Rao-Blackwellization with Substructuring for Identification of a Class of Noisy Nonlinear Dynamical Systems

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Abstract: *While particle filters are powerful tools for state or parameter estimations of highly nonlinear dynamical systems, they become quite inefficient for higher dimensional systems as simulations over large ensembles of samples are required to obtain a desirable accuracy in the estimations. Rao-Blackwellization is a technique that exploits the structure of the model to analytically marginalize a subset of the state vector so as to reduce the dimension of the state space over which particle filters need to be applied. In this study, a novel procedure for implementing Rao-Blackwellization for state and parameter estimations of uncertain dynamical systems of engineering interest is proposed. The strategy is based on decomposing the system to be estimated into mutually coupled linear and nonlinear substructures and then putting in place a framework to account for coupling between the substructures. While particle filters are applied to the nonlinear substructures, estimation of linear substructures proceeds using a bank of Kalman filters. Numerical illustrations are provided for state/ parameter estimations of a few linear and nonlinear oscillators with noise in both the process and measurements. The proposed procedure is notably efficient in state and parameter estimations of many engineering systems with localized nonlinearity.*

Keywords: *Particle filters, Rao-Blackwellization, substructuring, stochastic Taylor expansions, state and parameter estimations*

1. Introduction

Estimation of state and parameters of uncertain dynamical systems is of distinctive significance in many engineering applications. For instance, feedback control applications of dynamical systems require the states of the system to be estimated from the available noisy measurements. Similarly, performance evaluation and health monitoring of mechanical and structural systems require estimation of both states and parameters of the system. For systems described by linear-Gaussian state space models, the classical Kalman filter provides the optimal estimate. For nonlinear state estimation, there are two main classes of methods, viz., the suboptimal filter strategies, such as those based on the extended Kalman filter (EKF) or its variants [1,2] and those based on Monte Carlo simulations, popularly known as particle filters [3-7]. Unlike the EKF, which is limited to capturing a Gaussian approximation of the states using the first two moments, the particle filters obtain the conditional probability density function (pdf) through a set of random particles (simulated trajectories at discrete time instants) with associated weights. Hence, they are enabled to treat system nonlinearity and thus the non-Gaussian nature of the response and even non-Gaussian nature of noises. Different versions of particle filters, their development and potential applications can be found in [8,9]. However, despite the stated advantages, particle filters become increasingly inefficient with increasing system dimensionality as a large number of samples may have to be simulated in order to appropriately represent the conditional pdf. However, if the model has a tractable (linear) substructure with the associated variables amenable for treatment with the Kalman filter, the particle filter may only be applied to a reduced state space corresponding to the complimentary (possibly nonlinear) substructure. Other than an improvement in the computational efficiency, this has also an attendant benefit of a reduced variance of the estimator as a direct consequence of Rao-Blackwell theorem. The essence of this theorem is that, given a pair of random variables X and Y, we have the conditional inequality var $\{E[h(X) \mid Y]\} \leq$ var $[h(X)]$ (var{.} denotes variance and $E[.]$ the mathematical expectation) [10]. In the context of Monte Carlo simulation, this reflects a basic principle - one should carry out analytical computations as much as possible [11]. Combining particle filters with analytical computations (through Kalman filters) is generally referred to as Rao-Blackwellization [10, 11], resulting in Rao-Blackwellized Particle filters (RBPF) [8, 12]. It has been shown that Rao-Blackwellization, if possible, reduces the risk of divergence in state estimation problems, even when a small number of samples are employed [13].

Particle filters are widely used in navigation and target tracking applications. Their use in state/parameter estimation of mechanical systems has however not been attempted until recently. Manohar and Roy [14] have demonstrated the potential of particle filters for parameter identification of noisy nonlinear dynamical systems. Ching *et al*. [15, 16] have implemented particle filters to estimate the states and parameters of linear and nonlinear systems with time varying parameters. Li *et al*. [17] have used it for identifying nonlinear hysteretic systems. The RBPF has been used for different applications such as target tracking [18], navigation [13], fault diagnosis [19] and mobile robotics [20]. The existing forms of the RBPF crucially depend on a specific form of coupling between the two subsystems with information cascading from only one subsystem to the other, with no mutual interactions. Unfortunately, with this restriction in place, the potential of the RBPF in the context of state and parameter estimations of mechanical/structural systems cannot be entirely realized.

More often than not, engineering structures are designed to behave as linear systems and hence a substantially major part of the structure is likely to remain strictly linear even under extreme loading conditions. In other words, the nonlinearity in most nonlinear dynamical systems of engineering interest is spatially localized. To cite instances, mention may be made of nonlinear joints in an otherwise linear structure, base isolated structures or the localized plastic deformation of a mechanical/structural system. Thus, given that linear substructures (with the corresponding vector fields being linear in the state variables) may be readily traced out in mathematical models of most mechanical/structural systems, it is natural to ask if an adaptation of the RBPF is possible in the state and parameter estimations of these systems. Even if the system is decomposable into linear and nonlinear substructures, we again emphasize that a direct application of the RBPF is not possible owing to the two-way nature of the coupling existing between the substructures of engineering dynamical systems. In the present paper, we propose a novel variation of the RBPF that indeed enables state and parameter estimations of mechanical systems (e.g. oscillators). Following this, we also demonstrate the performance of the method using a few linear and nonlinear mechanical oscillators with uncertainty in both the process and measurements. The governing stochastic differential equations (SDE-s) of these oscillators are discretized using explicit forms of Ito-Taylor expansions. The numerical results adequately bring forth the superiority of the novel RBPF over standard particle filters in state and parameter estimations of such systems.

2. The State Estimation Using Particle Filters

Let $x_k \in \Re^{n_x}$ denote the state vector of a dynamical system and $y_k \in \Re^{n_y}$ the instantaneous measurement vector at time instant t_k. From the governing SDE-s for the dynamical system as well as the observation process, the discrete model of the system may be obtained as:

$$x_{k+1} = f_k(x_k, w_k) \tag{1a}$$

$$y_k = h_k(x_k, v_k) \quad k = 1, 2, ..., N_k \tag{1b}$$

Here $f_k : \Re^{n_x} \times \Re^{n_w} \to \Re^{n_x}$ and $h_k : \Re^{n_x} \times \Re^{n_v} \to \Re^{n_y}$ are linear/nonlinear functions of the state, $w_k \in \Re^{n_w}$ and $v_k \in \Re^{n_v}$ are sequence of zero-mean mutually independent random variables, independent of current and past states. It may be emphasized that x_k would possess strong Markov properties, since the state vector sequence is derived through a strong discretization of the governing SDE-s and that the conditional pdfs $p(x_k \mid x_{k-1})$ and $p(y_k \mid x_k)$ are deducible from Eqs. (1). For convenience, we introduce the following sequences of random variables: $x_{0:k} := \{x_i\}_{i=0}^{k}$ and $y_{1:k} := \{y_i\}_{i=1}^{k}$. The objective is to estimate the conditional pdf $p(x_{0:k} \mid y_{1:k})$ recursively in time or, more conveniently, only the marginal pdf $p(x_k \mid y_{1:k})$ and thus find the expectation of a function $\psi(x_k)$ as:

$$I(\psi) = E[\psi(x_k)] = \int \psi(x_k)\, p(x_k \mid y_{1:k})\, dx_k \tag{2}$$

The integral in Eq. (2) is multi-dimensional and its analytical evaluation is generally not possible. However, when process and measurement equations are linear and noises are Gaussian and additive, Kalman filter provides the exact (optimal) solution. However, for a more general case, a formal solution is derivable as follows (Gordon *et al*., [3]). First obtain:

$$p(x_k \mid y_{1:k-1}) = \int p(x_k, x_{k-1} \mid \; y_{1:k-1})\, dx_{k-1} = \int p(x_k \mid x_{k-1})\, p(x_{k-1} \mid \; y_{1:k-1})\, dx_{k-1} \tag{3}$$

This equation represents the prediction equation. When the measurement vector y_k becomes available, one may derive the updation equation based on Bayes' theorem as follows:

$$p(x_k \mid y_{1:k}) = \frac{p(x_k, y_{1:k})}{p(y_{1:k})} = \frac{p(y_k \mid x_k)\, p(x_k \mid y_{1:k-1})}{p(y_k \mid y_{1:k-1})} = \frac{p(y_k \mid x_k)\, p(x_k \mid y_{1:k-1})}{\int p(y_k \mid x_k)\, p(x_k \mid y_{1:k-1}) dx_k} \tag{4}$$

Eqs. (3) and (4) constitute a formal solution to the estimation problem. The general principle of the particle filter is to use Monte Carlo simulation strategies to approximately obtain the above integrals and hence the associated conditional pdf-s.

3. Rao-Blackwellized Particle Filters

The standard particle filter becomes computationally inefficient for higher dimensional systems as one has to choose very large number of particles to approximate the conditional pdf-s with acceptable accuracy and thereby to reduce the variance (that occurs due to the finiteness of the ensemble size) associated with the estimated quantities. However, when the model has a tractable substructure such that a significantly large subset of the state variables can be marginalized out analytically (using the Kalman filter, say), then we only need to sample from a reduced state space. This improves the efficiency of the sampling technique and reduces the variance of the estimator. This technique is called Rao-Blackellization. The principle may be further explained as follows.

