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Narrowband random excitation of a limit cycle system

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Summary: The response of the Van der Pol oscillator to stationary narrowband Gaussian excitation is considered. The central frequency of excitation is taken to be in the neighborhood of the system limit cycle frequency. The solution is obtained using a non-Gaussian closure approximation on the probability density function of the response. The validity of the solution is examined with the help of a stochastic stability analysis. Solution based on Stratonovich's quasistatic averaging technique is also obtained. The comparison of the theoretical solutions with the digital simulations shows that the theoretical estimates are reasonably good.

Zufallserregung eines Systems mit Grenzzyklus in einem schmalen Frequenzband

Übersicht: Gegenstand der Untersuchung ist die Antwort des Van der Pol-Schwingers auf eine stationäre Gaußsche Erregung in einem schmalen Frequenzband. Die zentrale Frequenz der Erregung wird in der Nachbarschaft zur Frequenz des Grenzzyklus gewählt. Die Lösung wird durch eine nicht-Gaußsche approximierende Einschließung für die Wahrscheinlichkeitsdichte der Antwort gewonnen. Die Gültigkeit dieser Lösung wird mit Hilfe einer stochastischen Stabilitätsanalyse überprüft. Darüber wird eine Lösung nach Stratonovichs quasistatischer Mittelungsmethode bestimmt. Der Vergleich der theoretischen Lösungen mit zahlenmäßigen Simulationen ergibt eine befriedigende Güte der theoretischen Abschätzungen.

1 Introduction

Limit cycle systems form an important class of nonlinear dynamical systems. The study of behaviour of these systems under different types of excitations is of considerable importance in nonlinear vibrations. Under a strict harmonic excitation these systems exhibit a wide variety of nonlinear behaviour, notably, the phenomenon of frequency entrainment. Simulation and analytical studies on the response of a typical limit cycle system, namely, the Van der Pol (VDP) oscillator to broad band random excitation has revealed that the response in this case is distinctly non-Gaussian with the displacement and velocity processes having bimodal probability distributions [12, 4]. The present authors have recently studied the effect of addition of a Gaussian white noise on the response of a periodically excited Van der Pol's oscillator [3, 7]. The methods of equivalent linearization, stochastic averaging and equivalent nonlinearization have been employed to study the response in the primary harmonic entrainment region. In the present paper attention is focused on the response of the oscillator to a narrowband excitation. The response of nonlinear systems to this type of excitations has many interesting features. The effect of narrowband noise on Duffing's oscillator has been studied by several authors, notable among them being Lennox and Kuak [5], Richard and Anand [9] and Iyengar [1]. However, the literature on narrowband excitation of limit cycle systems is very limited. Stratonovich [10] has studied the response of Rayleigh's oscillator to narrowband noise and obtained the stationary amplitude distribution valid in a limited region of parameter space using the quasistatic averaging technique. Windrich, Müller and Popp [11] have examined the scope of equivalent linearization methods in analysing stochastically disturbed limit cycle systems.

In the study on response of the VDP oscillator to white noise excitation the present authors have developed a non-Gaussian closure approximation [4]. In the present study this method is extended to investigate the response under narrowband random excitation. The acceptability of the approximate solution is further examined with the help of a stochastic stability analysis. The theoretical solutions are also compared with digital simulations.

2 Closure Approximation

The equation of motion of the forced VDP oscillator is given by

$$\ddot{x} - \varepsilon\dot{x}(1 - 4x^2) + x = f(t). \tag{1}$$

It is assumed that $0 < \varepsilon \ll 1$. The excitation $f(t)$ is taken to be a zero mean narrowband process obtained as the stationary output of a lightly damped linear single degree system excited by a Gaussian white noise process, i.e.

$$\begin{aligned} &\ddot{f} + 2\eta\lambda\dot{f} + \lambda^2 f = w(t), \\ &\langle w(t_1)\, w(t_2)\rangle = 2D\delta(t_1 - t_2). \end{aligned} \tag{2}$$

Here $\langle . \rangle$ denotes the expectation operator and $\delta(.)$ is the Dirac delta function. The damping coefficient η in the above equation is assumed to be small compared to unity. This would ensure that $f(t)$ is a narrow band process with central frequency λ. For the case of $f(t) = 0$, (1) possesses a stable periodic equilibrium solution given by [8]

$$x(t) = \sin(t + \phi) + O(\varepsilon). \tag{3}$$

Here ϕ is a constant depending on the initial conditions $x(0)$ and $\dot{x}(0)$ [4]. When $f(t) \neq 0$, the response would be a mixture of the system limit cycle and the response to the external excitation. Before constructing an approximate solution for this case it is useful to recall the qualitative features of response of the system to broad band excitation [4].

