been removed and the response is normalized to have unit standard deviation.

3.4 Discussion

The spring supported structure shown in figure 8 approaches a simply supported beam as the spring rates $k_1 \to \infty$ and $k_2 \to \infty$. Consequently for large values of spring rates, the system behaves as a linear system no matter what values the nonlinear spring rate α_1 and α_2 take. This feature can be observed in figure 9, where it is observed that for large values of $k_1 = k_2 = k$, the response variance reaches a constant value. It has been verified that this constant value indeed corresponds to the response variance of a simply supported beam suffering differential support motions $u(t)$ and $v(t)$. This verification is based dynamic stiffness matrix analysis of a simply supported beam and the calculations used are independent of the response analysis for spring supported beam. The influence of increase in α for a fixed value of k is to reduce the response variance; see figure 10. The simulation and analytical results agree well for α less than about 2.0E+07 N/m^3, beyond which, the accuracy of the linearization solution is found to deteriorate. The probability density functions shown in figure 12 (with mean removed and standard deviation normalized to unity) indicate that the response has unimodal probability density functions and it retains the broad features of Gaussian function. This would mean that the method of equivalent linearization is, in principle, applicable to analyze the problem.

4.0 PARTIALLY SPECIFIED INPUTS AND CRITICAL CROSS psd MODELS

4.1 Background

The formulation developed in the previous section enables the response analysis to be made when information on the psd matrix of the input vector random process is available. In this section we consider the situations in which the input psd matrix is only partially specified. Specifically, we assume that the diagonal terms in this matrix which represent the auto-psd functions are known while the off-diagonal terms which represent the cross psd functions are not known. It is aimed to find the cross psd functions which produce the highest and the lowest response steady state variance. The motivation for considering this type of problems has been outlined in our earlier works (Sarkar and Manohar 1996,1998). Briefly stated, we are considering situations in which the earthquake loads on the structure are specified in terms of design response spectra of the various excitation components. These response spectra, by definition, do not encapsulate information on phase lag and coherency loss among excitation components and hence the method of specifying earthquake loads *via* a set of response spectra is at a disadvantage. We overcome this difficulty by recasting the problem within the framework of random vibration theory. This calls for establishing auto-psd functions compatible with the given set of response spectra. This can be achieved by using results from extreme value statistics that form the basis of standard random vibration theory. We would not, however, be able to determine the input cross psd functions by this means. Thus, we get the situation in which we have input random processes whose psd matrix is only partially specified. Consequently, the problem of finding the optimal cross psd functions that bound the response variance becomes of considerable interest.

4.2 Critical cross psd models

The determination of the critical and most favorable cross psd functions is based on the analysis using the equivalent linearization solution outlined in the previous sections. We envisage the following alternative scenarios in establishing the optimal cross psd functions:

Model I To find optimal $|S_{uv}(\omega)|$ given $S_{uu}(\omega)$, $S_{vv}(\omega)$ and $\phi_{uv}(\omega)$

Here we take $\phi_{uv}(\omega) = \omega\tau_0$ with $\tau_0 = L/V$, $L=$ distance between the supports and $V=$ apparent velocity of seismic wave propagation. With this assumption, using equation 27, the steady state variance of force in the left spring can be shown to be given by

$$\sigma_l^2 = \beta_1^2 \int_0^\infty \{H_1(\omega)S_{uu}(\omega) + H_2(\omega)S_{vv}(\omega) + H_5(\omega,\tau_0)|S_{uv}(\omega)|\}d\omega. \quad (34)$$

Here the response functions $H_1(\omega)$ and $H_2(\omega)$ are as in equation 28. Using equation 28, and noting that $\phi_{uv}(\omega) = \omega\tau_0$, the function $H_5(\omega,\tau_0)$ is given by

$$H_5(\omega,\tau_0) = g_1(\omega)\cos(\omega\tau_0) + g_2(\omega)\sin(\omega\tau_0) \quad (35)$$

Here the functions g_1 and g_2 are as in equation 28. We note that the functions $H_1(\omega)$ and $H_2(\omega)$ are non-negative while $H_5(\omega,\tau_0)$ can take either negative or positive values depending on ω and τ_0.