Let the states x_k be partitioned into two groups as x_k^I and x_k^{II} such that, conditioned on x_k^I, the conditional posterior distribution $p(x_k^{II} \mid x_k^I, y_{1:k})$ is analytically tractable. Then making use of the decomposition of the posterior as

$$p(x_k^I, x_k^{II} \mid y_{1:k}) = p(x_k^{II} \mid x_{k,}^I y_{1:k})\, p(x_k^I \mid y_{1:k}), \tag{5}$$

x_k^{II} can be marginalized out and we need to focus on estimating $p(x_k^I \mid y_{1:k})$ which corresponds to a state space of reduced dimension. In particular, if $p(x_k^{II} \mid x_k^I, y_{1:k})$ is a linear-Gaussian state space model, the states x_k^{II} conditioned on x_k^I and $y_{1:k}$ can be estimated exactly using the Kalman filter. The expectation $I(\psi)$ in Eq. (2) may be re-written as

$$I(\psi) = \int \psi(x_k^I, x_k^{II})\, p(x_k^I, x_k^{II} \mid y_{1:k}) dx_k^I dx_k^{II} \tag{6}$$

Since the following identities hold

$$\begin{aligned} p(x_k^I, x_k^{II} \mid y_{1:k}) &= \frac{p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^I, x_k^{II})}{\int p(x_k^I, x_k^{II}, y_{1:k}) dx_k^I dx_k^{II}} \\ &= \frac{p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I)\, p(x_k^I)}{\int p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I)\, p(x_k^I) dx_k^I dx_k^{II}}, \end{aligned} \tag{7}$$

it follows that:

$$\begin{aligned} I(\psi) &= \frac{\int \psi(x_k^I, x_k^{II})\, p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I)\, p(x_k^I) dx_k^I dx_k^{II}}{\int p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I)\, p(x_k^I) dx_k^I dx_k^{II}} \\ &= \frac{\int g(x_k^I)\, p(x_k^I) dx_k^I}{\int \left[p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I) dx_k^{II} \right] p(x_k^I) dx_k^I} \end{aligned} \tag{8}$$

where $g(x_k^I) = \int \psi(x_k^I, x_k^{II})\, p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I)\, p(x_k^I) dx_k^{II}$. Noting that

$$\int p(y_{1:k} \mid x_k^I, x_k^{II})\, p(x_k^{II} \mid x_k^I) dx_k^{II} = \int \frac{p(x_k^I, x_k^{II}, y_k)}{p(x_k^I, x_k^{II})} \frac{p(x_k^I, x_k^{II})}{p(x_k^I)} dx_k^{II} = p(y_{1:k} \mid x_k^I),$$

$I(\psi)$ may be expressed as

$$I(\psi) = \frac{\int g(x_k^I)\, p(x_k^I) dx_k^I}{\int p(y_{1:k} \mid x_k^I)\, p(x_k^I) dx_k^I} \tag{9}$$

Assuming that $g(x_k^I)$ and $p(y_{1:k} \mid x_k^I)$ can be evaluated analytically, $I(\psi)$ may also be evaluated via the above expression.

The estimate of $I[\psi(x_k)]$ using a standard particle filter (bootstrap filter) involves the following steps.

(1) *Prediction:* Each sample from the pdf $p(x_{k-1} \mid y_{1:k-1})$ is propagated through the process equation to obtain sample from the prior at t_k: $\left\{x_k^*(i)\right\}_{i=1}^N$. Here N denotes the ensemble size.

(2) *Update:* When the measurement y_k arrives, the likelihood of each prior sample is evaluated and a normalized weight for each particle is obtained as

$$q_i = \frac{p(y_k \mid x_k^*(i))}{\sum_{j=1}^{N} p(y_k \mid x_k^*(j))} \tag{10}$$

Define a probability mass function $P\left[x_k(j) = x_k^*(i)\right] = q_i$ and generate N samples $\{x_k(i)\}_{i=1}^{N}$ from this discrete distribution. These samples are approximately distributed as the required pdf $p(x_k \mid y_{1:k})$ [3]. This procedure is repeated for all the time steps. Now the estimate of $I[\psi(x_k)]$ may be obtained as

$$\hat{I}(\psi) = \frac{1}{N}\sum_{i=1}^{N} \psi(x_k(i)) \tag{11}$$

The Rao-Blackwellized estimate of $I(\psi)$ may be obtained as

$$\hat{I}^R(\psi) = \frac{1}{N}\sum_{i=1}^{N} E\left[\psi(x_k^I(i), x_k^{II})\right] \tag{12}$$

Here $E[.]$ represents the expectation with respect to the pdf $p\left(x_k^{II} \mid x_k^I(i), y_{1:k}\right)$. Note that efficient particle filters using sequential importance sampling are available and they perform better than the bootstrap filter provided the importance sampling density function is appropriately chosen. Since the present focus is on evolving a novel Rao-Blackwellization in the context of substructured mechanical oscillators, we continue to use the bootstrap filter.

4. Implementation of Rao-Blackwellized Particle Filters for Mechanical Oscillators

Before discussing the implementation of Rao-Blackwellization for mechanical oscillators, it would be useful to examine the type of systems that are amenable for the available form of Rao-Blackwellization as outlined in the last section. When a system is conceived as an assemblage of two or more substructures, the arrangement may be either cascaded or coupled as schematically shown in Fig. 1. Referring to Eq. (5), we readily comprehend that, for implementing Rao-Blackwellization, the vector field corresponding to the nonlinear state vector x_k^I must be functionally independent of the state vector x_k^{II} of the conditionally linear subsystem. Thus we conclude that Rao-Blackwellization is directly applicable to systems with cascaded substructures only. However, the present focus is on estimations of mechanical oscillators that generally represent spatially discretized forms of partial differential equations (PDE-s) governing the motion of a system of relevance in solid mechanics. Substructures of such oscillators are almost always coupled (Fig. 1(b)), due to the action/reaction between any pair of them, and hence they do not directly admit Rao-Blackwellization even if some of the substructures are linear. In the present work, the issue of coupling is tackled by expressing x_k^{II} in the model of the nonlinear part ($x_k^I \sim p(x_k^I \mid x_k^{II}, y_{1:k})$) in terms of the known values of x_{k-1}^I and x_{k-1}^{II} (corresponding to a specific simulation or particle) using explicit forms of Ito-Taylor expansions. In the procedure developed here for implementing Rao-Blackwellization for mechanical systems, a standard particle filter (bootstrap filter) is used for estimating the states of the nonlinear part and the discrete Kalman filter is employed for the linear part. Further details of the procedure are provided in the following.

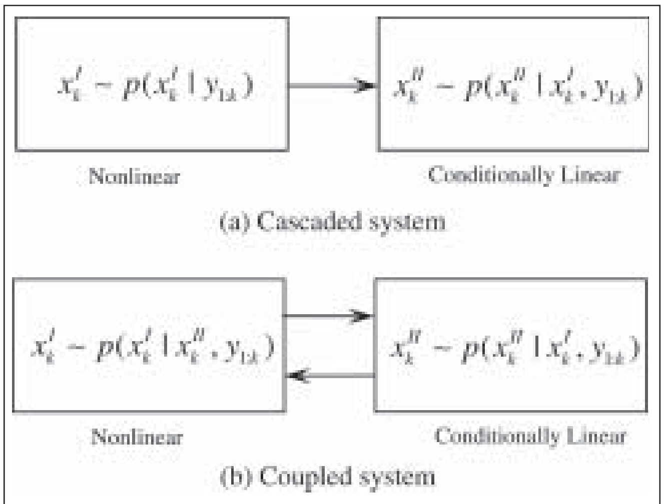

Figure 1: A Schematic Representation of Cascaded and Coupled Substructures

The process and measurement equations of the nonlinear part may be expressed as:

$$x_{k+1}^I = f_k(x_k^I, x_{c,k}, w_k^I) \tag{13a}$$

$$y_k^I = h_k^I(x_k^I, v_k^I) \tag{13b}$$

where $x_{c,k}$ is the displacement at the interface degrees of freedom common to both substructures and w_k^I is the random variable representing the process noise. y_k^I is the measurement and v_k^I is the measurement noise. The equation of motion of the linear part may be expressed in state space form as

$$\dot{x}^{II} = Ax^{II} + BF + \sigma\xi \tag{14}$$

where A and B are system matrices and the vector F contains the forces acting in the system including the interaction forces between the substructures. σ is a matrix whose elements determine the strength of the process noise and the elements of ξ are white noise (to be interpreted formally as the derivative of a standard Brownian motion). The equation may be discretized as

$$x_{k+1}^{II} = \phi_k x_k^{II} + \Gamma_k + w_k^{II} \tag{15}$$

where $\phi_k = e^{A(t_{k+1}-t_k)}$ is the fundamental solution matrix, $\Gamma_k = \int_{t_k}^{t_{k+1}} e^{A(t_{k+1}-\tau)} BF(\tau)d\tau$ denotes the particular solution corresponding to the forcing vector BF and w_k^{II} is a zero mean Gaussian random variable representing the process noise. The measurement equation of the linear part may be expressed as

$$y_k^{II} = h_k^{II}(x_k^{II}, v_k^{II}) \tag{16}$$

where v_k^{II} represents the measurement noise.