Firstly, it is noted that the stable limit cycle of the unforced system has a strong influence on the stochastic response. The limit cycle can be expected to be the most probable state in the presence of weak noise. Secondly, in the steady state the mean response would be zero. For this result to be consistent in the limit of the noise level going to zero, the basic limit cycle given by (3) must also be a stationary random process with zero mean. This would impose additional constraints on the nature of ϕ in (3) which in turn constrains the initial conditions $x(0)$ and $\dot{x}(0)$. A sufficient condition for the limit cycle oscillation to be a stationary stochastic process is that the parameter ϕ be a random variable distributed uniformly in $(-\pi, \pi)$. Based on these considerations the response to broadband noise was taken to be of the form

$$x(t) = F\sin(\omega t + \phi) + z(t). \tag{4}$$

Here F and ω are inknown deterministic constants, ϕ is a random variable distributed uniformly in $(-\pi, \pi)$ and $z(t)$ is a Gaussian stationary random process which is statistically independent of ϕ. The qualitative features of the response mentioned above are valid even when the band width of the excitation is reduced. Thus (4) can be taken as the form of the solution for (1) when $f(t)$ is given by (2). The first term in this solution represents the response component due to limit cycle effects and $z(t)$ is the effect due to external force. In the primary harmonic region, i.e. when λ is close to the basic limit cycle frequency, the frequency of oscillation will be close to λ. Further, in the limit of $D \to 0$, it is required that the response asymptotically approaches the basic limit cycle oscillation in which case $F = 1$. Thus for the narrow band excitation it is expedient to take $F = 1$ and $\omega = \lambda$. The assumed solution of (1) is thus given by

$$x(t) = \sin(\lambda t + \phi) + z(t). \tag{5}$$

To determine the properties of $z(t)$, (5) is substituted into (1) to get

$$\begin{aligned} &\ddot{z} - \varepsilon[\lambda\cos(\lambda t + \phi) + \dot{z}]\,[1 - 4z^2 - 4\sin^2(\lambda t + \phi) - 8z\sin(\lambda t + \phi)] \\ &+ z + (1 - \lambda^2)\sin(\lambda t + \phi) = f(t). \end{aligned} \tag{6}$$

This equation is multiplied by $z_1 = z(t_1)$ and averaged under the closure assumption that z, $\dot{z}$ and f are jointly Gaussian [2]. Further, noting that $z(t)$ and ϕ are independent, one gets

$$\langle \ddot{z}z_1 \rangle + \varepsilon \langle \dot{z}z_1 \rangle (1 + 4\langle z^2 \rangle) + \langle zz_1 \rangle (1 + 8\varepsilon \langle z\dot{z} \rangle) = \langle fz_1 \rangle . \tag{7}$$

This moment equation is derivable from the linear system

$$\ddot{z} + \varepsilon \dot{z}(1 + 4\langle z^2 \rangle) + z(1 + 8\varepsilon \langle z\dot{z} \rangle) = f . \tag{8}$$

Thus, the above equation can be regarded as the equivalent linear system associated with (1). It may be noted that (6) can also be handled using the classical statistical linearization technique. Here an equivalent linear system is obtained which will be different from that one in (8). However, it is easy to show that the difference in the corresponding solutions would be only in the transient regime and both the solutions would lead to the same steady state solutions. Equation (8) together with (2) can now be solved using the Fokker-Planck-Kolmogorov (FPK) equation approach. With the notations

$$\begin{aligned}
&s_1 = \langle z^2 \rangle, \quad s_2 = \langle z\dot{z} \rangle, \quad s_3 = \langle zf \rangle, \quad s_4 = \langle z\dot{f} \rangle,\\
&\qquad\qquad s_5 = \langle \dot{z}^2 \rangle, \quad s_6 = \langle \dot{z}f \rangle, \quad s_7 = \langle \dot{z}\dot{f} \rangle,\\
&\qquad\qquad\qquad\qquad s_8 = \langle f^2 \rangle, \quad s_9 = \langle f\dot{f} \rangle,\\
&\qquad\qquad\qquad\qquad\qquad\qquad s_{10} = \langle \dot{f}^2 \rangle .
\end{aligned} \tag{9}$$

