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Dynamic stiffness matrix for axially vibrating stochastic rods using Stratonovich's averaging principle

C. S. Manohar

Summary The problem of determining the dynamic stiffness matrix of a rod with broad band randomly varying mass and stiffness properties is considered. The governing stochastic boundary value problem is solved. First, a general solution to the field equation is obtained by using Stratonovich's stochastic averaging theorem. Subsequently, the elements of dynamic stiffness coefficients are evaluated by choosing appropriately the arbitrary constants of the general solution. The analytically determined statistics of the amplitude and phase of the stiffness coefficients are shown to compare favorably with digital simulation solutions.

Key words dynamic stiffness, stochastic averaging, structural mechanics, system stochasticity.

1 Introduction

The analysis of structural systems with stochastic stiffness, mass and damping properties is currently receiving notable attention in stochastic mechanics research. Developments in this area of research have been reviewed in [3], [7], [2], [1], [9]. A few research monographs on this topic addressing both static and dynamic problems have also appeared [6,8,16]. These studies find applications in the safety assessment of important structures [17] and also in the design of structures subjected to high frequency excitations using statistical energy analysis formalisms. The latter class of problems has been discussed in [10] and, more recently, in [5]. In an ongoing program of research, aimed at understanding the dynamical behavior of stochastic continuous systems, we have studied the free and forced vibration characteristics of random rod and beam elements [11–13]. Some of the major conclusions emerging from these studies are:

- Approaches based on Markov process theory are useful in characterizing eigensolutions of axially vibrating stochastic rods. In particular, the stochastic averaging theorem of Stratonovich provides a powerful means for generating acceptable free vibration solutions for a class of problems [12].
- The complex frequency response curves for stochastic rods and beams tend to become statistically stationary for large values of driving frequencies. A statistical overlap factor can be formed. Given the statistical properties of the system physical properties, it allows the response statistics to be characterized. A similar behavior at high frequencies in the spatial domain is also observed in regions away from boundaries [11,13].
- The frequency response functions of stochastic systems is pronouncedly non-Gaussian. In some cases, the probability density function of the frequency response functions has a very long upper tail. This limits the usefulness of the response mean and standard deviation as descriptors of the system behavior. Lognormal and gamma distributions fit digitally simulated data on frequency functions well [11].

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The aim of this paper is to develop an analytical method for evaluating the harmonic response of an axially vibrating rod with randomly varying mass and stiffness properties. The standard way of performing this analysis would be to do the free vibration analysis first. It is then followed by eigenfunction expansion procedure, leading to the evaluation of dynamic response. In the context of stochastically defined systems, however, this approach has two drawbacks. Firstly, it requires the determination of the joint statistics of natural frequencies and mode shapes, which is by no means an easy task. Secondly, the series expansion introduces a large number of random variables, which is, at least, twice as many as the number of modes retained in the modal expansion. This expansion, in turn, increases the size of integration on joint probability distributions while evaluating the response statistics. Besides, when the stochastic mode shapes are not determined exactly, which, most often, will be the case, the question of orthogonality of mode shapes requires careful interpretation. An alternative approach, which proves to be advantageous, would be to evaluate the dynamic stiffness matrix directly by solving the governing field equations [14]. This, as will be shown in this paper, not only eliminates the necessity of determining the random eigensolutions, also, restricts the number of random variables entering the formulations. An important step in this analysis consists of solving the governing field equation which, for axially vibrating stochastic rods, is a second-order ordinary differential equation with stochastic coefficients. Exact solutions to this problem are not possible except under highly specialized circumstances. Consequently, approximations become necessary. In the present study, it is proposed to employ the stochastic averaging principle to approximately analyze the field equation. The application of this principle requires that there be a clear-cut separation between the characteristic length of the stochastic variations and that of the system. It may be noted in this context that stochastic averaging principles are widely used in the time-domain random vibration problems [15] while, in problems of system stochasticity, their applications, to the best of the author's knowledge, have been studied only to a limited extent[12]. Numerical results on the response statistics of a harmonically driven, discretely damped stochastic rod as a function of driving frequency are presented. Satisfactory comparison with a limited amount of digital simulation results is also demonstrated.

2
Stochastic rods

The field equation for the axial vibration of an inhomogeneous, viscously damped rod element can be written as

$$\frac{\partial}{\partial x}\left[AE(x)\frac{\partial Y}{\partial x} + C_1(x)\frac{\partial^2 Y}{\partial x \partial t}\right] = \rho(x)\frac{\partial^2 Y}{\partial t^2} + C_2(x)\frac{\partial Y}{\partial t} - F(x,t) \ . \tag{1}$$

This equation arises not only in the study of axially vibrating rods, but also is relevant to the vibration analysis of inhomogeneous shafts, strings and soil layers modeled as shear beams. The stiffness $AE(x)$, mass per unit length $\rho(x)$, viscous damping coefficients $C_1(x)$ and $C_2(x)$ are obtained by randomly perturbing the respective constant mean values as follows:

$$AE(x) = AE_0[1 + \delta g(x)] \ , \tag{2}$$

$$\rho(x) = \rho_0[1 + \epsilon f(x)] \ , \tag{3}$$

$$C_1(x) = C_{10}[1 + v_1 q_1(x)] \ , \tag{4}$$

$$C_2(x) = C_{20}[1 + v_2 q_2(x)] \ . \tag{5}$$

Here, AE_0 – denotes the mean axial stiffness, ρ_0 – mean mass per unit length, C_{10}, C_{20} – mean damping coefficients. Furthermore, $g(x), f(x), q_1(x)$ and $q_2(x)$ are modeled as jointly stationary, meansquare bounded, zero-mean random processes with unit standard deviations. An alternative way of modelling $AE(x)$, which proves to be useful in this study, would be to take

$$\frac{1}{AE(x)} = \frac{1}{AE_0}[1 + \gamma h(x)] \ , \tag{6}$$

where $h(x)$ is again taken to have zero mean and to be meansquare bounded and jointly stationary with $f(x), q_1(x)$ and $q_2(x)$. The parameters $\epsilon, \delta, \gamma, v_1$ and v_2 are deterministic quantities indicating the strengths of stochastic variations, and can generally be taken to be small compared to unity. The external excitation $F(x,t)$ can, in general, be a random field.