The critical cross psd function function which maximizes σ_l^2 under the constraint that $0 \le |S_{uv}(\omega)| \le \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$ is given by

$$\begin{aligned} |S_{uv}(\omega)| &= 0 \qquad \text{if } H_5(\omega,\tau_0) < 0 \\ |S_{uv}(\omega)| &= \sqrt{S_{uu}(\omega)S_{vv}(\omega)} \ \ \text{if } H_5(\omega,\tau_0) > 0. \end{aligned} \quad (36)$$

Conversely, the most favorable cross psd function function which minimizes σ_l^2 under the constraint that $0 \le |S_{uv}(\omega)| \le \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$ is given by

$$\begin{aligned} |S_{uv}(\omega)| &= 0 \qquad \text{if } H_5(\omega,\tau_0) > 0 \\ |S_{uv}(\omega)| &= \sqrt{S_{uu}(\omega)S_{vv}(\omega)} \ \ \text{if } H_5(\omega,\tau_0) < 0. \end{aligned} \quad (37)$$

Model II To find optimal $|S_{uv}(\omega)|$ and $\phi_{uv}(\omega)$ given $S_{uu}(\omega)$ and $S_{vv}(\omega)$

Here we adopt the alternative version of response representation given in equation 29. Using this, the variance of the force in the left spring is given by

$$\sigma_l^2 = \beta_1^2 \int_0^\infty \{H_1(\omega)S_{uu}(\omega)+H_2(\omega)S_{vv}(\omega)+R_1(\omega)|S_{uv}(\omega)|\cos[\alpha_1(\omega)-\phi_{uv}(\omega)]\}d\omega \quad (38)$$

The functions H_1, H_2, $\alpha_1(\omega)$ and $R_1(\omega)$ are as in equations 28 and 30.

Noting, as before, that H_1 and H_2 are non-negative, it can be deduced that the highest σ_l^2 is produced when $|S_{uv}(\omega)| = \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$ and $\cos[\alpha_1(\omega) - \phi_{uv}(\omega)] = 1$. Thus the critical $|S_{uv}(\omega)|$ and $\phi_{uv}(\omega)$ are given by

$$|S_{uv}(\omega)| = \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$$

$$\phi_{uv}(\omega) = \alpha_1(\omega) = \tan^{-1}\{\frac{g_2(\omega)}{g_1(\omega)}\}. \tag{39}$$

Conversely, the lowest σ_l^2 is produced when $|S_{uv}(\omega)| = \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$ and $\cos[\alpha_1(\omega) - \phi_{uv}(\omega)] = -1$. Thus the most favorable $|S_{uv}(\omega)|$ and $\phi_{uv}(\omega)$ are given by

$$|S_{uv}(\omega)| = \sqrt{S_{uu}(\omega)S_{vv}(\omega)}$$

$$\phi_{uv}(\omega) = \pi + \alpha_1(\omega) = \pi + \tan^{-1}\{\frac{g_2(\omega)}{g_1(\omega)}\}. \tag{40}$$

In applying the formulations presented above, as has been already noted, the equivalent linearization method outlined in the previous section is employed. Here the iterative method of determining the equivalent linear constants need to be modified to handle the additional complexity arising from the system dependent optimal cross psd functions. Thus an additional iteration loop need to be introduced to determine the unknown cross psd functions. The following strategy was found to be satisfactory in the numerical work:

1. Assuming that the supports are acted upon by two independent excitations, $\beta_1 = \beta_{1a}$ and $\beta_2 = \beta_{2a}$ are calculated.

2. Assuming the supports are acted upon by two fully coherent motions, $\beta_1 = \beta_{1b}$ and $\beta_2 = \beta_{2b}$ are found.

3. Taking the mean values of β_1 and β_2 from the above steps as the initial estimates for β_1 and β_2, in the equivalent linear model, the first approximation to the optimal cross psd function is found.

4. The β_1 and β_2 values are recalculated for the new estimate of optimal cross psd function.

5. The procedure is repeated until the parameters β_1 and β_2 and the optimal response converge.

4.3 Numerical Results and Discussion

Results on optimal input cross psd functions and the extreme responses that these excitations produce have been obtained for both sdof and beam examples considered in sections 2 and 3.

Figures 2-4 contain results on the highest response variance for sdof systems. The results correspond to the critical input model I. These figures show results for the case when excitations are fully correlated (Case 1), excitations are mutually independent (Case 2) and when the excitations are deduced from the critical excitation models (Case 3). Results from both theory and simulations are shown in these figures. In figure 2, it may be noted that when the excitations are mutually independent (Case 2), the inputs are independent of time lag τ. Consequently, the simulation results for this case is shown only for $\tau = 0$. As may be expected, the critical responses are invariably higher than those produced by fully correlated or independent support motions. The ratio of critical response standard deviations to the corresponding results produced by independent excitations ranges from 1.0-1.06 for variation of τ, 1.05-1.07 for the variation of α and 1.03-1.05 for the variation of ω_g. Similarly, the ratio of critical response standard deviation to that of fully correlated excitations ranges from 1.05-1.77 for the variation of τ, 1.05-1.1 for the variation of α and 1.09-1.1 for the variation of ω_g.This range of variation indicates that for the sdof system under consideration, a 10 % higher design margins are needed, over and above that for the fully correlated and independent support motion cases, if the critical excitations are to be accommodated, except for the case of variation of τ, (for the case of fully correlated case), where the margins go up to 77 %.