the following moment equations can easily be derived from the governing FPK equation:

$$\dot{s}_1 = 2s_2 , \tag{10}$$

$$\dot{s}_2 = s_5 - s_2\varepsilon(1 + 4s_1) - s_1(1 + 8\varepsilon s_2) + s_3 , \tag{11}$$

$$\dot{s}_3 = s_6 + s_4 , \tag{12}$$

$$\dot{s}_4 = s_1 - 2\eta\lambda s_4 - \lambda^2 s_3 , \tag{13}$$

$$\dot{s}_5 = -2s_5\varepsilon(1 + 4s_1) - 2s_2(1 + 8\varepsilon s_2) + 2s_6 , \tag{14}$$

$$\dot{s}_6 = -s_6\varepsilon(1 + 4s_1) - s_3(1 + 8\varepsilon s_2) + s_8 + s_7 , \tag{15}$$

$$\dot{s}_7 = -s_7\varepsilon(1 + 4s_1) - s_4(1 + 8\varepsilon s_2) + s_9 - 2\eta\lambda s_7 - \lambda^2 s_6 , \tag{16}$$

$$\dot{s}_8 = 2s_9 , \tag{17}$$

$$\dot{s}_9 = s_{10} - 2\eta\lambda s_9 - \lambda^2 s_8 , \tag{18}$$

$$\dot{s}_{10} = -4\eta\lambda s_{10} - 2\lambda^2 s_9 + 2D . \tag{19}$$

In the stochastic steady state, the time derivatives of the moments vanish leading to the following equations for the response variance:

$$s_1 = \frac{D[\lambda^2(\nu + 2\eta\lambda) - \nu(\lambda^2 - 1) + 2\eta\lambda\nu(\nu + 2\eta\lambda)]}{(2\eta\lambda^3)\,[\lambda^2(\nu + 2\eta\lambda)^2 + (\lambda^2 - 1)^2 - 2\eta\lambda(\lambda^2 - 1)\,(\nu + 2\eta\lambda)]}, \tag{20}$$

$$s_5 = \frac{D(\nu + 2\eta\lambda)}{(2\eta\lambda^3)\,[\lambda^2(\nu + 2\eta\lambda)^2 + (\lambda^2 - 1)^2 - 2\eta\lambda(\lambda^2 - 1)\,(\lambda + 2\eta\lambda)]}, \tag{21}$$

where

$$\nu = \varepsilon(1 + 4s_1) . \tag{22}$$

Further, it follows from (5) that

$$p(x, \dot{x}) = 1/\left(4\pi^2 \sqrt{s_1 s_5}\right) \int_{-\pi}^{\pi} \exp\left[-0.5(x - \sin u)^2/s_1 - 0.5(\dot{x} - \lambda \cos u)^2/s_5\right] \mathrm{d}u , \tag{23}$$

$$p(x) = 1/\left(2\pi \sqrt{2\pi s_1}\right) \int_{-\pi}^{\pi} \exp\left[-0.5(x - \sin u)^2/s_1\right] \mathrm{d}u , \tag{24}$$

$$p(\dot{x}) = 1/\left(2\pi\sqrt{2\pi s_1}\right)\int_{-\pi}^{\pi} \exp\left[-[0.5(\dot{x} - \lambda\cos u)^2/s_5\right] \mathrm{d}u. \tag{25}$$

It is to be noted that the above probability density functions are non-Gaussian. The probability density function of displacement and velocity are, infact, bimodal in nature.

3 Stochastic stability

The solution in (23) has been obtained under a set of assumptions and hence is expected to be valid only in specific regions of the parameter space. A necessary condition for any solution to be acceptable as an approximation is that the assumed solution should be stochastically stable. That is, any perturbation to the structure of the assumed solution must asymptotically die away. This condition of stochastic stability of a proposed solution as a criterion for its acceptability has earlier been employed by Iyengar [1]. This criterion is also used in the present study. In this connection it is also important to note that the frequency response curves given by (20)–(22) are nonlinear in nature and can result in multivalued response variance. The question now arises as to which of the solutions are to be accepted. The stability analysis provides an answer to this question also.

