Probability Distribution of Extremes of Von Mises Stress in Randomly Vibrating Structures

Sayan Gupta
Research Student
Department of Civil Engineering,
Indian Institute of Science,
Bangalore 560012, India
e-mail: gupta.sayan@gmail.com

C. S. Manohar[1]
Professor
Department of Civil Engineering,
Indian Institute of Science,
Bangalore 560012, India
e-mail: manohar@civil.iisc.ernet.in

The problem of determining the probability distribution function of extremes of Von Mises stress, over a specified duration, in linear vibrating structures subjected to stationary, Gaussian random excitations, is considered. In the steady state, the Von Mises stress is a stationary, non-Gaussian random process. The number of times the process crosses a specified threshold in a given duration, is modeled as a Poisson random variable. The determination of the parameter of this model, in turn, requires the knowledge of the joint probability density function of the Von Mises stress and its time derivative. Alternative models for this joint probability density function, based on the translation process model, combined Laguerre-Hermite polynomial expansion and the maximum entropy model are considered. In implementing the maximum entropy method, the unknown parameters of the model are derived by solving a set of linear algebraic equations, in terms of the marginal and joint moments of the process and its time derivative. This method is shown to be capable of taking into account non-Gaussian features of the Von Mises stress depicted via higher order expectations. For the purpose of illustration, the extremes of the Von Mises stress in a pipe support structure under random earthquake loads, are examined. The results based on maximum entropy model are shown to compare well with Monte Carlo simulation results. [DOI: 10.1115/1.2110865]

Keywords: extreme value distribution, Von Mises stress, non-Gaussian random process, maximum entropy method, Laguerre series expansion, Gram-Charlier series expansion

Introduction

According to the failure theory based on the Von Mises yield criterion, failure in ductile structures occur if the Von Mises stress exceeds a permissible limit which signals initiation of yielding. The safety of such structures can be, thus, quantified in terms of the probability of exceedance of the Von Mises stress across the yield limits. For structures under random dynamic loads, the Von Mises stress is a random process and the structural failure probability can be expressed in terms of the probability of exceedance of the extremes of Von Mises stress, in a specified time interval. The Von Mises stress is a non-Gaussian random process whose probability distribution is hard to determine, even in linear structures under Gaussian excitations. This makes it difficult to estimate the probability distribution of the extremes of Von Mises stress. This paper focusses on the development of approximations for the probability distribution of extreme values for the Von Mises stress in linear structures under Gaussian dynamic loads.

Extreme value distributions of random processes are closely related to the probability distribution of first passage time of the random processes across specified thresholds, in a specified time interval [1,2]. A key feature in this formulation is the assumption that for high thresholds, the number of times the process crosses the threshold in a given duration of time, can be modeled as a Poisson process, whose parameter is the mean outcrossing rate of the process across the specified threshold. In applying this formulation, one, in turn, needs to know the joint probability density function (PDF) of the random process and its derivative, at the same instant of time [3]. This knowledge is seldom available for non-Gaussian processes, such as the Von Mises stress. Techniques for obtaining approximations for outcrossing probabilities of non-Gaussian processes have been discussed in the literature. Bounds on the exceedance probabilities of non-Gaussian processes have been obtained by applying linearization schemes [4–6]. Grigoriu [7] obtained approximate estimates for the mean outcrossing rate of non-Gaussian translation processes by studying the outcrossing characteristics of a Gaussian process obtained from Nataf's transformation of the parent non-Gaussian process. Madsen [8] adopted a geometrical approach for developing analytical expressions for the mean outcrossing rate of Von Mises stress in linear structures, under stationary Gaussian excitations. It must be noted that the first order correlation $\langle X(t)\dot{X}(t)\rangle$, where $X(t)$ is a zero-mean, stationary, non-Gaussian random process and $\dot{X}(t)$ is its time derivative at the same time instant, is equal to zero. Here, $\langle\cdot\rangle$ is the expectation operator. However, this does not imply that $X(t)$ and $\dot{X}(t)$ are stochastically independent. The models for extreme value distribution for non-Gaussian processes, as discussed by Grigoriu and Madsen, while they correctly take into account the fact that $\langle X(t)\dot{X}(t)\rangle=0$, do not, however, allow for mutual dependence between $X(t)$ and $\dot{X}(t)$ at the same time instant. Naess [9] developed a numerical procedure for computing the mean outcrossing rate of quadratic transformations of Gaussian processes. Here, the joint characteristic function for the process and its derivative is used, rather than their joint PDF. Methods for computing the root mean square of Von Mises stress, resulting from zero-mean, stationary Gaussian loadings, and for estimating their instantaneous exceedance probabilities have also been studied [10–12]. The present authors, in an earlier work, have used a response surface based method to study exceedance probabilities of Von Mises stress in a nonlinear structure under random excitations [13].

In this study, we develop an approximation for the probability distribution of the extremes of Von Mises stress in a randomly driven linear structure. A key feature of this study lies in appro-

[1]Corresponding author. Telephone: +91 80 2293 3121; Fax: +91 80 2360 0404.

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priately modeling the marginal and joint PDF of the Von Mises stress and its time derivative, at a given time instant. This enables application of Rice's formula [3], given by,

$$\nu^+(\zeta) = \int_0^\infty \dot{v} p_{V\dot{V}}(\zeta,\dot{v};t)\mathrm{d}\dot{v}, \tag{1}$$

in determining the mean outcrossing rate $\nu^+(\zeta)$, across the specified threshold ζ, from which one can construct the extreme value distribution. It must be noted that the Von Mises stress, σ_v, is a strictly positive quantity. This implies that $P[\sigma_v < \eta] = P[\sigma_v^2 < \eta^2]$, where η is a specified threshold and $P[\cdot]$ is the probability measure. This property becomes useful as it is found to be mathematically simpler to construct the probability distribution of the extremes of σ_v^2, rather than σ_v, as the moments of the square of the Von Mises stress are easier to compute. For a given time t, the square of the Von Mises stress and its time derivative, are random variables and are denoted by V and $\dot{V}$. A model for the first order PDF for V, $p_V(v)$, may be constructed using either the Laguerre polynomial series expansions [14] or the maximum entropy method (MEM) [15]. Approximations for the mean outcrossing rate may then be obtained by adopting the translation procedure proposed by Grigoriu [7]. This method does not require modeling of the marginal PDF, $p_{\dot{V}}(\dot{v})$, or the joint PDF $p_{V\dot{V}}(v,\dot{v})$. This model, however, does not take into account the stochastic dependence that exist between V and $\dot{V}$. The effect of mutual dependence between V and $\dot{V}$ can be included in the model for $p_{V\dot{V}}(v,\dot{v})$, if models for $p_{\dot{V}}(\dot{v})$ and $p_{V\dot{V}}(v,\dot{v})$ are constructed additionally. Models for $p_{\dot{V}}(\dot{v})$ can be constructed using either Gram-Charlier series expansions [14] or MEM. Subsequently, approximations for $p_{V\dot{V}}(v,\dot{v})$ are constructed by assuming that it is permissible to expand $p_{V\dot{V}}(v,\dot{v})$ in terms of two sets of orthogonal polynomials. These orthogonal functions are obtained from Gram-Schmidt orthogonalization and are expressed in terms of the moments $\langle V^m \dot{V}^n \rangle, m,n=0,1,\ldots$ An alternative approach is to employ the method of maximum entropy for constructing the marginal as well as joint PDFs of V and $\dot{V}$. The underlying principle of this method lies in constructing a continuous PDF that maximizes entropy subject to the constraints of available partial information, such as moments and the range of the random variables [15–17]. The difficulties in the application of MEM are: (a) Usually, higher order moments are seldom available [18], and (b) the computational effort required for determining the maximum entropy PDF parameters can be quite expensive as it involves solving a set of simultaneous nonlinear equations. Numerical techniques discussed in the literature, include the Newton-Raphson method [19,20] and a Fourier based approach [21]. Recently, Volpe and Baganoff [22], in their studies on fluid dynamics problems, established a simple linear relationship between the process moments and parameters of a univariate maximum entropy PDF. These authors demonstrated the important result that if the first $2N-2$ moments of the process are available, the first N parameters of the maximum entropy PDF can be obtained by solving a set of N simultaneous linear equations.