3 Dynamic stiffness matrix

The dynamic stiffness coefficient $D_{ij}(\omega)$ is defined as the relation of the harmonic force of frequency ω at nodal coordinate i to the harmonic displacement of unit amplitude and of the same frequency at nodal coordinate j, all other coordinates being fixed [14]. For the case of the rod element shown in Fig. 1, in which the nodal harmonic forces $P_i \exp[i\omega t]$ and nodal harmonic displacements $\delta_i \exp[i\omega t]$ coexist, the matrix of dynamic stiffness coefficients $D_{ij}(\omega)$ relates the nodal displacement and force amplitudes through the relation $\mathbf{P} = \mathbf{D}(\omega)\boldsymbol{\delta}$. The formulation of dynamic stiffness matrix enables the determination of forced response characteristics of the system. To develop the dynamic stiffness matrix for the stochastic rods described by Eq. (1), we begin by noting that, for harmonic nodal excitations, the solution of Eq. (1) can be sought in the form

$$Y(x,t) = y(x)\exp(i\omega t); \quad i = \sqrt{-1} \ . \tag{7}$$

This separates the time and space variables, and leads to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{[1+\delta g(x)]\frac{\mathrm{d}y}{\mathrm{d}x} + i\beta_1[1+\nu_1 q_1(x)]\frac{\mathrm{d}y}{\mathrm{d}x}\right\} + \lambda^2[1+\epsilon f(x)]y - i\beta_2[1+\nu_2 q_2(x)]y = 0 \ , \tag{8}$$

where

$$\lambda^2 = \frac{\rho_0\omega^2}{AE_0}, \beta_1 = \frac{C_{10}\omega}{AE_0}, \beta_2 = \frac{C_{20}\omega}{AE_0} \ . \tag{9}$$

In order to derive the elements of the dynamic stiffness matrix, the above equation needs to be solved for the following two sets of boundary conditions:

$$y(0) = \delta_i, \ y(L) = \delta_2 \ , \tag{10}$$

and

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) = \frac{-P_1}{AE(0)}; \ \frac{\mathrm{d}y}{\mathrm{d}x}(L) = \frac{P_2}{AE(L)} \ . \tag{11}$$

Equation (8) together with the above boundary conditions constitute a pair of complex stochastic boundary value problems. An important property of solution to these boundary-value problems is that, even when the random variations $f(x)$, $g(x)$, $q_1(x)$ and $q_2(x)$ arise as filtered white noise processes, the extended solution vector would not have the Markovian character. This is due to restrictions placed on the solution vector through the imposition of boundary conditions. It is of interest, at this stage, to consider three special cases.

[i] **Deterministic case:** Here, the parameters ϵ, δ, ν_1 and ν_2 are identically equal to zero, and one can show that the elements of dynamic stiffness matrix are given by

$$D_{11}(\omega) = D_{22}(\omega) = AE_0\tilde{\lambda}\cot\tilde{\lambda}L \ , \tag{12}$$

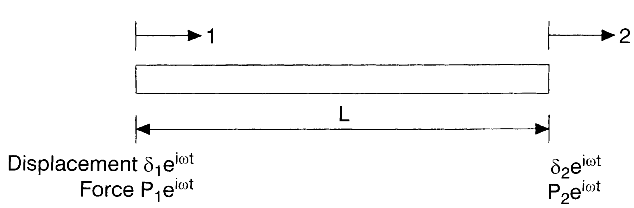

Fig. 1. Axillay vibrating rod element

$$D_{12}(\omega) = D_{21}(\omega) = -AE_0\tilde{\lambda}\text{cosec}\tilde{\lambda}L \ , \tag{13}$$

where

$$\tilde{\lambda}^2 = \frac{\rho_0\omega^2(1 - i\beta_2)}{AE_0(1 + i\beta_1)} \ . \tag{14}$$

Furthermore, as $\beta_1 \to 0$ and $\beta_2 \to 0$, the above results approach the known exact results for undamped systems [14].

[ii] **Static case:** For the case of $\omega = 0$, the elements of the dynamic stiffness matrix become equal to the respective static stiffness coefficients which, for the present case, are given by

$$D_{11} = D_{22} = \frac{AE_0}{\int_0^L \frac{1}{1+\delta g(s)} \mathrm{d}s} \tag{15}$$

$$D_{12} = D_{21} = \frac{-AE_0}{\int_0^L \frac{1}{1+\delta g(s)} \mathrm{d}s} \tag{16}$$

It may be noted that these quantities are random variables, the randomness being due to the stiffness uncertainties. If $\delta \to 0$, the stiffness coefficients approach the well-known static stiffness values of deterministic systems.

[iii] **Exactly solvable case:** Here, the stochastic variations in mass, stiffness and damping are taken to be such that

$$[1 + \delta g(x)] = [1 + \nu_1 q_1(x)] \ ,$$
$$[1 + \epsilon f(x)] = [1 + \nu_2 q_2(x)] \ ,$$
$$[1 + \delta g(x)] = \frac{1}{[1 + \epsilon f(x)]} \ . \tag{17}$$

The expressions for the system transfer functions for this system in terms of eigensolutions have been developed by [13]. Similar results *without* using modal expansions can be shown to be given by

$$D_{11}(\omega) = D_{22}(\omega) = AE_0\tilde{\lambda}\cot\left[\tilde{\lambda}L + \epsilon\tilde{\lambda}\int_0^L f(s)\mathrm{d}s\right] \ , \tag{18}$$

$$D_{12}(\omega) = D_{21}(\omega) = -AE_0\tilde{\lambda}\text{cosec}\left[\tilde{\lambda}L + \epsilon\tilde{\lambda}\int_0^L f(s)\mathrm{d}s\right] \ . \tag{19}$$

Obviously, systems of this type have little significance from a physical view point; these results, however, are clearly valuable as benchmarks to validate alternative analytical or numerical solutions.

For the more general class of problems given by Eqs. (8–11), no exact solutions are currently available. Consequently, to proceed further, one has to take recourse to either approximate analytical methods or to adopt Monte Carlo simulation procedures.