For carrying out the stochastic stability analysis a small perturbation $v(t)$ is imposed on $x(t)$ given by (5). The variational equation for $v(t)$ is derived from (1) as

$$\ddot{v} - \varepsilon\dot{v}(1 - 4x^2) + v(1 + 8\varepsilon x\dot{x}) = 0. \tag{26}$$

Here, $x(t)$ is a narrowband process which admits an envelope representation as

$$x(t) = a(t)\cos\theta(t), \qquad \dot{x}(t) = -a\omega_e\sin\theta(t), \qquad \theta(t) = \omega_e t + \phi(t) \tag{27}$$

where $a(t)$ and $\phi(t)$ are slowly varying random processes and ω_e is the central frequency, which, to a first approximation, is taken to be the average rate of upward zero crossing of the process $x(t)$, [6], i.e.

$$\omega_e = 2\pi\int_0^{\infty} \dot{x}p(0, \dot{x})\,\mathrm{d}\dot{x}. \tag{28}$$

Further, it follows from (23) and (27) that

$$p(a, \theta) = a\omega_e/\left(4\pi^2\sqrt{s_1 s_5}\right)\int_{-\pi}^{\pi} \exp\left[-0.5(a\cos\theta - \sin u)^2/s_1 - 0.5(-a\omega_e\sin\theta - \lambda\cos u)^2\right] \mathrm{d}u. \tag{29}$$

To analyse (26) further, the following transformations are introduced:

$$\tau = \omega_e t,$$
$$v(\tau) = u(\tau)\exp\left[0.5\varepsilon/\omega_e\int_0^{\tau} (1 - 4x^2)\,\mathrm{d}s\right]. \tag{30}$$

Now combining (26), (27) and (30) and retaining terms up to $O(\varepsilon)$, (26) can be rewritten as

$$u'' + \{1/\omega_e^2 - 2a^2/\omega_e\sin[2(\tau + \phi)]\}\,u = 0. \tag{31}$$

The primes here denote derivatives with respect to τ. Since the parametric excitation in this equation is $\sin[2(\tau + \phi)]$, the solution in the primary resonance region can be taken as

$$u(\tau) = A_1(\tau)\cos\tau + A_2(\tau)\sin\tau. \tag{32}$$

Further, since $x(\tau)$ is a narrowband process, following the quasistatic averaging outlined by Stratonovich [10], it can be shown that

$$A_1' = -C_{11}A_1 - C_{12}A_2, \qquad A_2' = C_{21}A_1 + C_{22}A_2, \tag{33), (34}$$

$$C_{11} = -0.5\varepsilon a^2/\omega_e \cos(2\phi), \qquad C_{12} = 0.5 - 0.5/\omega_e^2 - 0.5\varepsilon a^2/\omega_e \sin(2\phi), \qquad (35), (36)$$

$$C_{21} = 0.5 - 0.5/\omega_e^2 + 0.5\varepsilon a^2/\omega_e \sin(2\phi), \qquad C_{22} = C_{11}. \qquad (37), (38)$$

Since a and ϕ are slowly varying functions, the above coefficients are also slowly varying functions of time. Hence the solution for A_1 and A_2 can be taken as

$$A_1 = A_{01} \exp\left[\int_0^\tau \xi(s)\, ds\right], \qquad A_2 = A_{02} \exp\left[\int_0^\tau \xi(s)\, ds\right]. \qquad (39)$$

This leads to the condition

$$\xi^2 = C_{11}^2 - C_{12}C_{21}. \qquad (40)$$

Now, from (30) and (39) the condition for the almost sure asymptotic stability of $v(\tau)$ is

$$\lim_{\tau\to\infty} \left\langle \mathrm{Re} \left\{ 1/\tau \left[\int_0^\tau \xi(s)\, ds + 0.5\varepsilon/\omega_e \int_0^\tau (1 - 4x^2)\, ds \right] \right\} \right\rangle < 0. \qquad (41)$$