In order to implement Rao-Blackwellization, the following steps are used.

1. Set k = 0. Generate $\{x_{c,0}(i)\}_{i=1}^N$ and $\{x_0^I(i)\}_{i=1}^N$ from the initial pdf-s $p(x_{c,0})$ and $p(x_0^I)$ respectively. Also generate the noise vector $\{w_0^I(i)\}_{i=1}^N$ from the pdf $p(w^I)$.
2. Set $k = k + 1$. Obtain $\{x_k^I(i)\}_{i=1}^N$ using the process equation. When y_k^I arrives, estimate $\hat{x}_k^I$ using the standard particle filter algorithm. For all particles, obtain $x_c(t)$, $t \in (t_k, t_{k+1}]$, using Ito-Taylor expansions of the components of x_c of the linear substructure. Hence find the interaction force between the substructures in the time step considered.
3. Based on the approximation to the interaction effects computed in step (2), find Γ_k. Then, making use of the measurement y_k^{II}, estimate $\{\hat{x}_k^{II}(i)\}_{i=1}^N$ using N Kalman filters. Now find the displacements at the interface, $\{x_{c,k}(i)\}_{i=1}^N$.
4. Repeat steps 2 and 3 till the terminal time is reached.

The central idea of the proposed procedure may be observed to be the use of stochastic Taylor expansions (Ito-Taylor or even Stratonovich-Taylor) to extrapolate the displacements at the interface degrees of freedom. Thus the dependence of the state vector of the conditionally linear part on the process equation of the nonlinear part becomes computationally tractable within a Rao-Blackwellization framework. Any formal order of accuracy can be achieved, in principle, by considering an adequate number of terms in the Ito-Taylor expansions.

5. The Discretization of the System/Observation Equations

The governing SDE-s for the system/observations corresponding to the nonlinear substructure need to be discretized and brought to a form consistent with Eqs. 1(a) and 1(b) so that they may be processed further with the particle filter algorithm. Just as a time-marching algorithm for a deterministic ODE is often derived using variations of a Taylor expansion, the SDE-s may similarly be discretized using the stochastic Taylor (Ito-Taylor) expansion [21,22]. Details of Ito-Taylor expansions and related concepts in stochastic calculus may be found in [14, 21-23].

Towards numerical implementation, we presently consider a five degrees of freedom (5-DOF) linear oscillator, a 3-DOF linear oscillator and a 3-DOF nonlinear oscillator, subjected to support motion as shown in Figs. 2, 3 and 4 respectively. The decompositions of the oscillators into substructures are also indicated. Both the substructures of the 5-DOF linear system are linear. While the states of substructure 2 will be estimated by particle filter, substructure 1 will be handled by Kalman filter using the proposed methodology. In the 3-DOF nonlinear oscillator, only the 2nd substructure is nonlinear with hardening springs. Substructure 1 of the 3-DOF linear oscillator remains linear while the governing equation of substructure 2 becomes nonlinear when its parameters are declared as additional states.

The governing SDE-s of the 5-DOF oscillator may be expressed in the following incremental form.

$$\left.\begin{aligned} dx_1 &= x_6 dt \\ dx_2 &= x_7 dt \\ dx_3 &= x_8 dt \\ dx_4 &= x_9 dt \\ dx_5 &= x_{10} dt \\ dx_6 &= a_6 dt + \sigma_1 dB_1 \\ dx_7 &= a_7 dt + \sigma_2 dB_2 \\ dx_8 &= a_8 dt + \sigma_3 dB_3 \\ dx_9 &= a_9 dt + \sigma_4 dB_4 \\ dx_{10} &= a_{10} dt + \sigma_5 dB_5 \end{aligned}\right\} \tag{17}$$

with initial conditions $x_i(0) = x_{i0}, i \in [1,10]$. The diffusion coefficients σ_i, $i \in [1,5]$ represent the intensities of the associated additive noise processes and dB_i, $i \in [1,5]$ represent increments of Brownian motion process. The drift coefficient functions are given by

$$\left.\begin{aligned} a_6 &= -\frac{1}{m_1}\left[(k_1+k_2)x_1 - k_2x_2 + (c_1+c_2)x_6 - c_2x_7\right] - \ddot{x}_g \\ a_7 &= -\frac{1}{m_2}\left[-k_2x_1 + (k_2+k_3)x_2 - k_3x_3 - c_2x_6 + (c_2+c_3)x_7 - c_3x_8\right] - \ddot{x}_g \\ a_8 &= -\frac{1}{m_3}\left[-k_3x_2 + (k_3+k_4)x_3 - k_4x_4 - c_3x_7 + (c_3+c_4)x_8 - c_4x_9\right] - \ddot{x}_g \\ a_9 &= -\frac{1}{m_4}\left[k_4(x_4-x_3) + c_4(x_9-x_8) + k_5(x_4-x_5) + c_5(x_9-x_{10})\right] - \ddot{x}_g \\ a_{10} &= -\frac{1}{m_5}\left[k_5(x_5-x_4) + c_5(x_{10}-x_9)\right] - \ddot{x}_g \end{aligned}\right\} \tag{18}$$

The SDE-s for the 3-DOF nonlinear oscillator with the parameters α, β and c declared as additional states may be expressed as

$$\left.\begin{aligned} dx_1 &= x_4 dt \\ dx_2 &= x_5 dt \\ dx_3 &= x_6 dt \\ dx_4 &= a_4 dt + \sigma_l dB_1 \\ dx_5 &= a_5 dt + \sigma dB_2 \\ dx_6 &= a_6 dt + \sigma_r dB_3 \\ dx_7 &= \sigma_\alpha dB_4 \\ dx_8 &= \sigma_\beta dB_5 \\ dx_9 &= \sigma_c dB_5 \end{aligned}\right\} \tag{19}$$

The initial conditions are given by $x_i(0) = x_{i0}, i \in [1,6]$ and the drift coefficient functions are:

$$\left.\begin{aligned} a_4 &= -\frac{1}{m_l}\left[k_l x_1 + c_l x_4 + x_7(x_1-x_2) + x_8(x_1-x_2)^3 + x_9(x_4-x_5)\right] - \ddot{x}_g \\ a_5 &= -\frac{1}{m}\left\{x_7\left[2x_2 - (x_1+x_3)\right] + x_8\left[(x_2-x_1)^3 + (x_2-x_3)^3\right] + x_9\left[2x_5 - (x_4+x_6)\right]\right\} - \ddot{x}_g \\ a_6 &= -\frac{1}{m_r}\left[k_r x_3 + c_r x_6 + x_7(x_3-x_2) + x_8(x_3-x_2)^3 + x_9(x_6-x_5)\right] - \ddot{x}_g \end{aligned}\right\} \tag{20}$$

σ_l, σ and σ_r are the intensities of the additive process noises. It may be noted here that extending the state vector with parameters, to transform the problem to an optimal filtering problem, is a commonly used technique in fixed parameter estimation. The parameters will be declared as stochastic processes evolving in time. In Eq. 19, σ_α, σ_β and σ_c represent the assumed diffusion coefficients associated with σ, β and c respectively. The incremental form of the SDE-s corresponding to the 3-DOF linear oscillator, after declaring k_3 and c_3 as additional state variables, is given by

$$\left.\begin{aligned} dx_1 &= x_4 dt \\ dx_2 &= x_5 dt \\ dx_3 &= x_6 dt \\ dx_4 &= a_4 dt + \sigma_1 dB_1 \\ dx_5 &= a_5 dt + \sigma_2 dB_2 \\ dx_6 &= a_6 dt + \sigma_3 dB_3 \\ dx_7 &= \sigma_{k_3} dB_4 \\ dx_8 &= \sigma_{c_3} dB_5 \end{aligned}\right\} \tag{21}$$

with initial conditions $x_i(0) = x_{i0}, i \in [1,6]$. The drift coefficient functions are given by

$$\left.\begin{aligned} a_4 &= -\frac{1}{m_1}\left[(k_1+k_2)x_1 - k_2x_2 + (c_1+c_2)x_4 - c_2x_5\right] - \ddot{x}_g \\ a_5 &= -\frac{1}{m_2}\left[-k_2x_1 + k_2x_2 - c_2x_4 + c_2x_5 + x_7(x_2-x_3) + x_8(x_5-x_6)\right] - \ddot{x}_g \\ a_6 &= -\frac{1}{m_3}\left[x_7(x_3-x_2) + x_8(x_6-x_5)\right] - \ddot{x}_g \end{aligned}\right\} \tag{22}$$

σ_i, $i \in [1, 3]$, represent the diffusion coefficients of the associated additive noise processes; and, σ_{k_3} and σ_{c_3} represent the assumed diffusion coefficients of the SDE-s governing the evolutions of the parameter states k_3 and c_3 respectively. Note that the displacement and velocity components in Eqs. (17-22) are relative with respect to the support.