Results on optimal cross psd models and the associated responses for the case of the beam example are shown in figures 13-16. Figure 13 shows the optimal $|S_{uv}(\omega)|$ (model I) for time lag of $\tau = 0.5s$ together with the associated frequency response function $H_5(\omega)$. The plots of the optimal $\phi_{uv}(\omega)$

(model II) are shown in figure 14. The nature of critical and most favorable phase models is studied in figure 15. This figure shows the psd of the force in the left spring when input components $u(t)$ and $v(t)$ are fully coherent but have four different phase characteristics. The four phase models considered include the critical and most favorable phase spectra and also the cases of $\phi_{uv}(\omega) = 0$ (in-phase motions) and $\phi_{uv}(\omega) = \pi$ (out of phase motions). The psd functions of the force in the left spring for different models for the input cross psd functions are shown together in figure 16. Table 1 summarizes the response variance for different models for input cross psd functions and for different values of the nonlinearity parameter α. This table also provides the details of results based on digital simulations. The following observations are made based on results contained in figures 13-16 and table 1.

1. The critical and most favorable $|S_{uv}(\omega)|$ functions, when it is assumed that $\phi_{uv}(\omega) = \omega\tau_0$ (model I), consist of alternating sequence of frequency windows in which $|S_{uv}(\omega)|$ assumes its admissible extreme values of 0 and $\sqrt{S_{uu}(\omega)S_{vv}(\omega)}$. The locations of these windows in turn is governed by zeros of the response function $H_3(\omega, \tau_0)$. The highest and lowest responses are *not* produced by either when $u(t)$ and $v(t)$ are fully correlated nor when they are statistically independent.

2. When no assumption is made on the nature of $\phi_{uv}(\omega)$, the optimal responses are produced by fully coherent motions but with specific system dependent phase spectra (figures 14 and 15). Again, the highest and lowest responses are *not* produced either when $u(t)$ and $v(t)$ are perfectly in phase or when they are perfectly out of phase (figure 15).

3. The highest response is produced by cross psd models corresponding to model II. This feature is observed to be present over a wide range of α (Table 1). The lowest response is produced by the most favorable model II for small nonlinearities and, for larger values of nonlinearity, the most favorable response is produced by model I; see figure 16 and table 1. The highest response produced is substantially higher than the most favorable response. This highlights the important influence that the input cross correlation function has on structural response.

4. It is not obvious as to how the simulation strategy can be used to ascertain if the optimal cross psd functions developed in section 4 indeed

produce the extreme responses. Here one would need to prove that there exists no other cross psd functions which can lead to responses which are higher/lower than those produced by the optimal cross psd functions. On the other hand, the simulation results, shown in Table 1, indeed corroborate that the responses produced by critical cross psd models are optimal at least in relations to the responses produced by independent and fully correlated support motions.

5.0 CONCLUDING REMARKS

The method of equivalent linearization has been employed to analyze the response of nonlinearly supported sdof and single span beam structures subjected to stationary random differential support motions. The response analysis is carried out in the frequency domain based on the use of dynamic stiffness matrices. An iterative method to evaluate the equivalent linear parameters has been outlined. The performance of the approximations made is assessed by conducting digital simulations studies based on finite element method. Satisfactory agreement between theoretical and simulated results has been demonstrated over a wide range of system parameters. Furthermore, the nature of cross psd functions which lead to the highest and lowest response variance has been established. The extreme responses are produced neither by fully correlated motions nor by independent motions. Instead specific forms of cross psd functions are shown to exist which depend on system parameters and response variables of interest. The response variance is shown to be significantly influenced by the choice of input cross psd functions. The upper bound on the response variance is substantially higher than the lower bound. The question of stochastic stability of equivalent linearization solution, as a means to ascertain the acceptability of the approximate solution, has been investigated in this study only to a limited extent using digital simulation technique. Further work is needed to establish more easy-to-use analytical stability criterion to judge the acceptability of the equivalent linearization solution.