4
General case: solution by stochastic averaging

When the coefficient processes $f(x), g(x), q_1(x)$ and $q_2(x)$ are modeled as broad-band random processes, an approximate solution to Eq. (8) can be obtained by using the Stratonovich-Khasminskii averaging theorem [15]. This method consists of eliminating minor rapid variations in the response, and derives simplified equations for dominant slowly varying components. Furthermore, the broad-band stochastic variations are replaced by equivalent delta-correlated random processes which, consequently, enables the application of methods of the Markov process theory to obtain the solutions. This method is extensively used in random vibration studies, and its application to problems of system stochasticity has been attempted, to the best of the author's knowledge, only recently [12]. This type of problems, as has already been noted, are not amenable for treatment by the methods of the Markov process theory.

However, given the powerful nature of these methods, it is clearly desirable to overcome this difficulty. This is achieved in this study essentially by treating the solution of the stochastic boundary-value problem as a superposition of two solutions of the governing field equation under two independent sets of *initial* conditions. It is important to note that, in obtaining solutions to the associated pair of initial value problems, one can utilize the Markovian methods of solution.

The investigation in the present study is limited to undamped field equation. Therefore, the equation for further study reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{[1+\delta g(x)]\frac{\mathrm{d}y}{\mathrm{d}x}\right\}+\lambda^2[1+\epsilon f(x)]\ y=0\ . \tag{20}$$

To proceed further, it is found advantageous to adopt the alternative representation of stiffness process as given by Eq. (6), and seek the solution of the above equation in the form

$$y(x)=\exp[a(x)]\sin[\lambda x+\theta(x)]\ , \tag{21}$$

$$\frac{1}{[1+\gamma h(x)]}\frac{\mathrm{d}y}{\mathrm{d}x}=\exp[a(x)]\lambda\cos[\lambda x+\theta(x)]\ . \tag{22}$$

Here, $\exp[a(x)]$ and $\theta(x)$ are, respectively, the amplitude and phase functions which are governed by

$$\frac{\mathrm{d}a}{\mathrm{d}x}=0.5\lambda\sin 2(\theta+\lambda x)[\gamma h(x)-\epsilon f(x)]\ , \tag{23}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=\lambda\gamma h(x)\cos^2(\theta+\lambda x)+\epsilon\lambda f(x)\sin^2(\theta+\lambda x)\ . \tag{24}$$

These equations are exactly equivalent to Eq. (20) and are in a form suitable for applying the averaging theorem. This consists of a combination of a spatial averaging and ensemble averaging. The details of the formulation are well-known and, hence, will not be repeated here, see, for example, [15]. The application of the averaging theorem leads to a two-dimensional Markovian approximation to the solution vector $\{a(x),\theta(x)\}$ which can be shown to be governed by

$$\frac{\mathrm{d}a}{\mathrm{d}x}=m_1+\sigma_1 W_1(x)\ , \tag{25}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}x}=m_2+\sigma_2 W_2(x)\ , \tag{26}$$

Here, $W_1(x)$ and $W_2(x)$ are independent Gaussian white noise processes with unit strength. The drift and diffusion coefficients appearing in the above equations can be shown to be given by

$$\begin{aligned} m_1 =& \frac{1}{4}\lambda^2\int_{-\infty}^{0}[\gamma^2<h(x)h(x+\tau)>+\epsilon^2<f(x)f(x+\tau)> \\ &-\epsilon\gamma<h(x)f(x+\tau)>-\epsilon\gamma<f(x)h(x+\tau)>]\cos 2\lambda\tau\,\mathrm{d}\tau\ , \end{aligned} \tag{27}$$

$$m_2=\frac{1}{4}\lambda^2\int_{-\infty}^{0}<[\gamma h(x)-\epsilon f(x)][\gamma h(x+\tau)-\epsilon f(x+\tau)]>\sin 2\lambda\tau\,\mathrm{d}\tau\ , \tag{28}$$

$$\sigma_1^2=\frac{1}{8}\lambda^2\int_{-\infty}^{\infty}<[\gamma h(x)-\epsilon f(x)][\gamma h(x+\tau)-\epsilon f(x+\tau)]>\cos 2\lambda\tau\,\mathrm{d}\tau\ , \tag{29}$$

$$\sigma_2^2 = \frac{1}{4}\lambda^2 \int_{-\infty}^{\infty} < [\gamma h(x) - \epsilon f(x)][\gamma h(x+\tau) - \epsilon f(x+\tau)] > \mathrm{d}\tau$$
$$+ \frac{1}{8}\lambda^2 \int_{-\infty}^{\infty} < [\gamma h(x) - \epsilon f(x)][\gamma h(x+\tau) - \epsilon f(x+\tau)] > \cos 2\lambda\tau \, \mathrm{d}\tau \ . \tag{30}$$

Here $< \cdot >$ denotes the mathematical expectation operator. Furthermore, it may be deduced that $a(x)$ and $\theta(x)$, in this approximation, are stochastically independent and Gaussian distributed. Substitution of solution of Eqs. (25) and (26) into Eqs. (21) and (22) leads to $y(x)$ as a linear superposition of two independent solutions given by

$$y(x) = Q_1 F_1(x) \sin F_2(x) + Q_2 F_1(x) \cos F_2(x) \ , \tag{31}$$

$$\frac{1}{[1+\gamma h(x)]} \frac{\mathrm{d}y}{\mathrm{d}x} = Q_1 \lambda F_1(x) \cos F_2(x) - \lambda Q_2 F_1(x) \sin F_2(x) \ , \tag{32}$$

$$Q_1 = \exp(a_0) \cos \theta_0 \ , \tag{33}$$

$$Q_2 = \exp(a_0) \sin \theta_0 \ , \tag{34}$$

$$F_1(x) = \exp[m_1 x + \sigma_1 G_1(x)] \ , \tag{35}$$

$$F_2(x) = \lambda_x + m_2 x + \sigma_2 G_2(x) \ , \tag{36}$$

$$G_1(x) = \int_0^x W_1(s) \ \mathrm{d}s; \quad G_2(x) = \int_0^x W_2(s) \ \mathrm{d}s \ , \tag{37}$$

$$a_0 = a(0); \quad \theta_0 = \theta(0) \ . \tag{38}$$

Here, Q_1 and Q_2 or, equivalently, a_0 and θ_0, are the constants of integration which are to be found to satisfy the specified boundary conditions. It may be emphasized that the two functions $F_1(x) \sin F_2(x)$ and $F_1(x) \cos F_2(x)$ are independent not in a statistical sense but in the sense that they satisfy two independent *initial* conditions, namely, $\left[y(0), \frac{\mathrm{d}y}{\mathrm{d}x}(0)\right] = (0, 1)$ and $(1, 0)$, respectively. It may further be noted that the field solution given above is non-Gaussian in nature.