Using explicit forms of truncated Ito-Taylor expansions, maps for the displacement and velocity components of the substructure 2 of the 5-DOF linear oscillator over the interval (t_k, t_{k+1}) (with a uniform step-size $h = t_{k+I} - t_k$) may be shown to be:

$$x_{5(k+1)} = x_{5k} + x_{10k}h + a_{10k}\frac{h^2}{2} - \frac{1}{m_5}\left[k_5(x_{10k} - x_{9k}) + c_5(a_{10k} - a_{9k})\right]\frac{h^3}{6} + \sigma_5 I_{50} - \frac{c_5}{m_5}\left[(\sigma_5 I_{500} - \sigma_4 I_{400})\right] \tag{23a}$$

$$x_{10(k+1)} = x_{10k} + a_{10k}h - \frac{1}{m_5}\left[k_5(x_{10k} - x_{9k}) + c_5(a_{10k} - a_{9k})\right]\frac{h^2}{2} + \sigma_5 I_5 - \frac{c_5}{m_5}\left[(\sigma_5 I_{50} - \sigma_4 I_{40})\right] \tag{23b}$$

Here I_5, I_{40}, I_{50}, I_{400} and I_{500} are multiple stochastic integrals (MSI-s) given by $I_r = \int_{t_k}^{t_k+h} dB_r(s)$, $I_{r0} = \int_{t_k}^{t_k+h}\int_{t_k}^{s} dB_r(s_1)ds$ and $I_{r00} = \int_{t_k}^{t_k+h}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_r(s_2)ds_1 ds$, $r =$ 4, 5. Following Ito's isometry, it may be shown that these MSI-s are normal random variables [14] of the form

$$\begin{Bmatrix} I_r \\ I_{r0} \\ I_{r00} \end{Bmatrix} \sim N\left[0_{3x1}\ \begin{bmatrix} h & \frac{h^2}{2} & \frac{h^3}{6} \\ \frac{h^2}{2} & \frac{h^3}{3} & \frac{h^4}{8} \\ \frac{h^3}{6} & \frac{h^4}{8} & \frac{h^5}{20} \end{bmatrix}\right] \tag{24}$$

In order to implement step (2) of the algorithm proposed in the previous section, (truncated) Ito-Taylor expansions for the displacement and velocity components of the substructure 1 at the interface degrees of freedom (dof-s) in the time interval $t \in (t_k, t_{k+1}]$ are required. These expansions presently take the following forms:

$$\begin{aligned} x_4(t_k + t) &= x_{4k} + x_{9k}t + a_{9k}\frac{t^2}{2} - \frac{1}{m_4}\left[-k_4 x_{8k} + (k_4 + k_5)x_{9k} - k_5 x_{10k} - c_4 a_{8k} + (c_4 + c_5)a_{9k} - c_5 a_{10k}\right]\frac{t^3}{6} \\ &+ \sigma_4\left(\int_{t_k}^{t_k+t}\int_{t_k}^{s} dB_4(s_1)ds - \frac{c_4 + c_5}{m_4}\int_{t_k}^{t_k+t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_4(s_2)ds_1 ds\right) + \sigma_3\frac{c_4}{m_4}\int_{t_k}^{t_k+t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_3(s_2)ds_1 ds \\ &+ \sigma_5\frac{c_5}{m_4}\int_{t_k}^{t_k+t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_5(s_2)ds_1 ds \end{aligned} \tag{25a}$$

$$\begin{aligned} x_9(t_k + t) &= x_{9k} + a_{9k}t - \frac{1}{m_4}\left[-k_4 x_{8k} + (k_4 + k_5)x_{9k} - k_5 x_{10k} - c_4 a_{8k} + (c_4 + c_5)a_{9k} - c_5 a_{10k}\right]\frac{t^2}{2} \\ &+ \sigma_4\left(\int_{t_k}^{t_k+t} dB_4(s_1) - \frac{c_4 + c_5}{m_4}\int_{t_k}^{t_k+t}\int_{t_k}^{s} dB_4(s_1)ds\right) + \sigma_3\frac{c_4}{m_4}\int_{t_k}^{t_k+t}\int_{t_k}^{s} dB_3(s_1)ds + \sigma_5\frac{c_5}{m_4}\int_{t_k}^{t_{k+t}}\int_{t_k}^{s} dB_5(s_1)ds \end{aligned} \tag{25b}$$

The MSI-s can be modeled, as before, as zero mean normal random variables with readily derivable variances.

For the 3-DOF nonlinear oscillator, the truncated forms of the stochastic Taylor expansion for the displacement and velocity components of the nonlinear substructure over the interval $\left(t_k, t_{k+1}\right]$ are given by:

$$\begin{aligned} x_{2(k+1)} &= x_{2k} + x_{5k}h + a_{5k}\frac{h^2}{2} - \frac{1}{m}\begin{Bmatrix} x_{7k}\left[2x_{5k} - (x_{4k} + x_{6k})\right] + x_{9k}\left[2a_{5k} - (a_{4k} + a_{6k})\right] + \\ 3x_{8k}\left[(x_{2k} - x_{1k})^2(x_{5k} - x_{4k}) + (x_{2k} - x_{3k})^2(x_{5k} - x_{6k})\right] \end{Bmatrix}\frac{h^3}{6} \\ &+ \sigma I_{20} - \frac{1}{m}\begin{Bmatrix} x_{9k}(2\sigma I_{200} + \sigma_l I_{100} + \sigma_r I_{300}) + \left[2x_{2k} - (x_{1k} + x_{3k})\right]\sigma_\alpha I_{400} + \\ \left[(x_{2k} - x_{1k})^3 + (x_{2k} - x_{3k})^3\right]\sigma_\beta I_{500} + \left[2x_{5k} - (x_{4k} + x_{6k})\right]\sigma_c I_{600} \end{Bmatrix} \end{aligned} \tag{26a}$$

$$
\begin{aligned}
x_{5(k+1)} &= x_{5k} + a_{5k}h - \frac{1}{m}\left\{\begin{matrix} x_{7k}\left[2x_{5k} - (x_{4k} + x_{6k})\right] + x_{9k}\left[2a_{5k} - (a_{4k} + a_{6k})\right] + \\ 3x_{8k}\left[(x_{2k} - x_{1k})^2 (x_{5k} - x_{4k}) + (x_{2k} - x_{3k})^2 (x_{5k} - x_{6k})\right] \end{matrix}\right\}\frac{h^2}{2} \\
&+ \sigma I_2 - \frac{1}{m}\left\{\begin{matrix} x_{9k}(2\sigma I_{20} + \sigma_l I_{10} + \sigma_r I_{30}) + \left[2x_{2k} - (x_{1k} + x_{3k})\right]\sigma_\alpha I_{40} + \\ \left[(x_{2k} - x_{1k})^3 + (x_{2k} - x_{3k})^3\right]\sigma_\beta I_{50} + \left[2x_{5k} - (x_{4k} + x_{6k})\right]\sigma_c I_{60} \end{matrix}\right\}
\end{aligned} \tag{26b}
$$

Displacement and velocity at the interface dof-s of substructure 1 of the 3-DOF nonlinear oscillator may now be written as

$$
\begin{aligned}
x_1(t) &= x_{1k} + x_{4k}(t - t_k) + a_{4k}\frac{(t - t_k)^2}{2} - \frac{1}{m_l}\left[\begin{matrix} k_l x_{4k} + c_l a_{4k} + x_{7k}(x_{4k} - x_{5k}) + x_{9k}(a_{4k} - a_{5k}) + \\ 3x_{8k}(x_{1k} - x_{2k})^2 (x_{4k} - x_{5k}) \end{matrix}\right]\frac{(t - t_k)^3}{6} \\
&+ \sigma_l\left(\int_{t_k}^{t}\int_{t_k}^{s} dB_1(s_1)ds - \frac{c_l + x_{9k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_1(s_2)ds_1 ds\right) + \sigma_\alpha\frac{x_{1k} - x_{2k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_4(s_2)ds_1 ds \\
&+ \sigma_\beta\frac{(x_{1k} - x_{2k})^3}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_5(s_2)ds_1 ds + \sigma_c\frac{x_{4k} - x_{5k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_6(s_2)ds_1 ds + \sigma\frac{x_{9k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_2(s_2)ds_1 ds
\end{aligned} \tag{27a}
$$

$$
\begin{aligned}
x_4(t) &= x_{4k} + a_{4k}(t - t_k) - \frac{1}{m_l}\left[\begin{matrix} k_l x_{4k} + c_l a_{4k} + x_{7k}(x_{4k} - x_{5k}) + x_{9k}(a_{4k} - a_{5k}) + \\ 3x_{8k}(x_{1k} - x_{2k})^2 (x_{4k} - x_{5k}) \end{matrix}\right]\frac{(t - t_k)^2}{2} \\
&+ \sigma_l\left(\int_{t_k}^{t} dB_1(s) - \frac{c_l + x_{9k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s} dB_1(s_1)ds\right) + \sigma_\alpha\frac{x_{1k} - x_{2k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s} dB_4(s_1)ds \\
&+ \sigma_\beta\frac{(x_{1k} - x_{2k})^3}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_5(s_1)ds + \sigma_c\frac{x_{4k} - x_{5k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_6(s_1)ds + \sigma\frac{x_{9k}}{m_l}\int_{t_k}^{t}\int_{t_k}^{s} dB_2(s_1)ds
\end{aligned} \tag{27b}
$$