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APPENDIX A REDUCED DYNAMIC STIFFNESS MATRIX FOR THE LINEARIZED STRUCTURE

Figure A.1 shows the beam resting on linear spring supports. We define the dynamic stiffness coefficient $D_{ij}(\omega)$ for a linear structural element as the amplitude of harmonic displacement with frequency ω at node i due to an unit harmonic force with frequency ω applied at node j with all other nodal displacements held fixed to zero. The dynamic stiffness matrix for a Euler Bernoulli beam element is exactly determinable (Paz 1984) and is given by

$$D = B \begin{bmatrix} a^2(cS+sC) & asS & -a^2(s+S) & a(C-c) \\ asS & (sC-cS) & a(c-C) & (S-s) \\ -a^2(s+S) & a(c-C) & a^2(cS+sC) & -asS \\ a(C-c) & (S-s) & -asS & (sC-cS) \end{bmatrix} \qquad A.1$$

where, $s = \sin(aL)$, $S = \sinh(aL)$, $c = \cos(aL)$, $C = \cosh(aL)$, $B = \frac{aEI}{(1-cC)}$ and $a^4 = \frac{m\omega^2 - \mathrm{i}\omega c}{EI}$ The dynamic stiffness matrix of a spring element with an axial stiffness k is identical to its static stiffness matrix and is given by

$$\begin{bmatrix} K \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \qquad A.2$$

The rules for analyzing built-up structures using dynamic stiffness matrices are identical to those followed in analyzing these structures under static loads. This calls for the determination of the global dynamic stiffness matrix and reduced dynamic stiffness matrix for the structure. The equilibrium equation for the structure under consideration in frequency domain can be

$$\{\ P\ \} = \begin{bmatrix} D_1 \end{bmatrix} \ \{\Delta\} \qquad A.3$$

Here, P =vector of nodal forces, Δ=vector of nodal displacements and D_1=global dynamic stiffness matrix. These quantities are given by,

$$\{\Delta\}^T = [u\ v\ y_3\ y_4\ y_5\ y_6]$$

$$[P]^T = [p_1\ p_2\ 0\ 0\ 0\ 0]$$

$$
\left[\begin{array}{c} D_1 \end{array}\right] = B \begin{bmatrix} \frac{k_1}{B} & 0 & -\frac{k_1}{B} & 0 & 0 & 0 \\ 0 & \frac{k_2}{B} & 0 & 0 & -\frac{k_2}{B} & 0 \\ -\frac{k_1}{B} & 0 & \frac{k_1}{B} + a^2(cS + sC) & asS & -a^2(s+S) & a(C-c) \\ 0 & 0 & asS & (sC - cS) & a(c-C) & (S-s) \\ 0 & -\frac{k_2}{B} & -a^2(s+S) & a(c-C) & \frac{k_2}{B} + a^2(cS+sC) & -asS \\ 0 & 0 & a(C-c) & (S-s) & -asS & (sC-cS) \end{bmatrix} \quad A.4
$$

The next step in the analysis is to derive reduced equations for the unknown displacement and forces. To do this, we partition the global stiffness matrix, displacement and force vector as follows:

$$
\left[\begin{array}{c} D_1 \end{array}\right] = \left[\begin{array}{c} D_{1a} \; D_{1b} \\ D_{1c} \; D_{1d} \end{array}\right]
$$

$$
[\Delta]^T = [\Delta_1 \;\; \Delta_2]\,; \quad \Delta_1^T = [u \;\, v]\,; \quad \Delta_2^T = [y_3 \; y_4 \; y_5 \; y_6]
$$

$$
\left[\begin{array}{c} P \end{array}\right]^T = [P_1 \;\, P_2]\,; \quad [P_1]^T = [p_1 \; p_2]; \quad [P_2]^T = [0 \; 0 \; 0 \; 0\,]
$$

$$
D_{1a} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}
$$

$$
D_{1b} = \begin{bmatrix} -k_1 & 0 & 0 & 0 \\ 0 & 0 & -k_2 & 0 \end{bmatrix}
$$

$$
D_{1c} = D_{1b}^T
$$

$$
D_{1d} = B \begin{bmatrix} \frac{k_1}{B} + a^2(cS+sC) & asS & -a^2(s+S) & a(C-c) \\ asS & (sC-cS) & a(c-C) & (S-s) \\ -a^2(s+S) & a(c-C) & \frac{k_2}{B} + a^2(cS+sC) & -asS \\ a(C-c) & (S-s) & -asS & (sC-cS) \end{bmatrix} \quad A.5
$$