If as a further approximation $x(\tau)$ and hence $\xi(\tau)$ is taken to be ergodic, the time averages can be replaced by ensemble averages to get

$$\mathrm{Re}\, (\langle \xi \rangle + 0.5\varepsilon/\omega_e \langle 1 - 4x^2 \rangle) < 0. \qquad (42)$$

From (40) and (35)–(38) it can be shown that

$$\langle \xi \rangle = 0.5/\omega_e \langle [\varepsilon^2 a^4 - (\omega_e^2 - 1)^2]^{12} \rangle. \qquad (43)$$

The stability criterion of (42) now reads

$$\int_R [\varepsilon^2 a^4 - (\omega_e^2 - 1)^2]^{1/2} p(a)\, da + \varepsilon(1 - 4\langle x^2 \rangle) < 0. \qquad (44)$$

The integration in the above has to be carried out over only the real values of the integrand.

4 Stratonovich's solution

In many nonlinear random vibration problems the method of stochastic averaging has been observed to be a useful technique for analysing the response under broadband inputs. A prerequisite for applying this method has been the condition that the relaxation time of the system be much greater than the correlation time of the excitation. In the present problem, however, $f(t)$ is a narrowband process and hence this technique is clearly not applicable. On the other hand, if the parameters ε and η of (1) and (2) are assumed to satisfy the condition $\varepsilon \gg \eta$, then a clear cut separation between the time constants of system and excitation will again exist, but the condition would be the exact opposite of the requirement stipulated for applying the stochastic averaging method. Stratonovich [10] has proposed a variation of the standard stochastic averaging which is applicable to this case. This method is called the quasistatic averaging method. Stratonovich has applied this technique to a system which is similar to the one considered in (1). His results are reproduced here to facilitate a comparison with the solution developed in the earlier section. The method consists of only temporal averaging. The ensemble averaging with its attendant Markov approximation is dispensed with. In applying this method both the input and the response are expressed as

$$\begin{aligned} f(t) &= P(t) \cos[\lambda t + \gamma(t)], \\ x(t) &= A(t) \cos[\lambda t + \psi(t)] \end{aligned} \qquad (45)$$

where P, A, γ and ψ are slowly varying random processes. During temporal averaging these quantities are treated as random variables and hence remain constants. This leads to a memoryless transformation relating the input and the output amplitudes and phase angles. The solution of this transformation is further required to be stable. This requirement imposes additional constraints on the range of input and output variables. In cases where no stable amplitude and phase are possible, Stratonovich has proposed a second level of time averaging of the equations which have already been averaged once. Together with this additional averaging, the method leads to the probability density function of the response amplitude given by

$$p(A) = \begin{cases} 0 \quad \text{for} \quad 0 < A < \sqrt{Z} \\ (1 - P_1)\,\delta(A - 1) + (\varepsilon\lambda/\sigma_f)^2\,|A[(1 - A^2)^2 + \Delta^2] - 2A^3(1 - A^2)| \end{cases}$$
$$\times \exp\{-(A\varepsilon\lambda/\sigma_f)^2\,[(1 - A^2)^2 + \Delta^2]\} \quad \text{for } \sqrt{Z} < A < \infty,$$
$$P_1 = \exp\{-Z(\varepsilon\lambda/\sigma_f)^2\,[(1 - z)^2 + \Delta^2]\}, \qquad Z = \left(2 + \sqrt{1 - 3\Delta^2}\right)/3 \tag{46}$$
$$\sigma_f^2 = \langle f^2 \rangle, \qquad \Delta = (1 - \lambda^2)/(\varepsilon\lambda).$$

The solution has been obtained under the assumption that $|\Delta| < 0.5$. This assumption is reasonable, since, in the present approximation, interest is focused on the system behaviour near resonance. Although the case of $|\Delta| > 0.5$ has not been dealt by Stratonovich, it is possible to apply the method in this case also.