The focus of this study has been on the development of methods for constructing approximations for the probability density function for the extremes of a special class of non-Gaussian random process, namely, the von Mises stress in Gaussian excited, linear vibrating structures. The performance of the proposed methods have been compared with those existing in the literature, based on the translation process theory. In this study, we first extend on the work by Volpe and Baganoff and develop a system of moment equations for the maximum entropy joint PDF, $p_{V\dot{V}}(v,\dot{v})$. This requires the knowledge of the marginal and joint moments of V and $\dot{V}$. We assume that the structure is excited by zero-mean, stationary Gaussian loads and exhibits linear behavior. Consequently, the stress components, in the steady state, are stationary, Gaussian random processes, have zero mean and are completely specified through their covariance matrix. This enables computation of the joint moments of the stress components up to any desired order. The marginal and joint moments of V and $\dot{V}$, are expressible in terms of the joint moments of stress components and their time derivatives, and hence, can be computed up to any desired order. Once the parameters of the joint maximum entropy PDF are established, $\nu^+(\zeta)$ is subsequently computed from the Rice formula. This, in turn, leads to the solution to the first passage time problem and the problem of determining the extremes over a specified time duration. An alternative method for approximating the joint PDF using combined Laguerre-Hermite polynomial expansions from the knowledge of the joint moments, has also been studied. The proposed methods for obtaining the extreme value distribution of the Von Mises stress is illustrated through a numerical example and the results are compared with those obtained from Monte Carlo simulations. The new contributions that are made in this study can, thus, be summarized as follows:

1. Development of a formulation for constructing a bivariate maximum entropy probability density function. This is an extension of the earlier study in the area of fluid dynamics, by Volpe and Baganoff [22] on univariate distributions. Some of the key features of this formulation are:
   - (a) The fitting of the bivariate maximum entropy distributions involves the solution of a set of *linear* algebraic equations, and
   - (b) the method is capable of taking into account the presence of stochastic dependence between the Von Mises stress, in the steady state, and its time derivative at the same instant.

   The study also clarifies a few computational issues in the determination of these bivariate probability distributions, especially, with reference to the importance of choice of appropriate set of moment equations in successful development of these models.
2. The performance of the above model is compared with two alternative analytical models, namely, those based on a combined Laguerre-Hermite series expansion and translation process model. Relative merits of these alternatives, within a conceptual framework, as well as, in relation to their computational requirements and closeness of the model predictions to results from Monte Carlo simulations, have also been discussed.
3. The above studies have been conducted in the context of determination of extremes of an important class of non-Gaussian random processes in vibration engineering, namely, the Von Mises stress.

Problem Statement

We consider a linear structure under random dynamic loads. The loads are modeled as a vector of stationary, zero-mean, Gaussian random processes. The stress components at a specified location in the structure, $\Sigma(t)$, are linearly dependent on the excitations, and in the steady state, are thus, zero-mean stationary, Gaussian random processes. The state of stress at any point in the structure is defined through the vector $\Sigma(t) = [\sigma_1(t)\sigma_2(t)\sigma_3(t)\sigma_4(t)\sigma_5(t)\sigma_6(t)]^T$. Here, subscripts 1–3 denote the normal stress components, subscripts 4–6 denote the shear stress components, the superscript T denotes matrix transpose and t is the time. The power spectral density (PSD) matrix of the stress components is obtained from a random vibration analysis on the structure. The square of Von Mises stress, $V(t)$, at a specified location, is given by [10,23]

$$V(t) = \sum_{k=1}^{6} \sum_{l=1}^{6} A_{kl}\sigma_k(t)\sigma_l(t), \tag{2}$$

where,

$$\mathbf{A} = \begin{bmatrix} 1 & -0.5 & -0.5 & 0 & 0 & 0 \\ -0.5 & 1 & -0.5 & 0 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{3}$$

In this study, we define failure as the exceedance of $V(t)$, across a permissible threshold ζ, during a specified time interval $[t_0, t_0+T]$. The probability of structural failure is, thus, expressed as

$$P_f = 1 - P[V(t) \leq \zeta \;\forall\; t \in (t_0, t_0+T)]. \tag{4}$$

Note that for stationary random processes $t_0 \to \infty$. The evaluation of this probability constitutes a problem in time variant reliability analysis. Introducing the random variable $V_m = \max_{t_0 \leq t \leq t_0+T} V(t)$, and noting that $P_f = 1 - P[V_m \leq \zeta]$, the problem can be formulated in the time invariant format. The number of times $V(t)$ exceeds ζ in $[t_0, t_0+T]$, is a random variable and can be assumed to be Poisson distributed for high threshold levels of ζ. Expressions for the mean outcrossing rate of the stationary process $V(t)$, across ζ, is given by the well-known Rice's formula [Eq. (1)] and the failure probability is given by

$$P_f = P[V_m > \zeta] = 1 - \exp[-\nu^+(\zeta)T]. \tag{5}$$

For a given time instant t, the stress components, Σ, are Gaussian random variables and V and $\dot{V}$, are a pair of mutually dependent random variables. The calculation of $\nu^+(\zeta)$ requires the knowledge of $p_{V\dot{V}}(v,\dot{v})$ which, in general, is hard to obtain. On the other hand, the determination of the expectations, $\langle V^r \dot{V}^s \rangle$ is reasonably straightforward for $r=1,2,\ldots,N_1$ and $s=1,2,\ldots,N_2$. To determine $p_{V\dot{V}}(v,\dot{v})$, we have explored three possible options: (a) Model based on MEM, (b) model based on series representation for $p_{V\dot{V}}(v,\dot{v})$, and (c) translation process model for $V(t)$. The options (a) and (b) are capable of taking into account $\langle V(t)^r \dot{V}(t)^s \rangle$ for any desired N_1 and N_2. The third option is based on the work of Grigoriu. In this model, the process $V(t)$ and its time derivative $\dot{V}(t)$, turn out to be independent, at the same time instant. In the following sections, we discuss the procedure of developing the three models.