To find the dynamic stiffness coefficients, we select the arbitrary constants such that the displacement boundary conditions consistent with the definition of a given stiffness coefficient is satisfied. Thus, to find $D_{11}(\omega)$ and $D_{12}(\omega)$, we use $y(0) = 1$ and $y(L) = 0$, and accordingly get $Q_1 = -\cot F_2(L)$ and $Q_2 = 1$. From Eqs. (32) and (11), it follows that $D_{11}(\omega) = AE_0 \lambda \cot F_2(L)$ and $D_{12}(\omega) = -AE_0 \lambda F_1(L) \mathrm{cosec} F_2(L)$. Thus, when the nodal harmonic forces $P \exp[i\omega t]$ and nodal harmonic displacements $\delta \exp[i\omega t]$ coexist, the force and displacement amplitudes are related through the relation

$$P_1 = AE_0 \lambda [\delta_1 \cot F_2(L) - \delta_2 F_1(L) \mathrm{cosec} F_2(L)] \ , \tag{39}$$

$$P_2 = AE_0 \lambda [-\delta_1 F_1(L) \mathrm{cosec} F_2(L) + \delta_2 \cot F_2(L)] \ . \tag{40}$$

Notice that, as ϵ and δ tend to zero, the above result reduces to the exact expressions valid for uniform rods [14].

5
Digital simulations

The accuracy of the above results can be assessed by comparing the analytical results with digital simulations. For this purpose, Eq. (20) needs to be solved numerically for a large number of realizations of the processes $g(x)$ and $f(x)$ and for the boundary conditions stated in Eqs. (10) and (11). This problem can be solved within the frameworks of either finite element or transfer matrix methods. Alternatively, the problem can be first converted into a pair of equivalent initial-value problems and these, in turn, can be solved using the Runge-Kutta method. For this purpose, two independent solutions, denoted by $y_1(x)$ and $y_2(x)$, are obtained by solving Eq. (20) under two different initial conditions, namely, $\left[y(0), \frac{\mathrm{d}y}{\mathrm{d}x}(0)\right] = (1, 0)$ and

$(0,1)$, respectively. The expressions for the dynamics stiffness coefficients can be derived by taking the general solution in the form

$$y(x) = a_1 y_1(x) + a_2 y_2(x), \tag{41}$$

and are shown to be given by

$$D_{11}(\omega) = \frac{AE(0)y_1(L)}{y_2(L)} \quad , \tag{42}$$

$$D_{12}(\omega) = \frac{-AE(0)}{y_2(L)} \quad , \tag{43}$$

$$D_{21}(\omega) = AE(L)\left[y'_1(L) - \frac{y'_2(L)y_1(L)}{y_2(L)}\right] \quad , \tag{44}$$

$$D_{22}(\omega) = \frac{AE(L)y'_2(L)}{y_2(L)} \quad . \tag{45}$$

From the above equations apparently it is not evident that $D_{12}(\omega) = D_{21}(\omega)$. However, it can be shown, through specific examples, for instance, the exactly solvable case considered in Eq. (17), or by numerical simulations, that the condition $D_{12}(\omega) = D_{21}(\omega)$ is indeed satisfied.

6
Numerical example and discussions

For the purpose of illustration, the system shown in Fig. 2 is analyzed using the above results. The stiffness and mass along the rod length are modeled as independent, stationary random Gaussian processes; three alternative shapes for the autocovariance functions are considered, namely

Model I

$$R(\tau) = \exp[-\alpha_1|\tau|] \quad -\infty < \tau < \infty, \tag{46}$$

Model II

$$R(\tau) = \exp\left[-\alpha_2\tau^2\right] \quad -\infty < \tau < \infty, \tag{47}$$

and

Model III

$$R(\tau) = \begin{cases} 1.0 - \alpha_3|\tau| & \text{at } -\frac{1}{\alpha_3} < \tau < \frac{1}{\alpha_3} \\ 0 & \text{at } |\tau| > \frac{1}{\alpha_3}. \end{cases} \tag{48}$$

In this context it must be noted that the shapes of the autocovariance functions of the processes $f(x)$ and $g(x)$ enter the theoretical solution through the sine and cosine integrals appearing in Eqs. (27–30) for the drift and diffusion coefficients. These integrals, corresponding to the autocovariance functions listed above, are given in Table 1. It may be noted that the term $(2k-1)!!$ appearing in this Table stands for $(2k-1)!! = 1.3.5.\ldots(2k-1)$. The expressions in the last row in this Table stand for the correlation length of the random process [18].