5 Numerical results

The theoretical solutions based on the closure approximation and the quasistatic averaging method are shown in Figs. (1)–(4). The quasistatic averaging solution yields information on the moments and the probability density function of the response amplitude, while the closure solution characterizes the displacement, velocity and the amplitude processes. It should be noted here that the response amplitude process $a(t)$ of closure approximation (27) and $A(t)$ of quasistatic averaging (45) are not strictly identical. Nevertheless for ease in comparison the moments and the probability density functions of these two quantities are shown on the same Figs. (1), (2) and (4). In the numerical work the parameters ε and η are taken to be 0.05 and 0.01, respectively. The response moments as function of the detuning parameter $\Delta = (1 - \lambda^2)/(\varepsilon\lambda)$ and noise level D are shown in Figs. (1) and (2), respectively. The closure solution is obtained by solving (20)–(22). Although these equations are nonlinear, in the parameter range considered, the solutions are, however, found to be single valued. Further the stability criterion of (44) shows that these solutions are stable and hence acceptable over the entire range of the parameters considered. The theoretical probability density functions of displacement and amplitude processes for the case of $D = 2 \times 10^{-5}$, and several values of Δ are shown in Figs. (3) and (4). The probability density of the

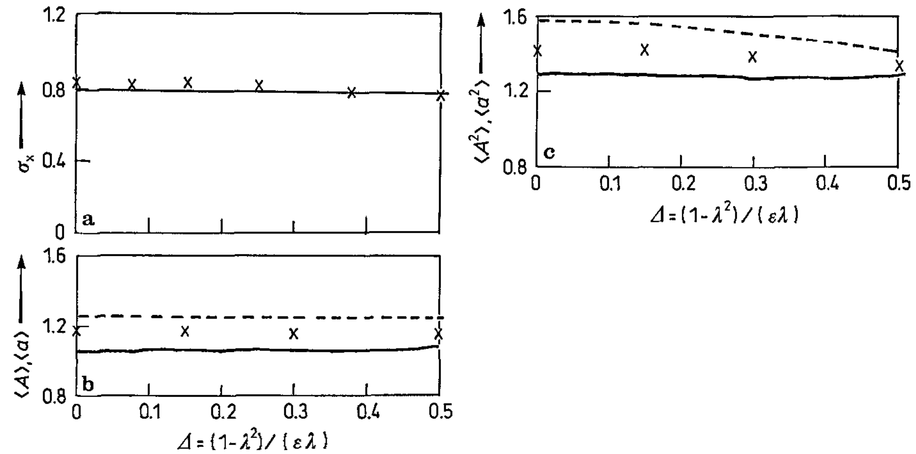

Fig. 1. a Steady state variance (—— closure, × × simulation), **b** Steady state mean amplitude (—— $\langle a \rangle$ closure, ----- $\langle A \rangle$ quasistatic averaging, × × $\langle A \rangle$ simulation), **c** Steady state mean square amplitude (—— $\langle a^2 \rangle$ closure, ----- $\langle A^2 \rangle$ quasistatic averaging, × × $\langle A^2 \rangle$ simulation); $D = 2 \times 10^{-5}$, $\varepsilon = 0.05$, $\eta = 0.01$

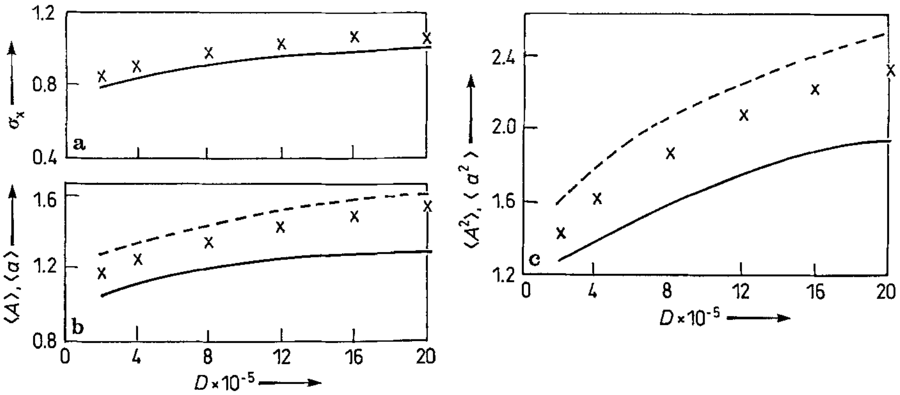

Fig. 2. a Steady state variance (—— closure, × × simulation), **b** Steady state mean amplitude (—— $\langle a\rangle$ closure, ----- $\langle A\rangle$ quasistatic averaging, × × $\langle A\rangle$ simulation), **c** Steady state mean square amplitude (—— $\langle a^2\rangle$ closure, ---- $\langle A^2\rangle$ quasistatic averaging, × × $\langle A^2\rangle$ simulation); $\Delta = 0$, $\varepsilon = 0.05$, $\eta = 0.01$