Similarly one may express the displacement and velocity at the interface dof-s of substructure: 2 also. The truncated Ito-Taylor expansions for the displacement and velocity components of substructure 2 of the 3-DOF linear oscillator may be shown to be of the form

$$
\begin{aligned}
x_{3(k+1)} &= x_{3k} + x_{6k}h + a_{6k}\frac{h^2}{2} - \frac{1}{m_3}\left[x_{7k}(x_{6k} - x_{5k}) + x_{8k}(a_{6k} - a_{5k})\right]\frac{h^3}{6} + \sigma_3 I_{30} \\
&- \frac{1}{m_3}\left[(x_{3k} - x_{2k})\sigma_{k_3} I_{400} - x_{8k}\sigma_2 I_{200} + x_{8k}\sigma_3 I_{300} + (x_{6k} - x_{5k})\sigma_{c_3} I_{500}\right]
\end{aligned} \tag{28a}
$$

$$
\begin{aligned}
x_{6(k+1)} &= x_{6k} + a_{6k}h - \frac{1}{m_3}\left[x_{7k}(x_{6k} - x_{5k}) + x_{8k}(a_{6k} - a_{5k})\right]\frac{h^2}{6} + \sigma_3 I_3 \\
&- \frac{1}{m_3}\left[(x_{3k} - x_{2k})\sigma_{k_3} I_{40} - x_{8k}\sigma_2 I_{20} + x_{8k}\sigma_3 I_{30} + (x_{6k} - x_{5k})\sigma_{c_3} I_{50}\right]
\end{aligned} \tag{28b}
$$

Expansions for displacement and velocity at the interface dof-s of substructure 1 of the 3-DOF linear oscillator are given by

$$x_2(t) = x_{2k} + x_{5k}(t-t_k) + a_{5k}\frac{(t-t_k)^2}{2} - \frac{1}{m_2}\left[k_2(x_{5k}-x_{4k}) + c_2(a_{5k}-a_{4k}) + x_{7k}(x_{5k}-x_{6k}) + x_{8k}(a_{5k}-a_{6k})\right]\frac{(t-t_k)^3}{6}$$
$$+\sigma_2\left(\int_{t_k}^{t}\int_{t_k}^{s} dB_2(s_1)ds - \frac{c_2+x_{8k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_2(s_2)ds_1 ds\right) + \sigma_1\frac{c_2}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_1(s_2)ds_1 ds$$
$$+\sigma_3\frac{x_{8k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_3(s_2)ds_1 ds + \sigma_{k_3}\frac{x_{3k}-x_{2k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_4(s_2)ds_1 ds + \sigma_{c_3}\frac{x_{6k}-x_{5k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s}\int_{t_k}^{s_1} dB_5(s_2)ds_1 ds \tag{29a}$$

$$x_5(t) = x_{5k} + a_{5k}(t-t_k) - \frac{1}{m_2}\left[k_2(x_{5k}-x_{4k}) + c_2(a_{5k}-a_{4k}) + x_{7k}(x_{5k}-x_{6k}) + x_{8k}(a_{5k}-a_{6k})\right]\frac{(t-t_k)^2}{2}$$
$$+\sigma_2\left(\int_{t_k}^{t} dB_2(s) - \frac{c_2+x_{8k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_2(s_1)ds\right) + \sigma_1\frac{c_2}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_1(s_1)ds$$
$$+\sigma_3\frac{x_{8k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_3(s_1)ds + \sigma_{k_3}\frac{x_{3k}-x_{2k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_4(s_1)ds + \sigma_{c_3}\frac{x_{6k}-x_{5k}}{m_2}\int_{t_k}^{t}\int_{t_k}^{s} dB_5(s_1)ds \tag{29b}$$

Finally we observe that truncations of the Ito-Taylor expansions have been so adjusted as to consistently ensure that local orders of accuracy of the displacement and velocity maps, in the above equations, are $O(h^3)$ and $O(h^2)$ respectively. Since the state of the system is presently being measured, the observation equation is in a discrete form: $y_k = h_k(x_k) + v_k$, v_k being the vector of measurement noise. However, if the measurement equation is in the form of an SDE, this equation may as well be discretized using Ito-Taylor expansions.

6. Numerical Illustrations

Consistent with the developments in the last section, we continue to work with the 5-DOF linear, 3-DOF nonlinear, and 3-DOF linear oscillators. In particular, we use the 5-DOF and 3-DOF nonlinear oscillators to bring out the superior performance of the proposed RBPF in state estimations vis-à-vis a full-fledged particle filter without substructuring. It is known that the Kalman filter provides the optimal solution for state estimation of linear oscillators with additive Gaussian noises. Hence, given the exact solution via Kalman filter, the 5-DOF oscillator may be well exploited to compare the relative numerical performances of the RBPF and the standard particle filter for state estimation. Next, to demonstrate the potential of the proposed method in parameter estimations, we use the 3-DOF linear and nonlinear oscillators. The governing equations of the 3-DOF linear system become nonlinear when the parameters of substructure 2 are declared as additional states. For the 3-DOF nonlinear oscillator, on the other hand, nonlinearity is localized in substructure 2. These three examples together therefore cover a reasonable range of test cases of interest.

For all the state estimation problems, the support displacement is assumed to be harmonic, i.e., $x_g(t) = x_{g0}\sin(\lambda t)$. However, for parameter estimation problems, the support motion is taken to be a realization of the stochastic process $x_g(t) = \sum_{i=1}^{n} a_i \sin(\omega_i t + \theta_i)$ where the parameters a_i, ω_i and θ_i are appropriately chosen random variables. The following parameter values are considered for the 5-DOF linear oscillator: m_i = 100 kg, k_i = 30000 N/m and c_i = 100 Ns/m (i = 1, 2,..., 5). The parameters of the support displacement are assumed as x_{g0} = 0.01m and λ = 14 rad/s, which is close to the first natural frequency of the system. The process noise parameters are assumed as $\sigma_i = 0.01|\ddot{x}_g|_{\max}\ (i = 1,2,3,4)$ and $\sigma_5 = 0.03|\ddot{x}|_{\max}$ where $\left|\ddot{x}_g\right|_{\max}$ is the maximum value of the realization of $\left|\ddot{x}_g(t)\right|$ over the time interval of interest. Measurements on the velocities of the 4th and 5th mass points are assumed to be made. The following parameter values are considered for the 3-DOF nonlinear oscillator: $m_1 = m_2 = 100$ kg, $k_1 = 30000$ N/m, $k_2 = 25000$ N/m, $\alpha = 3000$ N/m, $\beta = 1\times10^6$ N/m^3, $c_1 = 60$ Ns/m, $c_2 = 50$ Ns/m and $c = 75$ Ns/m. For the state estimation problem, the parameters of the process noise are chosen as $\sigma_l = \sigma_r = 0.01|\ddot{x}_g|_{\max}$ and $\sigma = 0.03|\ddot{x}_g|_{\max}$, whereas for the parameter estimation problem, their values are taken as $\sigma_l = \sigma_r = \sigma = 0.01|\ddot{x}_g|_{\max}$. The parameters of the support displacement are taken to be $x_{g_0} = 0.01$m and $\lambda = 20$rad/s for the state estimation problem. The displacement and velocity maps, given by Eqs. (26a,b), may be used for the state estimation problem of the 3-DOF nonlinear system by setting $x_7 = \alpha$, $x_8 = \beta$, $x_9 = c$ and $\sigma_\alpha = \sigma_b = \sigma_c = 0$. Measurements of all velocity components are assumed to be made in the state estimation problem. All the systems considered here are assumed to start from rest. Moreover the initial conditions are treated as deterministic.