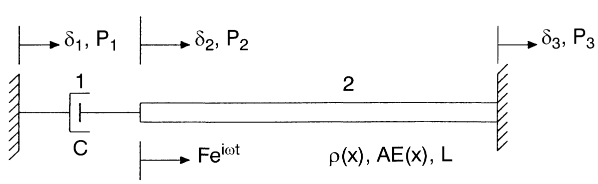

Fig. 2. Discretely damped axially vibrating rod

Table 1. Autocovariance models for random fields

Model	I	II	III
Autocovariance function $R(\tau)$	$\exp[-\alpha_1\lvert\tau\rvert]$ $-\infty<\tau<\infty$	$\exp[-\alpha_2\tau^2]$ $-\infty<\tau<\infty$	$1-\alpha_3\lvert\tau\rvert$ $-\frac{1}{\alpha_3}<\tau<\frac{1}{\alpha_3}$
$\int_{-\infty}^{\infty} R(\tau)\mathrm{d}\tau$	$\frac{2}{\alpha_1}$	$\sqrt{\frac{\pi}{\alpha_2}}$	$\frac{1}{\alpha_3}$
$\int_{-\infty}^{\infty} R(\tau)\cos 2\lambda\tau\mathrm{d}\tau$	$\frac{2\alpha_1}{\alpha_1^2+4\lambda^2}$	$\sqrt{\frac{\pi}{4\alpha_2}}\exp\left[\frac{-\lambda^2}{\alpha_2}\right]$	$\frac{\alpha_3}{2\lambda^2}\left[1-\cos\frac{2\lambda}{\alpha_2}\right]$
$\int_0^{\infty} R(\tau)\sin 2\lambda\tau\mathrm{d}\tau$	$\frac{2\lambda}{\alpha_1^2+4\lambda^2}$	$\frac{\lambda}{\alpha_2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)!!}\left[\frac{-2\lambda^2}{\alpha_2}\right]^{(k-1)}$	$\frac{1}{2\lambda}-\frac{\alpha_3}{4\lambda^2}\sin\frac{2\lambda}{\alpha_3}$
$\frac{1}{R(0)}\int_0^{\infty}\lvert R(\tau)\rvert\mathrm{d}\tau$	$\frac{1}{\alpha_1}$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha_2}}$	$\frac{1}{2\alpha_3}$

We begin by considering the use of Model I for the mass and stiffness processes and examine the nature of response statistics variations with respect to the driving frequency ω. Thus, we take the autocovariance of the processes $f(x)$ and $g(x)$ to be of the form

$$R_{ff}(\tau)=\exp(-\alpha\lvert\tau\rvert) \qquad \text{and} \qquad R_{gg}(\tau)=\exp(-\xi\lvert\tau\rvert) \ . \tag{49}$$

It must be noted in this context that the assumption of Gaussian distributions for strictly positive quantities such as mass and stiffness is, strictly speaking, inadmissible. However, for small ϵ and δ, the resulting errors can be expected to be negligible. It may also be noted that although the application of averaging principle requires that ϵ and δ be small compared to unity, it does not, however, require that the stochastic functions to be Gaussian-distributed. On the other hand, this assumption, in the present study, simplifies the generation of digital simulation results for comparison purposes. Combining Eqs. (39) and (40) with the dynamic stiffness matrix of a viscous damper element given by $D_{11}(\omega)=D_{22}(\omega)=C\omega i; D_{21}(\omega)=D_{12}(\omega)=-C\omega i$ and using the usual procedure for assembling stiffness matrices, it can be shown that

$$\delta_2(\omega)=\frac{F}{Ci\omega+AE_0\lambda\cot F_2(L)} \ , \tag{50}$$

$$P_3(\omega)=\frac{-FAE_0\lambda F_1(L)}{Ci\omega\sin F_2(L)+AE_0\lambda\cos F_2(L)} \ . \tag{51}$$

It follows that $\delta_2(\omega)$ and $P_3(\omega)$ are complex random processes evolving in ω. It must be noted that the above frequency functions are functions of only two random variables, namely, $F_1(L)$ and $F_2(L)$. This implies that for calculating statistics of these functions, not necessarily the first order statistics, one has to perform, at the most, a two-dimensional integration on the probability distributions. This, in the opinion of the author, is a major simplification of the problem. Figures 3–8 show a comparison of statistics of amplitude and phase of $\delta_2(\omega)$ and $P_3(\omega)$

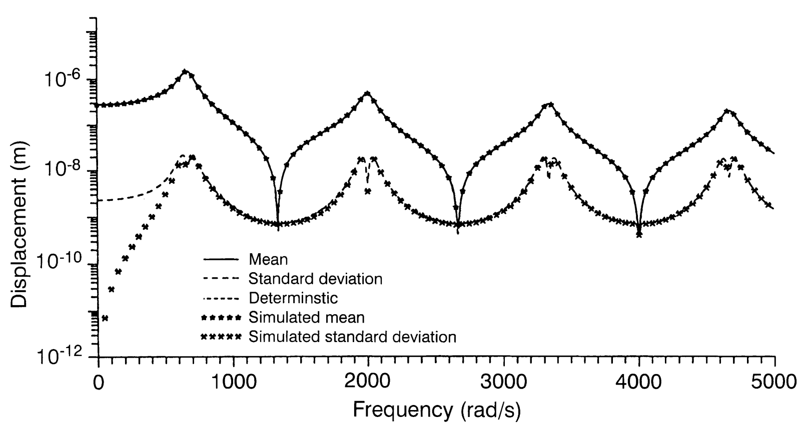

Fig. 3. Amplitude of displacement at node 2; $\epsilon=0.03; \gamma=0.0$

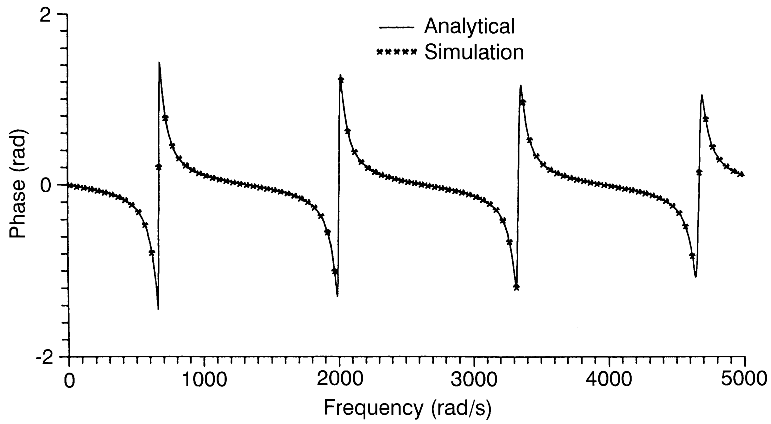

Fig. 4. Mean of the phase of displacement at node 2; $\epsilon = 0.03; \gamma = 0.0$

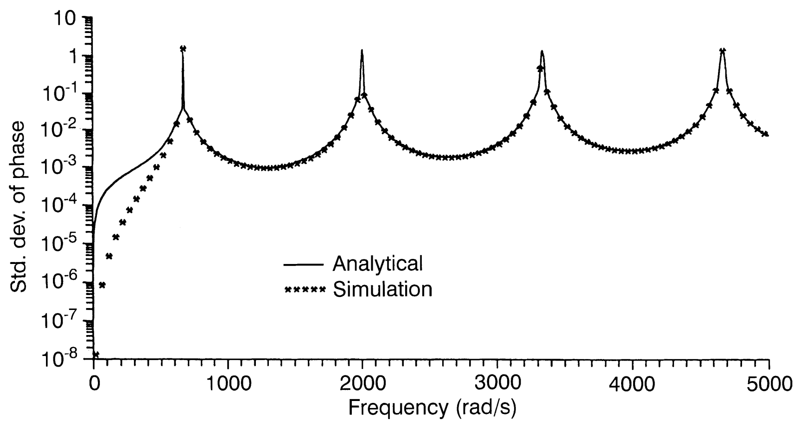

Fig. 5. Standard deviation of phase of displacement at node 2; $\epsilon = 0.03; \gamma = 0.0$