Maximum Entropy PDF

The principle in maximum entropy method for determining PDF of a random variable Y, lies in finding a continuous PDF, $p(\mathbf{y})$, that maximizes entropy $\mathcal{H}$, given by

$$\mathcal{H} = -\int_{-\infty}^{\infty} p(\mathbf{y}) \ln p(\mathbf{y}) \mathrm{d}p(\mathbf{y}), \tag{6}$$

subject to the constraints

$$\int_{-\infty}^{\infty} y_1^r y_2^s \cdots y_n^t p(\mathbf{y}) \mathrm{d}\mathbf{y} = m_{rs\cdots t}, \quad (\mathbf{r}, s, \ldots t = 0,1,2\ldots). \tag{7}$$

Here, $m_{rs\cdots t} = \langle y_1^r y_2^s \cdots y_n^t \rangle$ denotes the $(r+s+\cdots+t)$-th joint moment. Using calculus of variations, it can be shown that the optimal PDF is of the form

$$p(\mathbf{y}) = \exp\left[-\left(\lambda_{00} + \sum_k \lambda_k y_k + \sum_{k,j} \lambda_{kj} y_k y_j + \cdots + \sum_{k,j,\ldots,l} \lambda_{kj\ldots l} y_k y_j \ldots y_l\right)\right], \quad -\infty \leq \mathbf{Y} \leq \infty. \tag{8}$$

Determining the vector of unknown parameters $\{\lambda\}$ involves the solution of a set of simultaneous nonlinear equations and the associated difficulties have been a major deterrent in the use of this method. The recently developed moment equations by Volpe and Baganoff [22], however, provides a powerful alternative technique for avoiding expensive numerical computations.

Moment Equations. Volpe and Baganoff [22], in their studies on problems of fluid dynamics, have recently established that a simple linear relationship exists between the parameters of a univariate maximum entropy PDF and its moments. These linear equations have been termed as moment equations. This method, to the best of our knowledge, appears not to have been used in the context of problems of structural reliability analyses. Using a procedure similar to the one proposed by Volpe and Baganoff, we derive the moment equations for a second-order maximum entropy PDF. This we use to construct the joint PDF for the Von Mises stress and its time derivative. The procedure for the development of the moment equations is discussed next. For the sake of completeness, we first review the procedure for establishing the moment equations for the univariate maximum entropy PDF. Subsequently, the method for developing the bivariate maximum entropy PDF is presented.

Univariate PDF. Assuming that the random variable Y has a continuous PDF, the maximum entropy PDF for Y is of the form

$$p(y) = \lambda_0 \exp\left[-\sum_{k=1}^{N} \lambda_k y^k\right]. \tag{9}$$

The derivative of $p(y)$, with respect to y, is given by

$$\frac{\partial p(y)}{\partial y} = -\sum_{k=1}^{N} k\lambda_k y^{k-1} p(y). \tag{10}$$

Multiplying $\partial p(y)/\partial y$ with y^j and integrating with respect to y, in the interval $[-\infty, \infty]$, we get

$$\int_{-\infty}^{\infty} y^j \frac{\partial p(y)}{\partial y} \mathrm{d}y = -\sum_{k=1}^{N} k\lambda_k m_{k+j-1}. \tag{11}$$

Here, $\int_{-\infty}^{\infty} y^j p(y) \mathrm{d}y = m_j = \langle Y^j \rangle$. Alternatively, if the left hand side of Eq. (11) is integrated by parts, it can be shown that

$$\int_{-\infty}^{\infty} y^j \frac{\partial p(y)}{\partial y} \mathrm{d}y = -j m_{j-1}, \tag{12}$$

when N is even. From Eqs. (11) and (12), we get

$$\sum_{k=1}^{N} k\lambda_k m_{k+j-1} = j m_{j-1}. \tag{13}$$

For $j=0,1,2,\ldots$, the above formulation leads to a system of moment equations, of the form

$$\begin{bmatrix} m_0 & 2m_1 & 3m_2 & \ldots & Nm_{N-1} \\ m_1 & 2m_2 & 3m_3 & \ldots & Nm_N \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ m_{N-1} & 2m_N & 3m_{N+1} & \ldots & Nm_{2N-2} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \cdot \\ \cdot \\ \lambda_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ m_0 \\ \cdot \\ \cdot \\ (N-1)m_{N-2} \end{Bmatrix} \tag{14}$$

with $m_0 = 1$. These equations are exact and provide a simple linear relationship between the moments and the vector of unknown parameters $\{\lambda\}$. Inspection of Eq. (14) reveals that an infinite set of equations can be generated for a given set of parameters $\{\lambda\}$. This implies that $\{\lambda\}$ does not have a unique solution. However, it has been observed that if the vector of unknown parameters $\{\lambda\}$ is of dimension N, then the solution of the simultaneous equations generated by $j=0,\ldots,N-1$ lead to parameters which yield an acceptable model for the PDF. This would require information on the first $2N-2$ moments of Y. This is in contrast to other numerical techniques discussed in the literature, where the knowledge of the first N moments is sufficient to determine a N-dimensional parameter vector $\{\lambda\}$. However, during the course of this study, we have observed that for strictly positive random variables, acceptable maximum entropy PDF is obtained by solving for $\{\lambda\}$, the equations generated by $j=1,\ldots,N$. This, of course, requires knowledge of the first $2N-1$ moments of Y. The parameter λ_0 is the normalization parameter and is obtained as the inverse of $\int_{-\infty}^{\infty}\exp[-\Sigma_{k=1}^{N}\lambda_k y^k]\mathrm{d}y$.

Bivariate PDF. For a pair of correlated random variables X and Y, the maximum entropy joint PDF is of the form

$$p(x,y)=\lambda_0\exp\left[-\sum_{k+j=1}^{2N}\lambda_{kj}x^k y^j\right]. \tag{15}$$

Thus, the expression for $p(x,y)$, obtained from the above equation when $N=1$, is given by

$$p(x,y)=\lambda_0\exp[-\lambda_{10}x-\lambda_{20}x^2-\lambda_{01}y-\lambda_{02}y^2-\lambda_{11}xy]. \tag{16}$$

Similarly, the corresponding for $p(x,y)$ when $N=2$ is

$$\begin{aligned}p(x,y)=\lambda_0\exp[&-\lambda_{10}x-\lambda_{20}x^2-\lambda_{30}x^3-\lambda_{40}x^4-\lambda_{01}y-\lambda_{02}y^2\\&-\lambda_{03}y^3-\lambda_{04}y^4-\lambda_{11}xy-\lambda_{21}x^2y-\lambda_{12}xy^2-\lambda_{31}x^3y\\&-\lambda_{13}xy^3-\lambda_{22}x^2y^2].\end{aligned} \tag{17}$$

Differentiating $p(x,y)$ in Eq. (15) with respect to x and y, we get

$$\frac{\partial p(x,y)}{\partial x}=-\sum_{k+j=1}^{2N}k\lambda_{kj}x^{k-1}y^j p(x,y), \tag{18}$$

$$\frac{\partial p(x,y)}{\partial y}=-\sum_{k+j=1}^{2N}j\lambda_{kj}x^k y^{j-1}p(x,y). \tag{19}$$

We consider the integrals $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^r y^s\partial p(x,y)/\partial x\mathrm{d}x\mathrm{d}y$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^r y^s\partial p(x,y)/\partial y\mathrm{d}x\mathrm{d}y$. As for univariate case, we obtain expressions for the integrals in two different ways: (a) By substituting Eqs. (18) and (19) in the integrals and (b) by carrying out integration by parts. Equating the expressions obtained from (a) and (b), we get the following system of equations:

$$\sum_{k+j=1}^{2N}k\lambda_{kj}m_{k+r-1,j+s}=rm_{r-1,s}, \tag{20}$$

$$\sum_{k+j=1}^{2N}j\lambda_{kj}m_{k+r,j+s-1}=sm_{r,s-1}, \tag{21}$$

where, $m_{r,s}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^r y^s p(x,y)\mathrm{d}x\mathrm{d}y$, and $(r,s=0,1,2\ldots)$. We call Eqs. (20) and (21) as the *generating equations* and must be simultaneously used to construct the moment equations. As in the univariate case, here also we can construct an infinite set of equations to solve for the unknown parameters $\{\lambda\}$. However, acceptable models for the PDF are obtained only when the moment equations are generated by varying r and s from 0 in an ascending order. However, if X and/or Y is a strictly positive random variable, r and/or s is to be varied from 1 in an ascending order.