For parameter estimation of the 3-DOF linear oscillator, the chosen reference parameter values are: $m_1^* = m_2^* = 100$ kg, $m_3^* = 75$ kg, $k_1^* = k_2^* = 25000$ N/m, $k_3^* = 15000$ N/m, $c_1^* = c_2^* = 150$ Ns/m and $c_3^* = 100$ Ns/m. The values of the process noise parameters are taken as $\sigma_i = 0.01|\ddot{x}_g|_{\max}$ $(i = 1,2,3)$. Moreover the following random parameters of the support motion are assumed: a_i is uniformly distributed in the range [–0.005, 0.005]m, while ω_i and θ_i are uniformly distributed in the ranges [5, 30] rad/s and [0, 90]° respectively. Value of n is taken as 50. The velocities of the 3rd and 4th mass points constitute the measurements. The initial pdf of x_7(i.e., k_3) is assumed to be uniformly distributed in the range $[0.8k_3^*, 1.8k_3^*]$ and x_8(i.e., c_3) is assumed to be uniformly distributed in the range $[0.4c_3^*, 1.2c_3^*]$. It may be noted that the deviation of the mean of the initial pdf from the true value is 30% for x_7 and 20% for x_8. Finally, for the parameter estimation of the 3-DOF nonlinear oscillator, the following values are used for the components of the multi-frequency harmonic support motion: a_i, ω_i and θ_i are assumed to be uniformly distributed in the ranges [–0.005, 0.005]m, [10, 50]rad/s and [0, 90]° respectively. We use $n = 50$. In addition to the velocity measurements at all dof-s, the displacement of the substructure 2 is also measured. The initial pdf-s of x_7(i.e., α), x_8(i.e., β) and x_9(i.e., c) are assumed to be uniformly distributed in the ranges $[0.8\alpha^*, 1.4\alpha^*]$, $[0.8\beta^*, 1.4\beta^*]$ and $[0.6c^*, 1.2c^*]$ respectively. Note that α^*, β^* and c^* represent the reference parameter values of α, β and c respectively. Here, for all the parameters, the mean of the initial pdf is consistently off by 10% from the true value. In all the parameter estimation problems, the diffusion coefficients of the SDE-s corresponding to the parameter states (i.e., σ_{k_3} and σ_{c_3} for the 3-DOF linear oscillator; and, σ_α, σ_β and σ_c for the 3-DOF nonlinear oscillator) are assumed to be time varying with a slow decay given by the general form $\sigma(t) = 0.2\,P\exp(-0.5t)$, where P refers to the true value of the parameter. Due to the decaying nature of diffusion coefficients of the parameter SDE-s, subset of the state space containing the particles (representing the parameter states) shrinks as the estimation progresses and hence one may expect a converging estimate.

In the present study all the measurements are generated synthetically. The standard deviation of the measurement noise is assumed to be 7.5% of the maximum absolute value of the measured quantity for all the state estimation problems, while it is reduced to 5% for parameter estimation. In all the examples presented here, a uniform step size h is used. While we use $h = 0.005$ s for the 5-DOF linear and 3-DOF nonlinear oscillators, we increase the step size to $h = 0.01$ s for the 3-DOF linear system. The ensemble size used is a modest $N = 500$ for all state estimation problems. However, for the parameter estimation problems of the 3-DOF linear and 3-DOF nonlinear oscillators, the ensemble sizes are $N = 3000$ and $N = 6000$ respectively.

The results of state estimation of the 5-DOF linear system are shown in Figs. 5 and 6. Fig. 5 shows the measured quantities and their estimate using the proposed RBPF. Excellent convergence of the mean of the estimate can be observed. Fig. 6 shows a comparison of variances of the estimates of displacements and velocities at the 4th and 5th dof-s using the standard particle filter, the proposed RBPF and the Kalman filter. We observe that the RBPF estimates are consistently closer to the optimal estimate. A significantly superior performance of the proposed RBPF over the standard particle filter is clearly noticeable. Figs. 7 and 8 show the results of state estimations of the 3-DOF nonlinear oscillator. The measured velocities at all dof-s and their estimates using the novel RBPF are shown in Fig. 7. Variances of the states estimated using the standard particle filter and the RBPF are compared in Fig. 8. Once more, a substantial reduction in the variance of the RBPF-based simulation over that via the standard particle filter is observed. Figs

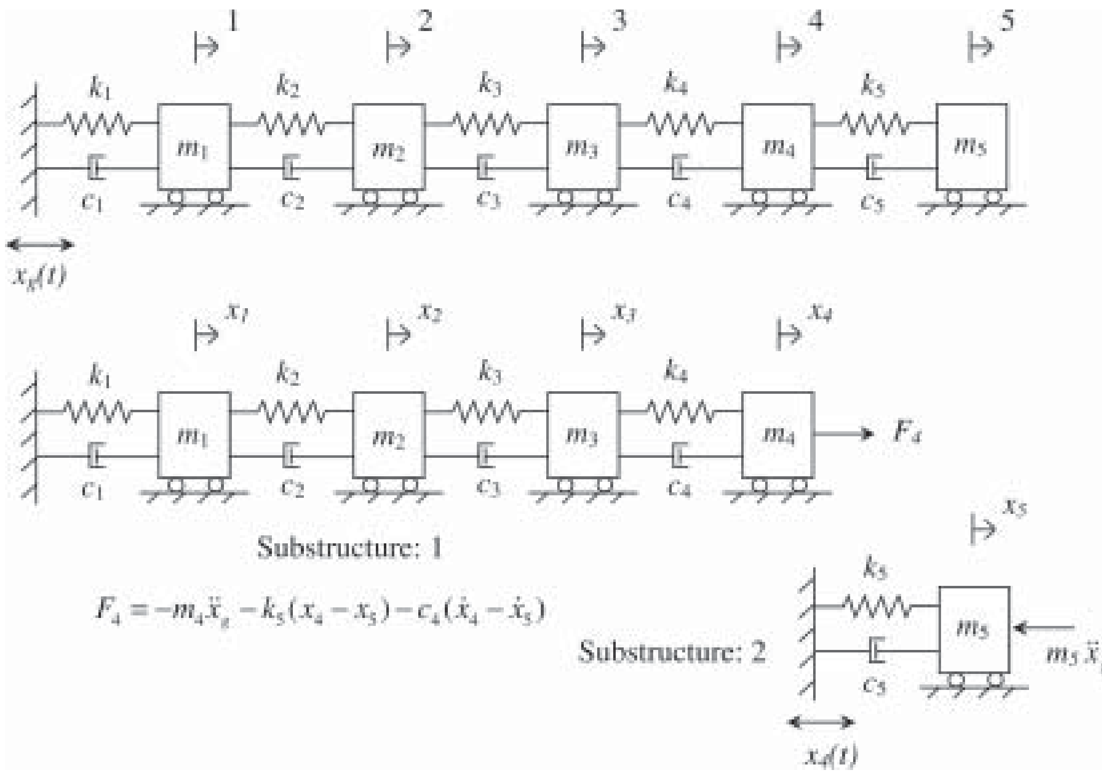

Figure 2: A 5-DOF Linear Oscillator with Linear Springs and its Decomposition into Substructures

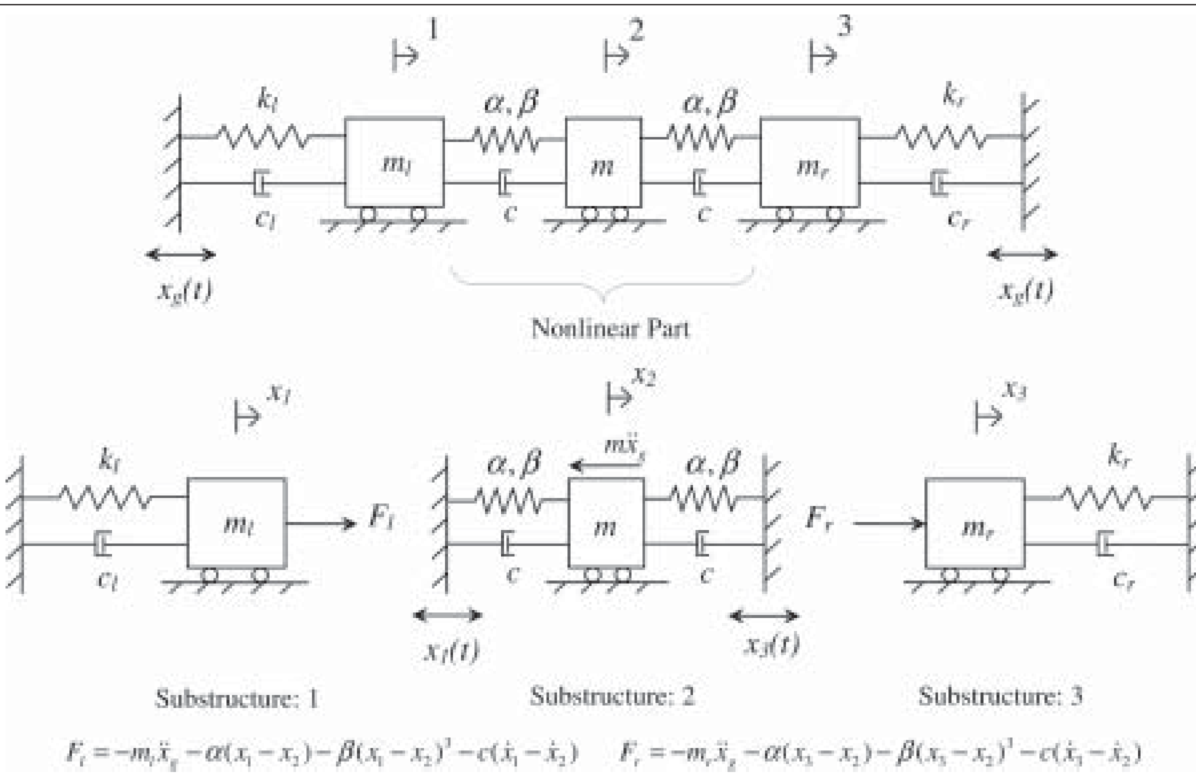

Figure 3: A 3-DOF Nonlinear Oscillator with Substructures (Note that the Spring Force in the Nonlinear Part is Given by $\alpha x+\beta x^3$, where x is the deformation of the spring)