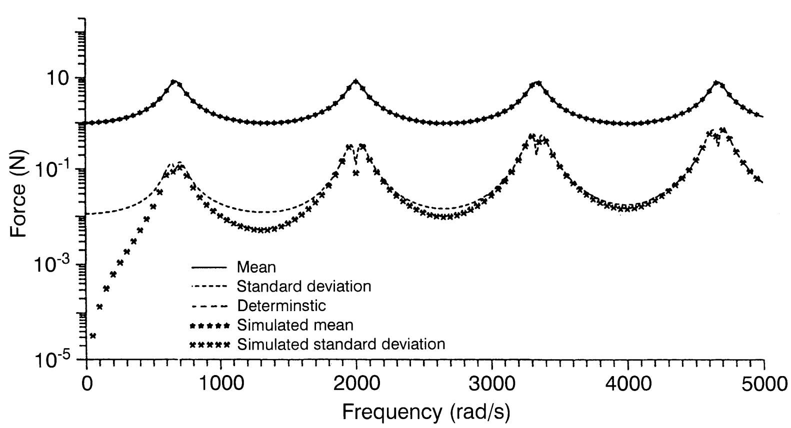

Fig. 6. Amplitude of force at node 3; $\epsilon = 0.03; \gamma = 0.0$

obtained using stochastic averaging method with those of a 500 samples digital simulations results. The system parameters chosen in this study are $L = 5.0\,\mathrm{m}, AE_0 = 18.00$ MN, $\rho_0 = 4.0$ kg/m, $C = 1000$ Nm/s, $\epsilon = 0.03, \gamma = 0.0, \alpha = 20.0\ \mathrm{m}^{-1}$ and $F = 1\,N$. Notice that in these results the only source of randomness is in specifying mass process. In the simulations, samples of $f(x)$ are generated as steady-state solutions of the first-order linear filter excited by samples of Gaussian white-noise process, that is

$$\frac{\mathrm{d}f}{\mathrm{d}x} + \alpha f = w(x) \quad . \tag{52}$$

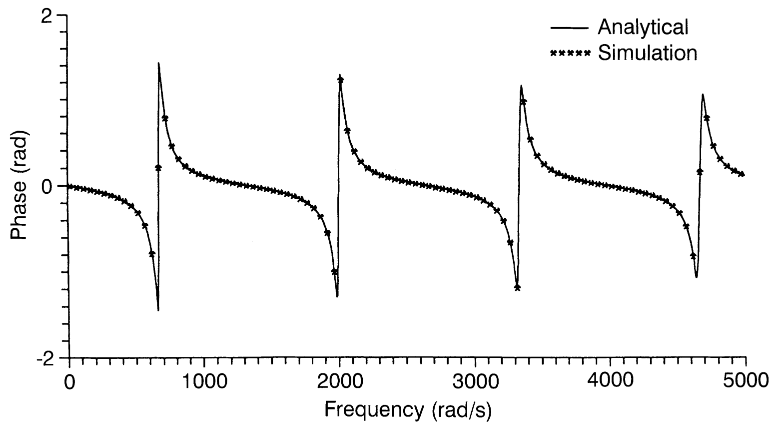

Fig. 7. Mean of the phase of the force at node 3; $\epsilon = 0.03; \gamma = 0.0$

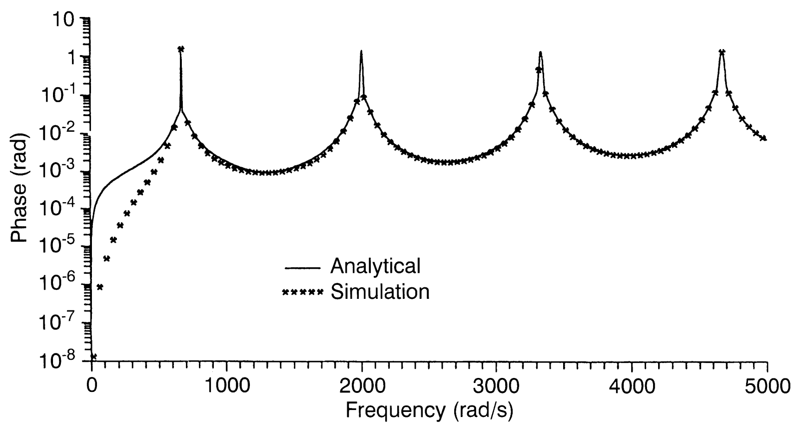

Fig. 8. Standard deviation of phase of force at node 3; $\epsilon = 0.03; \gamma = 0.0$

Samples of white-noise process were simulated using the procedure described in [4]. A fourth-order Runge Kutta algorithm was employed to integrate Eq. (20) using a step size of 0.01 m. Figures 9–11 show the analytical results on amplitude and phase of $\delta_2(\omega)$ for the same system with additional randomness also, in specifying stiffness with $\gamma = 0.05$ and $\xi = 20.0\ \mathrm{m}^{-1}$.