Maximum Entropy Bivariate PDF for V and $\dot{V}$. In this study, we employ the above principle for constructing the maximum entropy second-order PDF $p_{V\dot{V}}(v,\dot{v})$. Equations (20) and (21) are used for constructing the moment equations. We consider $N=2$ and consequently, the dimension of the vector of unknown parameters, M, is equal to 14. It must be noted that $V\geq 0$ and $-\infty\leq\dot{V}\leq\infty$. Thus, in constructing the moment equations r and s are taken to be integers, such that, $r\geq 1$ and $s\geq 0$. If V and $\dot{V}$ are considered to be independent, then $p_{V\dot{V}}(v,\dot{v})=p_V(v)p_{\dot{V}}(\dot{V})$ and the unknown parameters constitute a eight dimensional vector, whose elements are $[\lambda_{10},\lambda_{20},\lambda_{30},\lambda_{40},\lambda_{01},\lambda_{02},\lambda_{03},\lambda_{04}]$. Solution to these parameters are obtained from the system of equations generated for the (r,s) combinations (1, 0), (2, 0), (3, 0), (4, 0) for Eq. (20) and (0, 0), (0, 1), (0, 2), and (0, 3) for Eq. (21). It is observed that for Eq. (20), $r=j$ and $s=0$, where $\{\lambda_{j0}\}$ is the vector of unknown parameters for $p_V(v)$. However, for Eq. (21), $r=0$ and $s=j-1$, where $\{\lambda_{0j}\}$ are the vector of unknown parameters for $p_{\dot{V}}(\dot{v})$. This is because as $V\geq 0$, $r\geq 1$ and as $-\infty\leq\dot{V}\leq\infty$, $s\geq 0$. This essentially implies that two sets of four simultaneous equations, of the form in Eq. (14), with each set corresponding to V and $\dot{V}$, respectively, need to be solved separately. However, if the dependence characteristics of V and $\dot{V}$ are to be incorporated in the model for $p_{V\dot{V}}(v,\dot{v})$, one needs to consider, for $N=2$, six additional parameters $[\lambda_{11},\lambda_{21},\lambda_{12},\lambda_{31},\lambda_{13},\lambda_{22}]^T$. The presence of these terms ensure that all 14 equations are coupled. Of the six additional equations that need to be constructed, three each are obtained from Eqs. (20) and (21). This ensures that the moment equations constructed from the generating equations are in sequence. The additional (r,s) combinations are taken to be (1, 1), (2, 1), and (3, 1) for Eq. (20) and (1, 1), (1, 2), and (2, 2) for Eq. (21). If $\{\lambda_{kj}\}$ are the vector of the six additional parameters, the (r,s) combinations considered are $r=k$ and $s=j$ when Eq. (20) is used and $r=k$ and $s=j-1$ when Eq. (21) is used to construct the moment equations. It must be noted that Eqs. (20) and (21) are used in constructing the moment equations when $k>j$ and $k<j$, respectively. If $k=j$, either of the generating equations may be used, but care must be taken such that $M/2$ equations each are obtained from each of the generating equations, if M is even and for odd M, $(M-1)/2$ and $(M+1)/2$ equations are constructed from the generating equations.

Series Expansion Method

Alternatively, the joint PDF $p_{V\dot{V}}(v,\dot{v})$ may be constructed using a series expansion method. It is well known that many probability density functions can be approximated by a series of polynomials whose coefficients are determined from the moments of the process, for a given time. In particular, it has been shown that the PDF can be readily approximated by a few terms if a fortuitous choice is made for the expansion polynomials [14]. Here, we provide details for constructing a model for $p_{V\dot{V}}(v,\dot{v})$ using series expansions.

Model for $p_V(v)$. Since the probability distribution vanishes for negative values of $V(t)$, $p_V(v)$ can be approximated as

$$p_V(v)=\sum_{j=0}^{\infty}a_j\exp[-v]v^{\alpha}\mathcal{L}_j^{\alpha}(v). \tag{22}$$

where, $\mathcal{L}_j^{\alpha}(v)$ is the generalized Laguerre polynomial, defined by the Rodrigues's formula

$$\mathcal{L}_j^{\alpha}(v)=\frac{\exp[v]v^{-\alpha}}{j!}\frac{\partial^j}{\partial v^j}(\exp[-v]v^{j+\alpha}). \tag{23}$$

The orthogonality relation for the polynomial is given by

$$\int_0^\infty \exp[-v]v^\alpha \mathcal{L}_j^\alpha(v)\mathcal{L}_i^\alpha(v)\mathrm{d}v = \frac{\Gamma(\alpha+j+1)}{j!}\delta_{ij}. \qquad (24)$$

Here, δ_{ij} is the dirac-delta function. From Eqs. (22) and (24), the unknown coefficients a_j are obtained as

$$a_j = \frac{j!}{\Gamma(\alpha+j+1)}\int_0^\infty \mathcal{L}_j^\alpha(v)p_V(v)\mathrm{d}v. \qquad (25)$$

Applying the transformation $v=u/\beta$, it can be shown that $a_0 = 1/(\beta\Gamma(\alpha+1))$, $a_1=a_2=0$ and $a_3=(v_2/\beta^2(\alpha+3)-v_3/\beta^3)/(\beta\Gamma(\alpha+4))$. Here, α and β are arbitrary constants. It is found advantageous to take

$$\alpha = v_1^2/v_2^2 - 1, \qquad \beta = (v_2 - v_1^2)/v_1, \qquad (26)$$

where, v_n denotes the nth central moment of V. The coefficients other than a_0 and a_3 are too complicated. Retaining only the first term a_0 in the series expansion given by Eq. (22), yields a PDF which is identical to the gamma probability density function.

Model for $p_{\dot{V}}(\dot{V})$. Since $-\infty \leqslant \dot{V}(t) \leqslant \infty$, $p_{\dot{V}}(\dot{V})$ can be approximated using a Gram-Charlier series expansion, given by

$$p_{\dot{V}}(u) = \sum_{j=0}^{\infty} c_j\phi^{(j)}(u), \qquad (27)$$

where, $\phi(u)$ is the standard normal probability density function,

$$\phi^{(j)}(u) = \frac{\partial^j}{\partial u^j}\left[\frac{1}{2\pi}\exp(-u^2/2)H_j(u)\right], \qquad (28)$$

and $H_j(u)$ are the Hermite polynomials. $\phi(u)$ and $H_j(u)$ are orthogonal and satisfy the relation

$$\int_{-\infty}^{\infty} H_j(u)\phi^{(j)}(u)\mathrm{d}u = \delta_{ij}. \qquad (29)$$

The coefficients c_j are expressed in terms of the moments of $\dot{V}$. Investigations on the convergence of the Gram-Charlier series show that best results are achieved by grouping the coefficients as per the Edgeworth series [14]. Making the substitution $u=v-\langle v\rangle/(\langle v^2\rangle-\langle v\rangle^2)^{0.5}$, the coefficients c_1 and c_2 vanish and $c_0 = 1/(\langle v^2\rangle-\langle v\rangle^2)^{0.5}$ and $c_3=\langle v\rangle/(3!(\langle v^2\rangle-\langle v\rangle^2)^{0.5})$. Considering only one term expansion leads to a model for $p_{\dot{V}}(\dot{v})$ which is identical to the Gaussian distribution.