This leads to the equations,

$$
\begin{aligned} D_{1a}\Delta_1 + D_{1b}\Delta_2 &= P_1 \\ D_{1c}\Delta_1 + D_{1d}\Delta_2 &= P_2 \end{aligned} \quad A.6
$$

From this, one gets,

$$
\left[\begin{array}{c} \Delta_2 \end{array}\right] = \left[\begin{array}{c} D_{1d} \end{array}\right]^{-1} \left[\begin{array}{c} D_{1c} \end{array}\right] \left[\begin{array}{c} \Delta_1 \end{array}\right] \quad A.7
$$

Thus, the quantity $\mathcal{D}= [D_{1d}]^{-1}[D_{1c}]$ can be interpreted as the reduced dynamic stiffness matrix for the structure.

Table 1 Steady state variance of the force in the left spring (N^2)

	α=0 N/m^3	α=2e6 N/m^3		α=4e6 N/m^3	
	Theory	Theory	Simulation	Theory	Simulation
Independent	4.2546e4	4.1348e4	4.1532e4	4.1337e4	4.1162e4
Fully correlated	7.8203e4	7.6031e4	7.801e4	7.6007e4	7.7321e4
Critical					
Model I	8.0915e4	8.0013e4	8.019e4	7.9177e4	7.9720e4
Model II	8.3770e4	8.1474e4	8.2341e4	7.9667e4	8.0121e4
Most favorable					
Model I	3.9834e4	3.9222e4	4.0122e4	3.8668e4	3.9424e4
Model II	4.5347e3	4.5141e3	5.0231e3	4.4942e4	4.6101e4

FIGURE CAPTIONS

Figure 1 A doubly supported nonlinear sdof system under differential support motions.

Figure 2 Steady state standard deviation of the force in the left spring; α =6.1E+08 N/m^3, $\omega_g = 8\pi$ rad/s; Case 1 : $u(t)$ and $v(t)$ are fully correlated; Case 2: $u(t)$ and $v(t)$ are independent; Case 3: $u(t)$ and $v(t)$ are critically correlated.

Figure 3 Steady state standard deviation of the force in the left spring; τ =1 s, $\omega_g = 8\pi$ rad/s; α on x-axis has been normalized with respect to a reference value of 6.1E+08 N/m^3; Case 1 : $u(t)$ and $v(t)$ are fully correlated; Case 2: $u(t)$ and $v(t)$ are independent; Case 3: $u(t)$ and $v(t)$ are critically correlated.

Figure 4 Steady state standard deviation of the force in the left spring; $\tau = 1$ s, α =6.1E+08 N/m^3; Case 1 : $u(t)$ and $v(t)$ are fully correlated; Case 2: $u(t)$ and $v(t)$ are independent; Case 3: $u(t)$ and $v(t)$ are critically correlated.

Figure 5 Simulated PDF of the amplitude of response; $u(t)$ and $v(t)$ are independent; $\lambda_0^+ = \omega_g$; the values of ω_g are marked on the figures; ω_0= 4 π rad/s.

Figure 6 Time history of perturbation amplitude for different initial conditions; ω_g= 4 π rad/s; α =6.1E+08 N/m^3; $u(t)$ and $v(t)$ are independent.

Figure 7 Moments of perturbation amplitude using 250 samples simulations.

Figure 8 A nonlinearly supported Euler Bernoulli beam subjected to differential support motions.

Figure 9 Steady state standard deviation of the force in the left spring; α = 2.0E+06 N/m^3.

Figure 10 Steady state standard deviation of the force in the left spring; $u(t)$ and $v(t)$ are independent.

Figure 11 Steady state standard deviation of the force in the left spring; $u(t)$ and $v(t)$ are independent.

Figure 12 Simulated probability density functions of force in the left spring; $u(t)$ and $v(t)$ are independent; α =2.0+E06 N/m^3; note that the mean has been removed and standard deviation has been normalized to unity.

Figure 13(a) Frequency response function for the force in the left spring (b) critical cross psd model I τ_0=0.5 s.

Figure 14 Optimal phase spectra models for $u(t)$ and $v(t)$.

Figure 15 Response psd function of force in the left spring for different cases of fully coherent ground motions.

Figure 16 psd of force in the left spring for different cases of cross-correlations between $u(t)$ and $v(t)$.

Figure A.1 A linear Euler-Bernoulli beam subjected to differential support motions $u(t) = U \exp[\mathrm{i}\omega t]$ and $v(t) = V \exp[\mathrm{i}\omega t]$.