From Figs. 3–11 it can be observed that the passage of driving frequency through the system's resonant frequencies induces nonstationarity into the random processes $\delta_2(\omega)$, and $P_3(\omega)$. The analytical results capture this feature fairly accurately as evidenced by the satis-

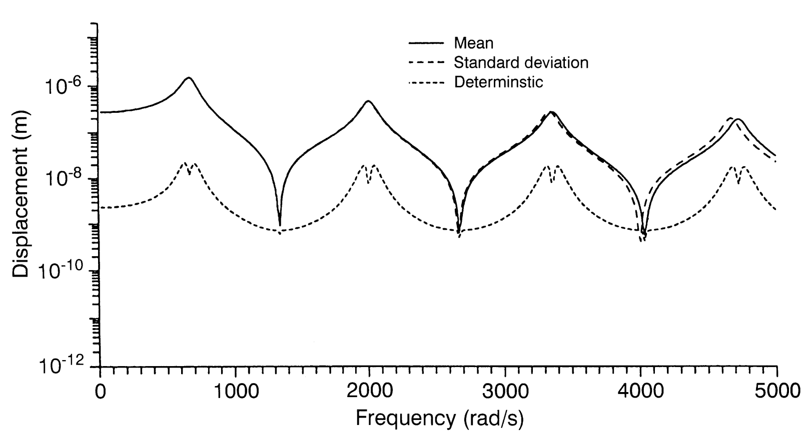

Fig. 9. Amplitude of displacement at node 2; $\epsilon = 0.03; \gamma = 0.05$

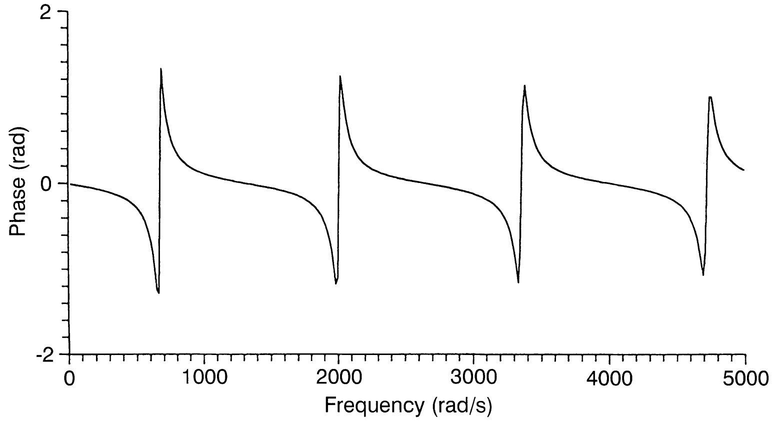

Fig. 10. Mean of the phase of displacement at node 2; $\epsilon = 0.03; \gamma = 0.05$

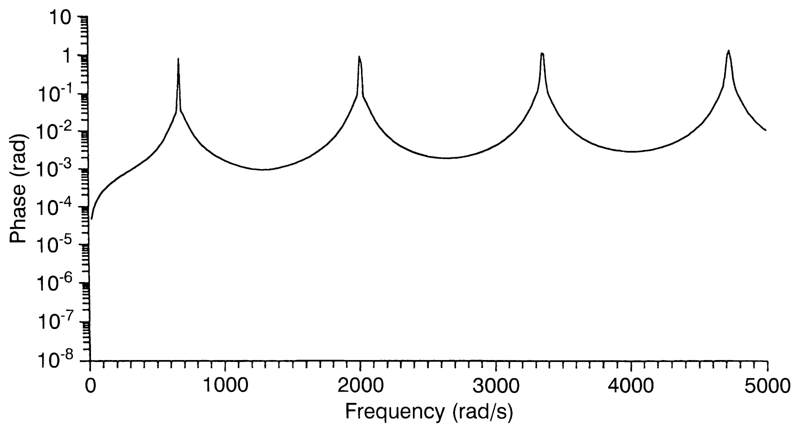

Fig. 11. Standard deviation of phase of displacement at node 2; $\epsilon = 0.03; \gamma = 0.05$

factory comparisons which are observed to exist between simulation and analytical solutions (Figs. 3–8). The response features when randomness is present in the mass process alone (Figs. 3–8) and when both mass and stiffness are random (Figs. 9–11) are observed to remain broadly similar. The standard deviations of the frequency response functions are greater near resonance and at the resonant frequency, the standard deviation of amplitude drops sharply (Figs. 3, 6 and 9), which the simulation results also corroborate, see also Figs. 12 and 14. This behavior is curious and remains unexplained in this study. For higher values of driving frequency, the characteristic length of the system becomes smaller as compared with correlation lengths of the mass and stiffness variations. Consequently, the accuracy of the averaging approximation deteriorates with increases in value of the driving frequency (Figs. 3–8). Conversely, at very low frequencies, that is, as $\omega \to 0$, in which case the wave length over which the averaging is done can exceed the length of the rod, the averaging results on standard deviations do not approach the respective static displacement/force limits satisfactorily. This can be seen by considering the behaviour of $\delta_2(\omega)$ given by Eq. (47) as $\omega \to 0$, that is

$$\lim_{\omega\to 0} \delta_2(\omega) = \lim_{\omega\to 0} \frac{F}{AE_0} \frac{\sin F_2(L)}{F_2(L)} \quad . \tag{53}$$

Thus, if one considers a system with randomness in the mass process alone, as $\omega \to 0$, the standard deviation of the response amplitudes must go to zero. This, however, does not happen, as can be easily verified by calculating the above limit. The answers on the mean, however, remain acceptable.

To illustrate the influence of the variation of the correlation length on the theoretical estimates of response statistics, we consider the case of $\gamma = 0$, that is, randomness in mass process alone, and $\epsilon = 0.03$. Figure 12 shows the influence of varying α on the standard deviation of

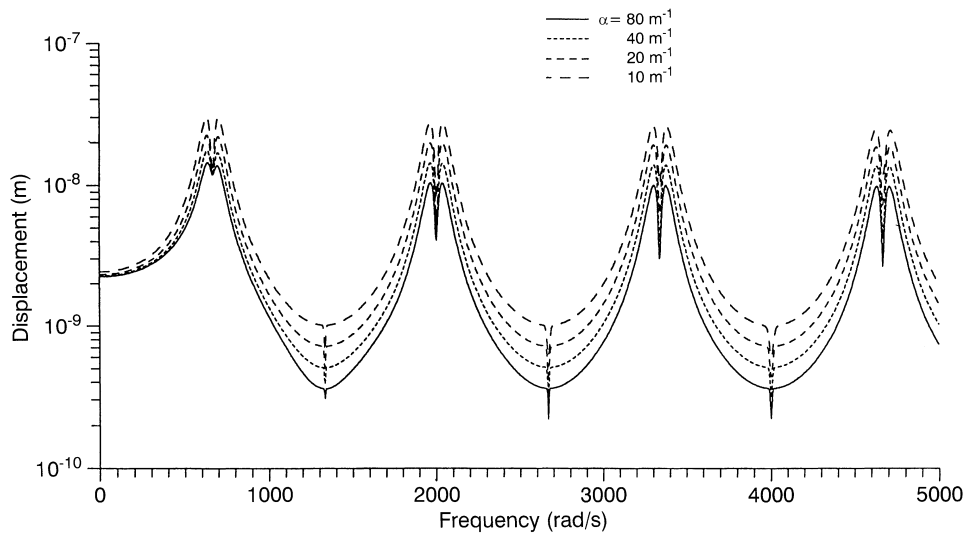

Fig. 12. Standard deviation of amplitude of displacement at node 2; $\epsilon = 0.03; \gamma = 0.0$