Model for $p_{V\dot{V}}(v,\dot{v})$. The joint probability density function, $p_{V\dot{V}}(v,\dot{v})$, can be expressed in terms of the first-order PDFs $p_V(v)$ and $p_{\dot{V}}(\dot{v})$ through the series expansion, given by

$$p_{V\dot{V}}(v,\dot{v}) = p_V(v)p_{\dot{V}}(\dot{v})\sum_{m,n=0}^{\infty}\alpha_{mn}\psi_m(u)\chi_n(\dot{v}), \qquad (30)$$

where, $\{\psi_n(v)\}_{n=1}^\infty$ and $\{\chi_m(\dot{v})\}_{m=1}^\infty$ are two sets of orthogonal polynomials, such that

$$\int_{-\infty}^{\infty} p_V(v)\psi_m(v)\psi_n(v)\mathrm{d}v = \delta_{mn},$$
$$\int_{-\infty}^{\infty} p_{\dot{V}}(\dot{v})\chi_m(\dot{v})\chi_n(\dot{v})\mathrm{d}\dot{v} = \delta_{mn}, \qquad (31)$$

and α_{mn} are constants, given by

$$\alpha_{mn} = \int_{-\infty}^{\infty} p_V(v)p_{\dot{V}}(\dot{v})\psi_m(v)\chi_n(\dot{v})\mathrm{d}v\mathrm{d}\dot{v}. \qquad (32)$$

The orthogonal polynomials $\{\psi_n(v)\}_{n=1}^\infty$ and $\{\chi_m(\dot{v})\}_{m=1}^\infty$ are constructed using Gram-Schmidt orthogonalization procedure and are expressed in terms of the moments $\langle V^r\dot{V}^s\rangle$.

Translation Process Model

The translation process model has been proposed earlier by Grigoriu [7]. In this method, estimates of the mean outcrossing rate are obtained by studying the statistics of a Gaussian distributed random process $X(t)$, obtained from a Nataf's transformation of the non-Gaussian translation process $V(t)$. This method thus, requires modeling of $p_V(v)$ only. Since it is assumed that $V(t)$ is a translation process, the outcrossing rate of $V(t)$ is identical to that of $X(t)$. Thus, Eq. (1) is written as

$$\nu^+(\zeta) = \frac{\sigma_{\dot{X}}}{\sqrt{2\pi}}\phi[g^{-1}(\zeta)]. \qquad (33)$$

Here, $\phi(x)$ is the normal PDF of X, and it has been shown [7] that

$$\sigma_{\dot{X}}^2 = \frac{\langle V^2\rangle - \langle V\rangle^2}{\displaystyle\int_{-\infty}^{\infty}\frac{\phi^3(x)}{\{p_V[g(x)]\}^2}\mathrm{d}x}, \qquad (34)$$

where, $V(t)=P_V^{-1}[\Phi(X(t))]=g(X(t))$, $P_V(\cdot)$ and $\Phi[\cdot]$ are, respectively, the probability distribution functions of $V(t)$ and $X(t)$ and $g(\cdot)$ is a nonlinear function.

Moment Computations for Von Mises Stress

For constructing models for $p_V(v)$, $p_{\dot{V}}(\dot{v})$, and $p(v,\dot{v})$ by the methods proposed in this paper, the concerned moments $\langle V^m\dot{V}^n\rangle$ need to be computed. $V(t)$ and its time derivative,

$$\dot{V} = \sum_{i=1}^{6}\sum_{j=1}^{6} A_{ij}(\sigma_i\dot{\sigma}_j + \dot{\sigma}_i\sigma_j), \qquad (35)$$

are expressed in terms of the components σ_i and their time derivatives $\dot{\sigma}_i$, which are Gaussian random variables. Thus, the moments $\langle V^m\dot{V}^n\rangle$ can be expressed in terms of the moments of $\langle\sigma_i^p\sigma_j^q\dot{\sigma}_k^r\dot{\sigma}_l^s\rangle$, where $(i,j,k,l=1,\ldots,6)$ and $(p+q+r+s=m+n)$. A two step procedure is adopted in computing $\langle V^m\dot{V}^n\rangle$.

1. We first identify all possible combinations of $\langle\sigma_i^p\sigma_j^q\dot{\sigma}_k^r\dot{\sigma}_l^s\rangle$, $(i,j,k,l=1,\ldots,6)$ and $(p+q+r+s=m+n)$, that are required for computing the moments $\langle V^m\dot{V}^m\rangle$.
2. Compute the moments $\langle\sigma_i^p\sigma_j^q\dot{\sigma}_k^r\dot{\sigma}_l^s\rangle$, $(i,j,k,l=1,\ldots,6)$ and $(p+q+r+s=m+n)$.

In computing the moments $\langle\sigma_i^p\sigma_j^q\dot{\sigma}_k^r\dot{\sigma}_l^s\rangle$, we use the moment generating function,

$$M(\theta) = \exp\left[\tfrac{1}{2}\theta^T\mathbf{K}\theta\right], \qquad (36)$$

where, θ is the 12-dimensional vector of random variables $[\mathbf{\Sigma},\dot{\mathbf{\Sigma}}]^T$ and $\mathbf{K}$ is the corresponding covariance matrix. Since the excitations are taken to be Gaussian and linear structure behavior is assumed, σ_i and $\dot{\sigma}_i$, $(i=1,\ldots,6)$, for a given t, are zero mean Gaussian random variables. Performing a random vibration analysis, the elements of the covariance matrix, in the steady state, are obtained from the following relations:

$$\langle\sigma_i\sigma_j\rangle = \int_{\omega_l}^{\omega_u} S_{\sigma_i\sigma_j}(\omega)\mathrm{d}\omega \qquad (37)$$

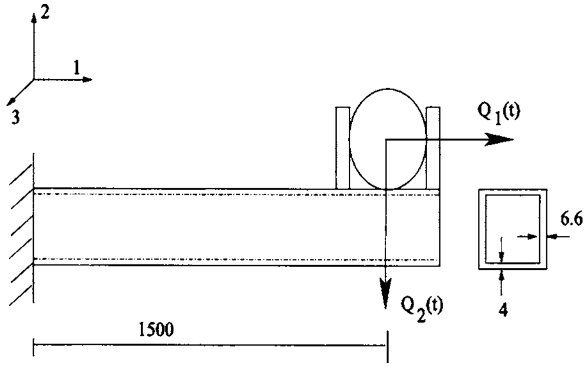

Fig. 1 Schematic diagram of the support for the fire-water system in a nuclear power plant; all dimensions are in mm

$$\langle \sigma_i \dot{\sigma}_j \rangle = \Re\left[\int_{\omega_l}^{\omega_u} - \mathrm{i}\omega S_{\sigma_i\sigma_j}(\omega)\mathrm{d}\omega \right] \tag{38}$$

$$\langle \dot{\sigma}_i \dot{\sigma}_j \rangle = \int_{\omega_l}^{\omega_u} \omega^2 S_{\sigma_i\sigma_j}(\omega)\mathrm{d}\omega. \tag{39}$$