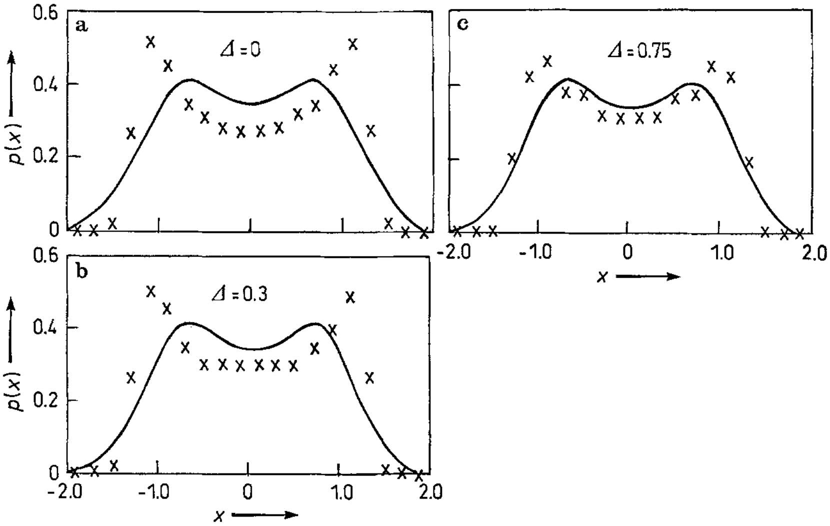

Fig. 3. a–c. Stationary probability density function (—— closure, × × simulation); $D = 2 \times 10^{-5}$, $\varepsilon = 0.05$, $\eta = 0.01$

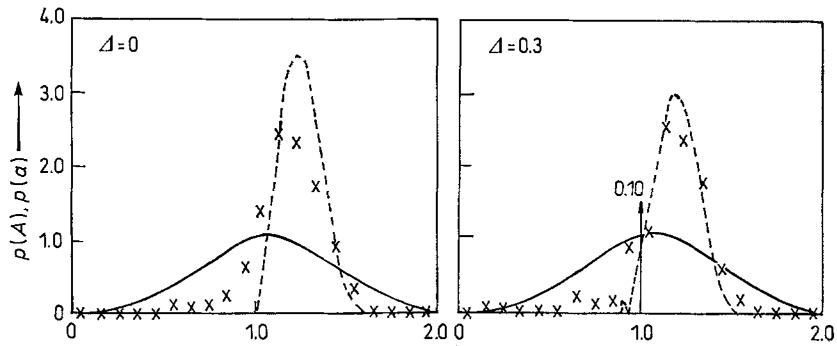

Fig. 4. a, b. Stationary probability density function (—— $p(a)$ closure, ---- $p(A)$ quasistatic averaging, × $p(A)$ simulation) $D = 2 \times 10^{-5}$, $\varepsilon = 0.05$, $\eta = 0.01$

response amplitude obtained using the quasistatic averaging technique has both discrete and continuous parts. For $\Delta \neq 0$, the discrete component assumes a non-zero value. Thus in Fig. 4b, for $\Delta = 0.3$ the theoretical probability density function has a discrete component of $P(A = 1) = 0.1$.

6 Digital simulations

The validity of the theoretical solutions can be checked with the help of numerical simulations. These studies are carried out on (1) and (2) after transforming the time variable t to $\tau = \lambda t/(2\pi)$. This leads to

$$x'' - 2\pi(\varepsilon/\lambda)\, x'(1 - 4x^2) + (2\pi/\lambda)^2\, x = (2\pi/\lambda)^2\, f, \tag{47}$$

$$f'' + 4\eta\pi f' + 4\pi^2 f = (2\pi/\lambda)^2\, W. \tag{48}$$

The stationary solution of (48) is fed as the input to (47). The integration is carried out using a fourth order Runge-Kutta algorithm with a step size of $\Delta\tau = 0.025$ and for a length of 50 cycles. To study the properties of response envelope process, the following quantities are simulated:

$$\begin{aligned} a(\tau) &= \{x^2(\tau) + (\lambda/\omega_e)^2\,[x'(\tau)]^2/(4\pi^2)\}^{1/2}, \\ A(\tau) &= \{x^2(\tau) + [x'(\tau)]^2/(4\pi^2)\}^{1/2}, \\ a_1(\tau) &= \{x^2(\tau) + \lambda^2[x'(\tau)]^2/(4\pi^2)\}^{1/2}, \\ a_2 &= \text{local extreme values of } x(\tau), \\ a_3 &= \text{positive local maxima of } x(\tau). \end{aligned} \tag{49}$$