$|\delta_2(\omega)|$. It must be noted that the variance of the process $f(x)$, for all values of α, is given by $R_{ff}(0) = 1.0$. As may be observed, the theory predicts that, for the same level of randomness in $f(x)$, as α becomes smaller, the response variability becomes higher. Finally, we consider the influence of change in the shape of the autocovariance function of system property random processes on the response statistics. For the purpose of illustration, plots of the three autocovariance functions listed in Table 1, are shown in Fig. 13. In these plots, the parameters α_1, α_2 and α_3 appearing in the three models are selected to be, respectively, 20.0, 100.0 π and 10.0; this ensures that the autocovariance functions would lead to identical correlation lengths in all the three cases. Figure 14 shows the standard deviation of $|\delta_2(\omega)|$ for the case of $\gamma = 0.0, \epsilon = 0$ and the process $f(x)$ assuming these three autocovariance functions. It follows from this plot that, for a given variance and correlation length of $f(x)$, the shape of the autocovariance function has little influence on the response standard deviations.

7 Conclusions

The paper outlines the development of a new methodology for deriving statistical properties of dynamic stiffness coefficients of an axially vibrating rod with randomly varying elastic and mass properties. Some notable features of the approach presented are as follows:

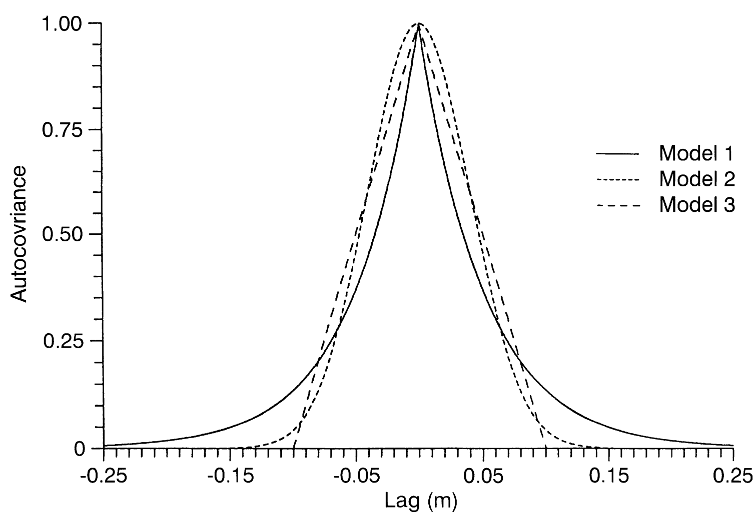

Fig. 13. Autocovariance functions with identical correlation lengths

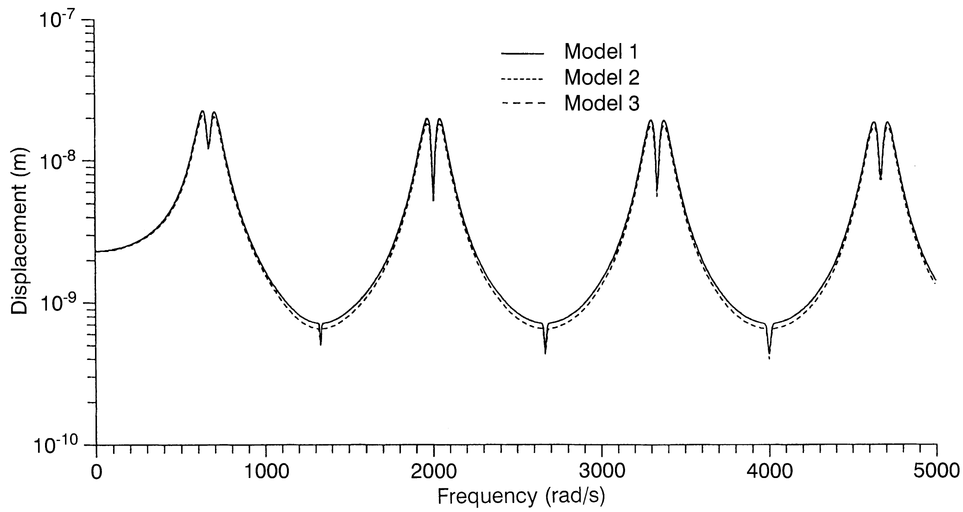

Fig. 14. Standard deviation of amplitude of displacement at node 2; $\epsilon = 0.03; \gamma = 0.0$

- The method enables determination of forced vibration characteristics without having to first find the probability distribution of random eigensolutions.
- Solution of the associated stochastic boundary value problem is obtained using techniques of Markov process theory, although the solution itself does not possess Markovian properties. This is made possible because the solution of the stochastic boundary value problem is obtained by superposing solutions of a pair of equivalent random initial-value problems.
- The method of stochastic averaging proves to be useful in the approximate analysis of the above mentioned initial-value problems. The solution obtained takes into account the mean, power-spectral density and cross-spectral density functions of the mass and stiffness processes. Satisfactory agreement is found to exist between the analytical results and a limited amount of digital simulations.

The paper also discusses an exactly solvable case which can serve as a benchmark for validating approximate procedures. Further studies on axially vibrating rods with distributed damping properties are currently being undertaken by the author. The field equation in this case becomes complex in nature and offers interesting challenges in analysis by stochastic averaging. Another question which requires further study is the relationship between stochastic dynamic stiffness matrix and the more commonly used modal analysis procedures. Application of the results reported in this study to the problem of seismic wave amplification through stochastic soil medium is also being currently investigated.

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