The required moments can be expressed through the relation

$$\langle \sigma_i^p \sigma_j^q \dot{\sigma}_k^r \dot{\sigma}_l^s \rangle = \left. \frac{\partial^{(p+q+r+s)} M(\theta)}{\partial\theta_i^p \partial\theta_j^q \partial\theta_k^r \partial\theta_l^s} \right|_{\theta=0}, \tag{40}$$

for $(i,j,k,l=1,\ldots,6)$. The higher order derivatives of $M(\theta)$ with respect to θ may be computed using symbolic software, such as MAPLE. Alternatively, rewriting $M(\theta)=\exp[z]$, where $z=\frac{1}{2}\theta^T\mathbf{K}\theta$, one can generate the following recursive relations

$$M_{ij} = Mz_{ij},$$

$$M_{ijkl} = M_{jk}z_{il} + M_{jl}z_{ik} + M_{jk}z_{ij}, \tag{41}$$

$$\cdot\cdot$$

$$M_{ijkl\cdot\cdot mn} = M_{jkl\cdot\cdot m}z_{in} + M_{jkl\cdot\cdot n}z_{im} + M_{jk\cdot\cdot mn}z_{il} + M_{jl\cdot\cdot mn}z_{ik} + M_{jk\cdot\cdot mn}z_{ij}.$$

Here, $M_{ij\cdot\cdot kl}=\partial^r z/\partial\theta_i\partial\theta_j\cdot\cdot\partial\theta_k\partial\theta_l$ denotes the rth derivative of M with respect to the components of θ, at $\theta=0$, $z_{ij}=\partial^2 z/\partial\theta_i\partial\theta_j=K_{ij}$ and $M=1$ for $\theta=0$. It must be noted that z being a quadratic function in θ, $\partial^k z/\partial\theta_i^k=0$, for $k>2$. It has been found that using Eq. (41), rather than Eq. (40), proves to be computationally much more economical.

Numerical Example and Discussions

For illustrating the proposed methods, we focus on developing the extreme value distribution of the square of Von Mises stress developed at the base of a support for a fire-water pipeline, in a seismically excited nuclear power plant. The support, built up of two channel sections (see Fig. 1), is modeled as a cantilever beam. $Q_1(t)$ and $Q_2(t)$ denote the reaction forces transmitted from the piping structure to the pipe support structure. The power spectral density functions of $Q_1(t)$ and $Q_2(t)$ are obtained from a random vibration analysis on the finite element model of the piping structure. The details of these calculations are presented elsewhere [24]. The fire water pipeline is considered to be the primary structure under earthquake excitations and the support is assumed to be the secondary structure. Consequently, the pipe is assumed to impart random forces $Q_1(t)$ and $Q_2(t)$ at the tip of the beam. The geometric dimensions of the support are illustrated in Fig. 1. The mass density, modulus of elasticity and Poisson's ratio have been taken to be 7.8333 kg/m^3, 2.018×10^5 N/m^2 and 0.33, respectively. A commercial finite element software is used to model the support structure. The structure is discretized into a 744 noded structure using 360 solid elements, with each node having 3 degrees of freedom. The first five natural frequencies for the support are, respectively, found to be 9.2, 10.1, 55.3, 59.3, and

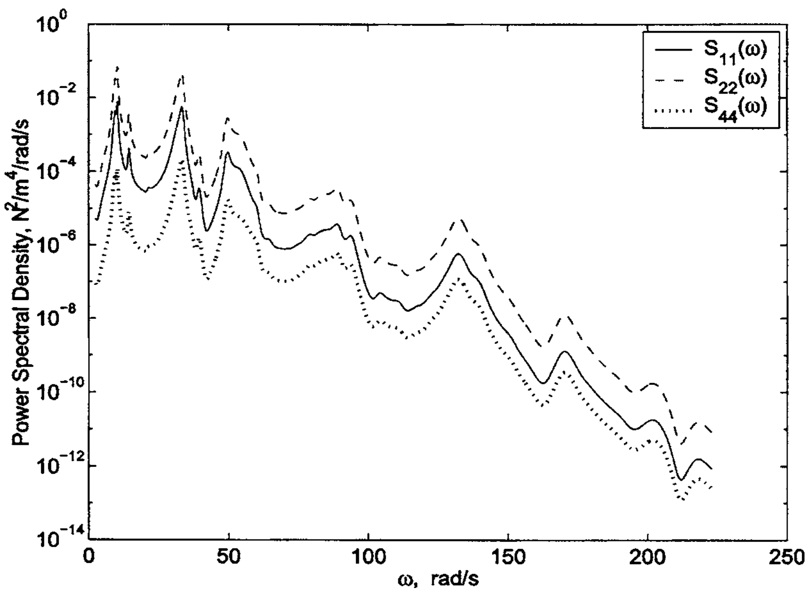

Fig. 2 Power spectral density functions for the stress components at the root of the cantilever; S_{ii} denotes the psd of the stress component σ_{ii}, ($i=1,2,4$)

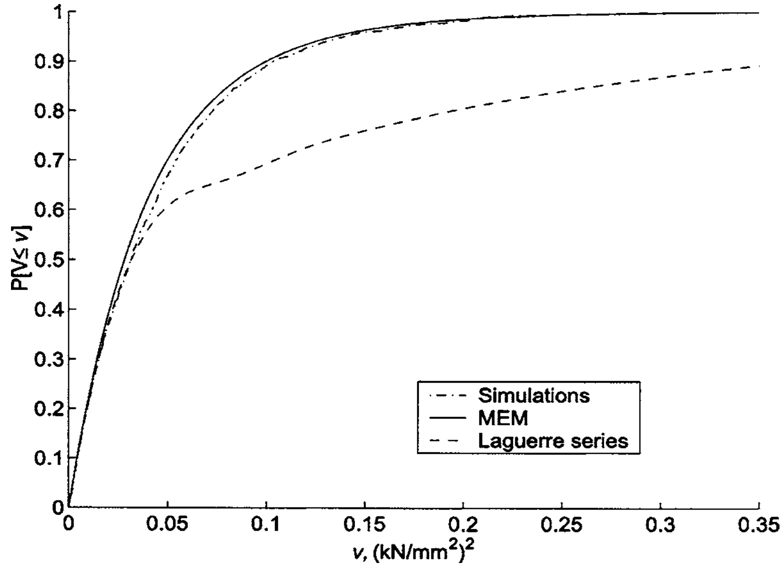

Fig. 3 Probability distribution function for $V(t)$

94.4 rad/s. Damping has been assumed to be viscous and proportional, with the mass and stiffness proportional constants taken to be 0.19 and 0.0021, respectively. This leads to damping in the first two modes to be equal to 2%. Since the pipes are not restrained axially, no forces act along the axis of pipe. Consequently, the stress components σ_3, σ_5, and σ_6 are small in comparison to the other three components and are taken to be zero. Thus, θ in Eq. (36) is taken to be a six-dimensional vector $[\sigma_1, \sigma_2, \sigma_4, \dot{\sigma}_1, \dot{\sigma}_2, \dot{\sigma}_4]^T$. The PSD functions of the stress components are obtained from a random vibration analysis. Figure 2 illustrates the PSD functions of the stress components $\sigma_1(t)$, $\sigma_2(t)$, and $\sigma_4(t)$. The shape factors for the PSD functions of $\sigma_1(t)$, $\sigma_2(t)$, and $\sigma_4(t)$ are found to be, respectively, 0.4796, 0.4762, and 0.4315.