The quantities $a(\tau)$, $A(\tau)$ and $a_1(\tau)$ are smooth functions of time τ, while a_2 and a_3 represent sets of discrete points. It can be observed that the points belonging to the sets a_2 and a_3 are contained in $a(\tau)$, $A(\tau)$ and $a_1(\tau)$. The definitions of $a(\tau)$ and $A(\tau)$ correspond to the envelope representation used in the closure solution (27) and the quasistatic averaging approximation (45), respectively. To estimate the desired statistics, the response quantities from $\tau = 49$ cycle to $\tau = 50$ cycle are picked up from each sample. This procedure is repeated for the entire ensemble of 100 samples. The numbers thus obtained from each sample are assembled together and processed to obtain the estimates for the moments and the probability density functions. The corresponding quantities for displacement process are shown in Figs. 1a, 2a and 3. The estimate of mean and mean square values of the response envelope obtained using different definitions of (49) are observed to be very close to each other. Thus, e.g. for $\Delta = 0.5$ and $D = 2 \times 10^{-5}$ the mean values using the five definitions of (49) are 1.17, 1.15, 1.15, 1.13 and 1.12, resp., and the corresponding mean square values read 1.41, 1.37, 1.35, 1.32 and 1.30. For sake of clarity only one set of these results, namely the at one corresponding to $A(\tau)$ is shown in Figs. 1b, 1c and 2c. Similarly the simulated histograms using different definitions were observed to be nearly identical. Again, only one of these results is plotted in Fig. 4.

7 Discussion and Conclusion

In this paper the response of Van der Pol's oscillator to quasiharmonic stochastic excitations is examined in the primary harmonic region using both the closure approximation and the quasistatic averaging technique. Under a strict harmonic excitation the system is known to exhibit the entrainment behaviour. When the amplitude and the frequency of the harmonic excitation are randomly perturbed, the response amplitude and phase, in turn, become slowly varying random processes. Thus, in any individual sample of the response, one can expect that the forming and breaking of entrained oscillations take place indefinitely. This interaction between the limit cycle oscillation and the dominantly periodic forcing can further be expected to produce multimodal response probability distributions. The closure approximation presented in this paper is based on a bimodal probability density function. The method leads to the moments and the probability distributions of displacement, velocity and amplitude processes. Further, the acceptability of this approximate solution is examined with the help of a stochastic stability analysis. In the parameter ranges considered, this analysis shows that the assumed solution is stochastically stable and hence acceptable. In the method of quasistatic averaging the solution is expressed as a memoryless nonlinear transformation of random variables. This method leads to the probability density of the response amplitude process. The numerical results presented in Figs. 1 and 2 show that the response moments obtained using the closure technique and the quasistatic averaging show qualitatively identical behaviour. The comparison with digital simulation results show that while the quasistatic averaging overestimates the response moments, the closure solution, on the other hand, underestimates the same. The bimodal nature of displacement distribution obtained using the closure technique qualitatively agrees well with the simulation results (Fig. 3). On the other hand, the comparison of amplitude distribution is poor (Fig. 4). In this case, the quasistatic averaging is found to give better results. When the detuning parameter Δ is not zero the quasistatic averaging solution predicts a discrete component in the probability density function of the amplitude located at the limit cycle amplitude of the system. Thus in Fig. 4b, for $\Delta = 0.3$, a component of $P(A = 1) = 0.1$ is present. A similar tendency appears to be present in the simulation results shown in this figure.

In conclusion, it can be said that the closure solution presented in this paper yields reasonably good approximation to the response moments but the predictions on the probability density functions are rather poor. The results based on quasistatic averaging technique, on the other hand, give fairly good estimates for both the response amplitude moments and the probability density function.

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