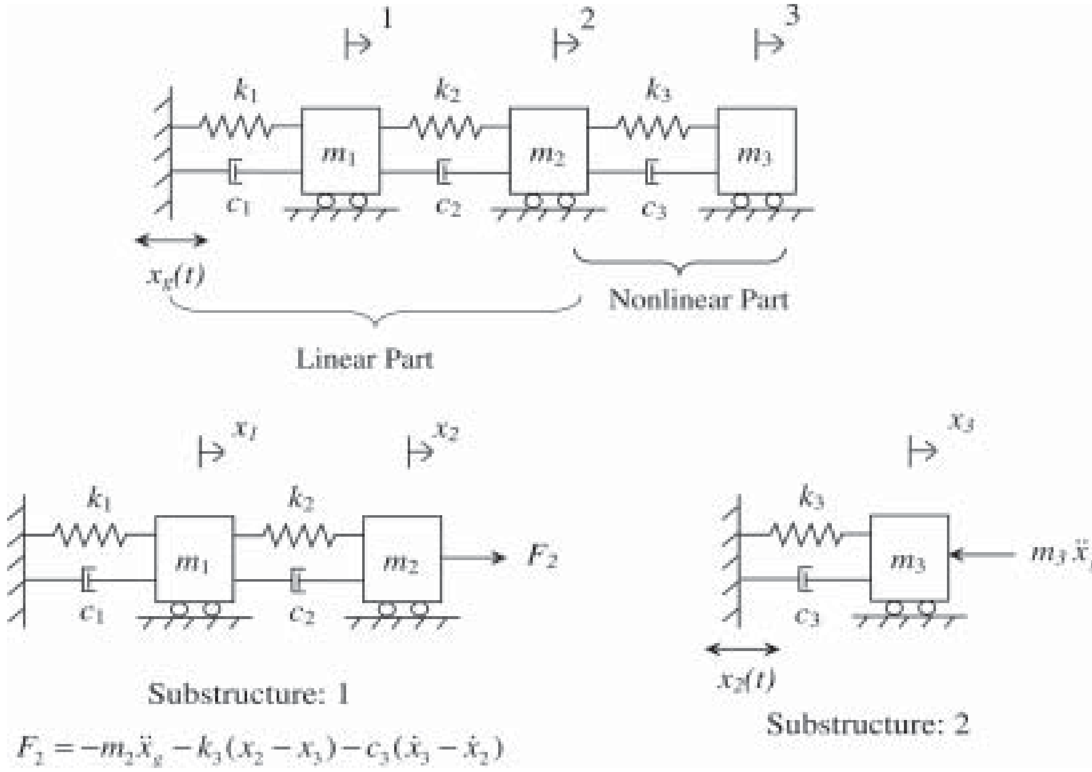

Figure 4: A 3-DOF Linear Oscillator with Substructures; Substructure 2 is Considered to be Nonlinear Since k_3 and c_3 are Declared as Additional State Variables

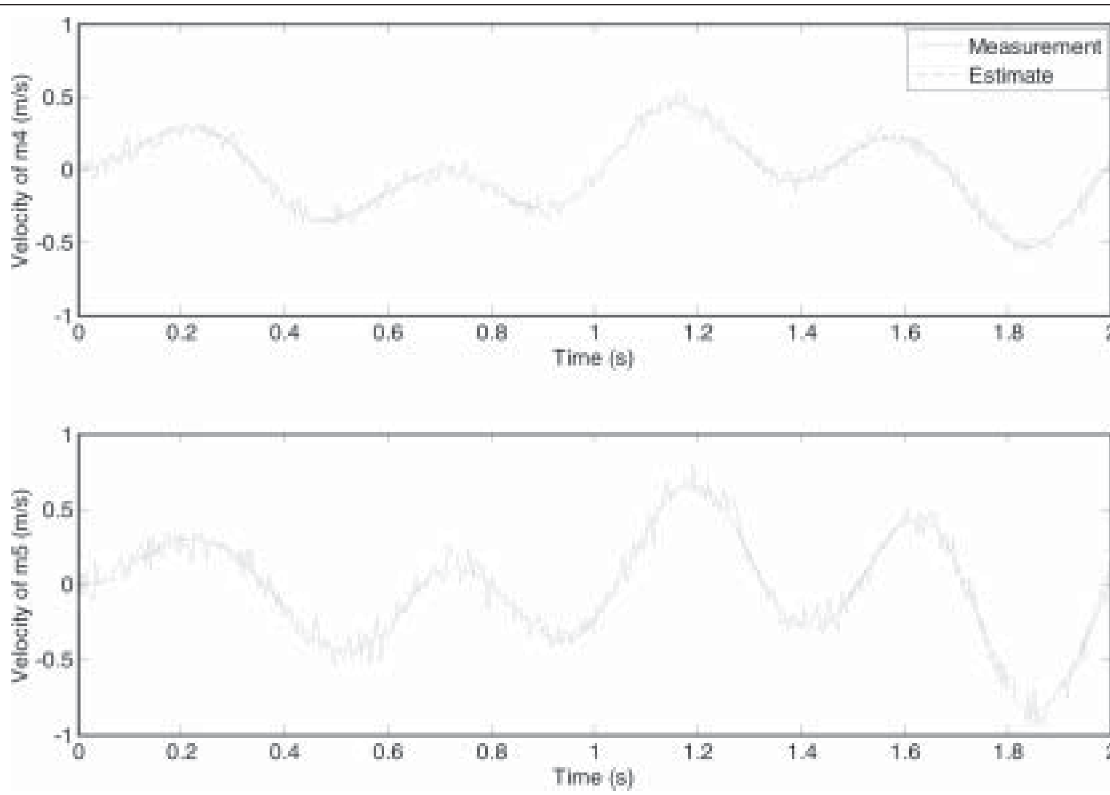

Figure 5: State Estimations of the 5-DOF Linear Oscillator using the RBPF -Measurements and Estimates

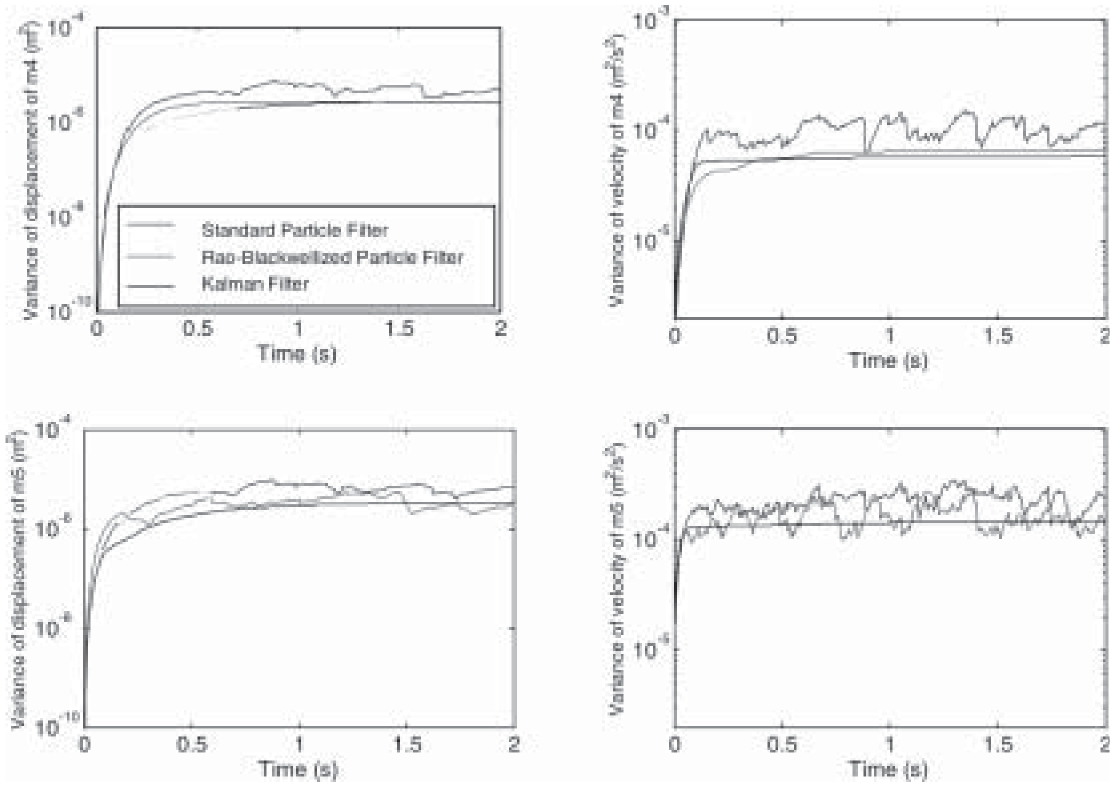

Figure 6: State Estimations of the 5-DOF Linear Oscillator - Comparison of Performance of a Standard Particle Filter and the RBPF with Kalman Filter