The moments $\langle V^k \rangle$, $\langle \dot{V}^r \rangle$, and $\langle V^m \dot{V}^n \rangle$, $(k, r, m, n \in I)$ are computed from the moment generating function and the parameters for the maximum entropy PDFs $p_V(v)$, $p_{\dot{V}}(\dot{v})$ and $p_{V\dot{V}}(v, \dot{v})$ are computed using the procedure discussed in this paper. The models for $p_V(v)$ and $p_{\dot{V}}(\dot{v})$ are of the form given in Eq. (9), with $N=4$. The unknown parameters for the maximum entropy distributions are determined using the procedure discussed in the paper and the resulting probability distribution functions for V and $\dot{V}$ are, respectively, illustrated in Figs. 3 and 4. Additionally, we construct models for $p_V(v)$ and $p_{\dot{V}}(\dot{v})$ using four term Laguerre series expansion and four term Gram-Charlier series expansions, respectively. It must be noted that the second and third term coefficients are equal to zero in both the series expansions, for an appropriate choice of the arbitrary parameters. The models for the probability distribution functions for V and $\dot{V}$ are compared with the corresponding probability distribution functions obtained from Monte Carlo simulations on a sample size of 2000 time histories; see Figs. 3 and 4. The fairly good agreement of the four parameter maximum entropy distribution functions with the simulation results illustrate the applicability of the maximum entropy models for $p_V(v)$ and $p_{\dot{V}}(\dot{v})$. From Fig. 3, it is observed that the four term Laguerre model for $P_V(v)$ is not acceptable, especially in the tails

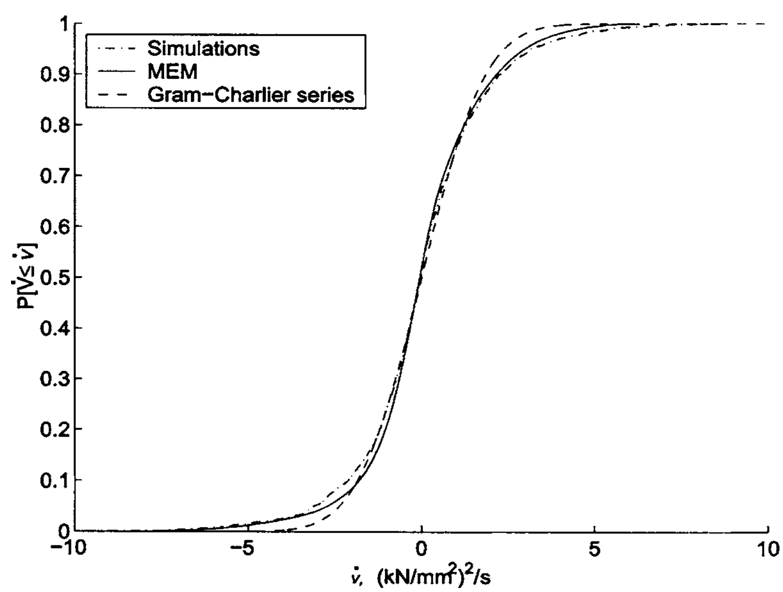

Fig. 4 Probability distribution function for $\dot{V}(t)$

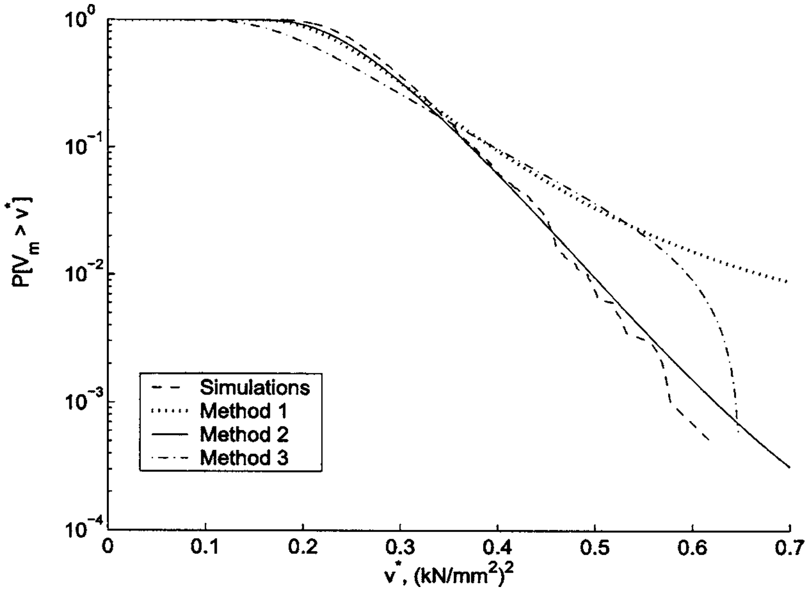

Fig. 5 Exceedance probability of Von Mises stress

of the distribution. On the other hand, including higher number of terms in the series expansion was found to lead to unacceptable behavior of probability density functions. It must be noted that the four term Laguerre series model for $p_V(v)$ require information only up to the third moment of V. In contrast, the four parameter maximum entropy model for $p_V(v)$ require information up to the seventh moment of V. This probably explains the acceptability of the maximum entropy model even at the tail regions. The construction of the four term Gram-Charlier series model for the probability distribution function of $\dot{V}$ requires information only up to the third moment of $\dot{V}$ in contrast to the corresponding four term maximum entropy model, which requires information up to the sixth moment. As expected, the maximum entropy model for the probability distribution function of $\dot{V}$ is observed to have a better resemblance with the distribution function obtained from simulations, especially at the tail regions, than the four term Gram-Charlier series model; see Fig. 4.

In constructing the probability distribution of extremes of $V(t)$, we consider the following three methods:

1. Method 1: Models for $p_V(v)$ and $p_{\dot{V}}(\dot{v})$ are constructed using Laguerre series and Gram-Charlier series expansions, respectively. Subsequently, we construct a model for $p_{V\dot{V}}(v,\dot{v})$ using Eq. (30), where $(m,n=1,2)$. Thus, the series expansion in Eq. (30) consists of four terms.
2. Method 2: The model for $p_{V\dot{V}}(v,\dot{v})$ is constructed using the maximum entropy procedure discussed in this paper. The number of parameters considered in the model is taken to be 14.
3. Method 3: We construct a model for $p_V(v)$ only, using the maximum entropy method. Subsequently, the mean outcrossing rate is estimated using the translation process model.

In methods 1 and 2, models for $p_{V\dot{V}}(v,\dot{v})$ are first constructed. Subsequently, estimates of $\nu^+(\zeta)$ are obtained from Rice's formula [Eq. (1)]. The integration in Eq. (1) is carried out numerically. On the other hand, in method 3, we need to construct a model only for $p_V(v)$ and estimates of $\nu^+(\zeta)$ are obtained by adopting the translation process theory. Once $\nu^+(\zeta)$ is determined, estimates of P_f are calculated from Eq. (5). In this example, we consider $T=10$ s. The failure probability curves obtained by the three methods are illustrated in Fig. 5. These are compared with the failure probability estimates from Monte Carlo simulations on a sample size of 2000 time histories. It is observed from Fig. 5 that the P_f estimates from method 2 are closest in agreement with those obtained from Monte Carlo simulations. On the other hand, the estimates obtained from method 3 have the largest deviations. This can be attributed to the fact that in method 3, the estimates are based only on the model for $p_V(v)$ and on the information of the variance of $\dot{V}$. This is in contrast to method 2 where the model for $p_{V\dot{V}}(v,\dot{v})$ is constructed based on the information of the first 7 moments of V, the first 6 moments of $\dot{V}$ and the cross moments $\langle V^r\dot{V}^s\rangle$ for all r and s, such that $2\leq(r+s)\leq 7$. Method 1 is observed to give better predictions than method 3 as separate models for $p_V(v)$, $p_{\dot{V}}(\dot{v})$, and $p_{V\dot{V}}(v,\dot{V})$ are constructed. Here, we consider only four term expansions for the PDF models. This requires information on the first three moments of V and $\dot{V}$ and the cross moments $\langle V^r\dot{V}^s\rangle$, for all r and s, such that $2\leq(r+s)\leq 4$. In order to exploit the available information on the higher moments, we need to consider higher number of terms in the series expansions for the models for the three PDFs. However, this introduced large numerical errors which lead to erroneous models for the probability density functions. It must be noted that since a sample size of 2000 time histories were considered in the Monte Carlo simulations, the failure probability estimates can be considered to be useful up to $P_f=10/N=5\times 10^{-3}$.

Conclusions

The present study considers the problem of approximating the probability density function of extreme values of a specific class of non-Gaussian random processes, namely, the Von Mises stress. The study is based on the assumption that the number of times the process crosses a specified threshold can be modeled as a Poisson random variable. This enables the solution to the problem of first passage times, which, in turn, leads to the approximation for the probability density function of extremes over a specified time duration. The determination of the average rate of crossing of a specified level by the random process is central to the implementation of this approach. This, in turn, requires the estimation of the joint probability density function of the process and its time derivative at the same time instant. This function itself is difficult to determine, especially for non-Gaussian processes, such as the Von

Mises stress. The focus of the present study has been in examining the issues related to the determination of this joint probability density function. The study has examined the relative merits of three alternative approaches for the determination of $p_{V\dot{V}}(v,\dot{v};t)$. These are based on a newly developed method that employs maximum entropy principle, series expansion based method on combined Laguerre-Hermite series expansion and results based on translation process models. The study has focused on stationary Von Mises stress process and, consequently, the process and its time derivative, are uncorrelated at the same time instant. While lack of correlation between two Gaussian random variables imply mutual independence, this, however, is not true for non-Gaussian quantities, such as the Von Mises stress and its time derivative at the same time instant. The study examines the relative merits of alternative methods for constructing $p_{V\dot{V}}(v,\dot{v};t)$ from the point of view of their ability to handle mutual dependencies between $V(t)$ and $\dot{V}(t)$, computational requirements and closeness of the resulting extreme value distribution to corresponding results from Monte Carlo simulations. Based on this study, we conclude that the method based on bivariate maximum entropy distribution for $p_{V\dot{V}}(v,\dot{v};t)$ provides relatively, the most robust and satisfactory means for determining the extreme value distributions. The models for extreme value distribution developed in this study are expected to be of value in the context of reliability analysis of vibrating structures in general, and in seismic fragility analysis of engineering structures, in particular.

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References

[1] Nigam, N. C., 1983, *Introduction to random vibrations*, MIT Press, Massachusetts.
[2] Lutes, L. D., and Sarkani, S., 1997, *Stochastic analysis of structural and mechanical vibrations*, Prentice Hall, New Jersey.
[3] Rice, S. O., 1944, “Mathematical Analysis of Random Noise,” Bell Syst. Tech. J., **23**, pp. 282–332. Reprinted in *Selected Papers in Noise and Stochastic Processes and Stochastic Processes*, edited by N. Wax, Dover, New York, 1954.
[4] Breitung, K., and Rackwitz, R., 1982, “Nonlinear Combination of Load Processes,” J. Struct. Mech., **10**, pp. 145–166.
[5] Pearce, H. T., and Wen, Y. K., 1985, “On Linearization Points for Nonlinear Combination of Stochastic Load Processes,” Struct. Safety, **2**, pp. 169–176.
[6] Wen, Y. K., 1990, *Structural load modeling and combination for performance and safety evaluation*, Elsevier, Amsterdam.
[7] Grigoriu, M., 1984, “Crossings of Non-Gaussian Translation Processes,” J. Eng. Mech., **110**(4), pp. 610–620.
[8] Madsen, H., 1985, “Extreme Value Statistics for Nonlinear Load Combination,” J. Eng. Mech., **111**, pp. 1121–1129.
[9] Naess, A., 2001, “Crossing Rate Statistics of Quadratic Transformations of Gaussian Processes,” Probab. Eng. Mech., **16**, pp. 209–217.
[10] Segalman, D., Fulcher, C., Reese, G., and Field, R., 2000, “An Efficient Method for Calculating r.m.s. Von Mises Stress in a Random Vibration Environment,” J. Sound Vib., **230**, pp. 393–410.
[11] Segalman, D., Reese, R., Field, C., and Fulcher, C., 2000, “Estimating the Probability Distribution of Von Mises Stress for Structures Undergoing Random Excitation,” J. Vibr. Acoust., **122**, pp. 42–48.
[12] Reese, G., Field, R., and Segalman, D., 2000, “A Tutorial on Design Analysis Using Von Mises Stress in Random Vibration Environments,” Shock Vib. Dig., **32**(6), pp. 466–474.
[13] Gupta, S., and Manohar, C. S., 2004, “Improved Response Surface Method For Time Variant Reliability Analysis of Nonlinear Random Structures Under Nonstationary Excitations,” Nonlinear Dyn., **36**, pp. 267–280.
[14] Deutsch, R., 1962, *Nonlinear transformations of random processes*, Engelwood-Cliffs: Prentice-Hall.
[15] Kapur, J. N., 1989, *Maximum entropy models in science and engineering*, John Wiley and Sons, New Delhi.
[16] Rosenblueth, E., Karmeshu, E., and Hong, H. P., 1986, “Maximum Entropy and Discretization of Probability Distributions,” Probab. Eng. Mech., **2**(2), pp. 58–63.
[17] Tagliani, A., 1989, “Principle of Maximum Entropy and Probability Distributions: Definition of Applicability Field,” Probab. Eng. Mech., **4**(2), pp. 99–104.
[18] Tagliani, A., 1990, “On the Existence of Maximum Entropy Distributions With Four and More Assigned Moments,” Probab. Eng. Mech., **5**(4), pp. 167–170.
[19] Sobczyk, K., and Trebicki, J., 1990, “Maximum Entropy Principle in Stochastic Dynamics,” Probab. Eng. Mech., **5**(3), pp. 102–110.
[20] Gupta, S., and Manohar, C. S., 2002, “Dynamic Stiffness Method for Circular Stochastic Timoshenko Beams: Response Variability and Reliability Analysis,” J. Sound Vib., **253**(5), pp. 1051–1085.
[21] Hurtado, J. E., and Barbat, A. H., 1998, “Fourier Based Maximum Entropy Method in Stochastic Dynamics,” Struct. Safety, **20**, pp. 221–235.
[22] Volpe, E. V., and Baganoff, D., 2003, “Maximum Entropy PDFs and the Moment Problem Under Near-Gaussian Conditions,” Probab. Eng. Mech., **18**, pp. 17–29.
[23] Chakrabarty, J., 1988, *Theory of plasticity*, McGraw-Hill, Singapore.
[24] Manohar, C. S., and Gupta, S., 2004, “Seismic Probabilistic Safety Assessment of Nuclear Power Plant Structures,” Project report, submitted to Board of Research in Nuclear Sciences, Government of India; Department of Civil Engineering, Indian Institute of Science, Bangalore.