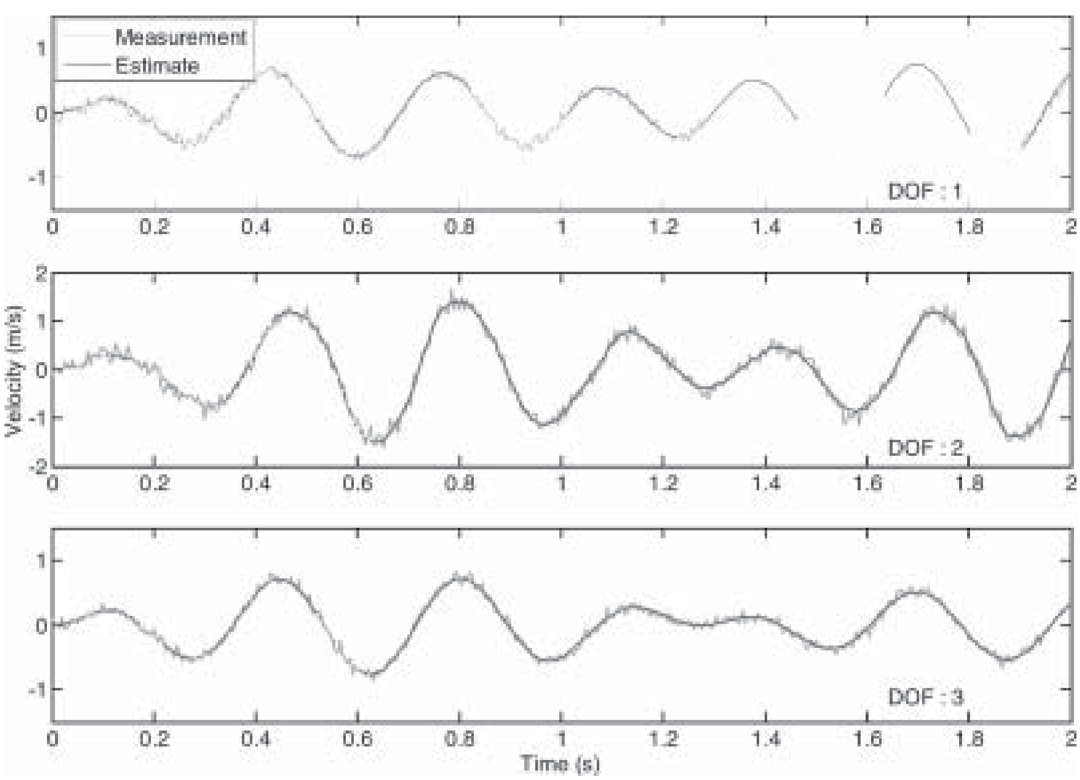

Figure 7: State Estimation of the 3-DOF Nonlinear Oscillator using the RBPF –Measurements and Estimates

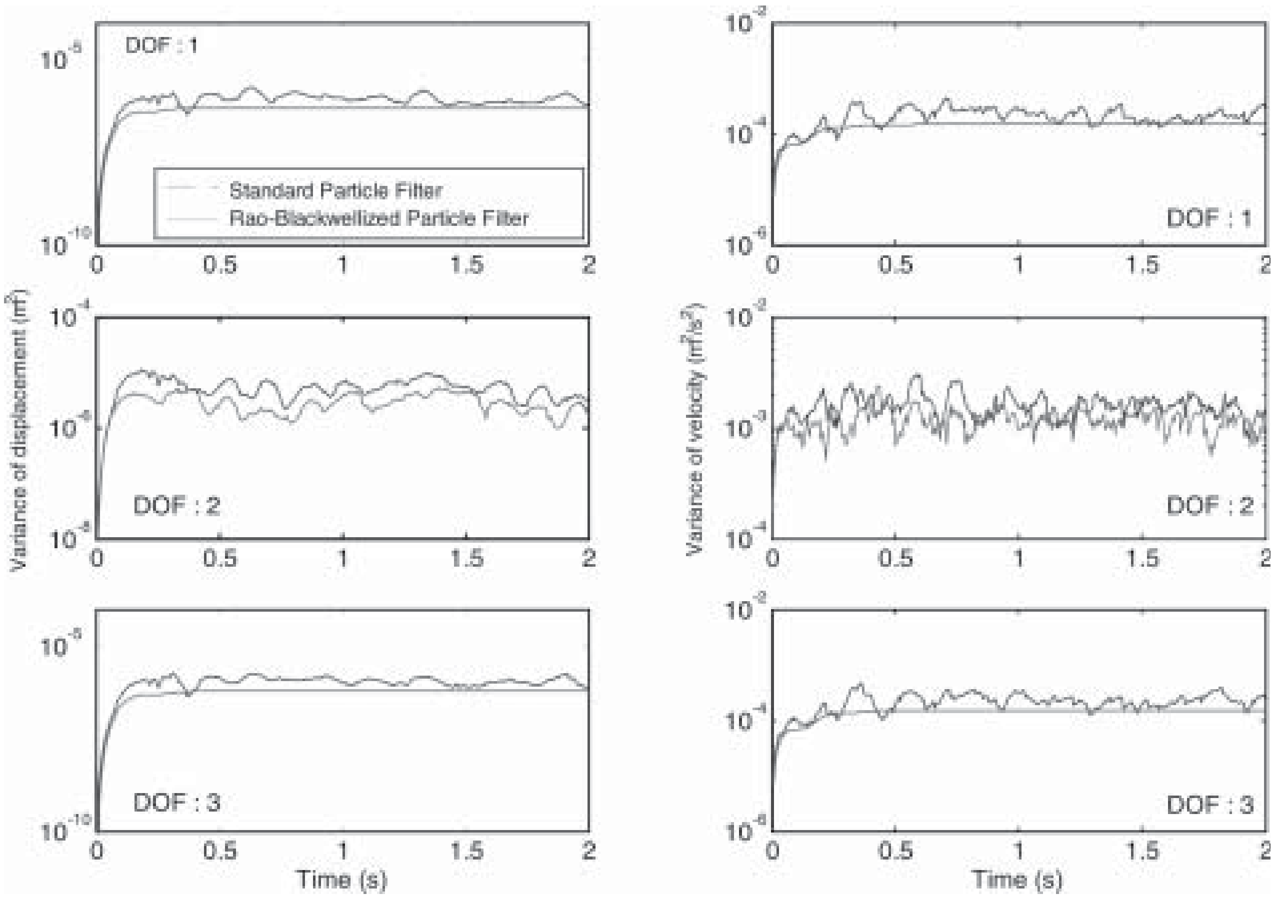

Figure 8: State Estimations of a 3-DOF Nonlinear Oscillator – Comparisons of Performance of a Standard Particle Filter and the RBPF

9 through 11 show the results of parameter estimation of the 3-DOF linear oscillator. Fig. 9 shows time histories of mean and standard deviation of the estimate of k_3 corresponding to five different runs of the same program. Note that all the random variables required for the simulation are generated independently in each run of the program. Figure 10 shows similar plots for the estimation of c_3. Even though these estimates do not converge to the true values of the associated parameters (which is only natural in the presence of noises), the robustness of the proposed RBPF is observable from these plots. Fig. 11 shows the initial and final pdf-s (stationary marginals) of the parameters. One readily observes the considerable reduction in the variance of the estimates from their initial (assumed) values; simultaneously as the mean values converge close to their true values. The results of parameter estimation of the 3-DOF nonlinear oscillator are given in Figs. 12-15. Time histories of means and standard deviations of the estimates of α, β and c for five different simulations are shown in Figs. 12, 13 and 14 respectively. The initial and final pdf-s of the parameters are shown in Fig. 15. The robustness and accuracy of the new RBPF in parameter estimation is again evident from these plots.

7. Concluding Remarks

A novel and numerically accurate form of Rao-Blackwellized particle filter is proposed for state and parameter estimations for a class of uncertain dynamical systems, typically with localized nonlinearity. The system to be identified is first decomposed into linear and nonlinear substructures which are mutually coupled. While the estimation using particle filters is restricted only to the states of nonlinear substructures, the states of complementary linear substructures are estimated using an ensemble of Kalman filters, thus enabling us to get a solution close to the optimal with lesser sampling variance for a given ensemble size. Here the sampling variance owes its origin not just to the finiteness of ensemble sizes, but also to truncation errors of the finite

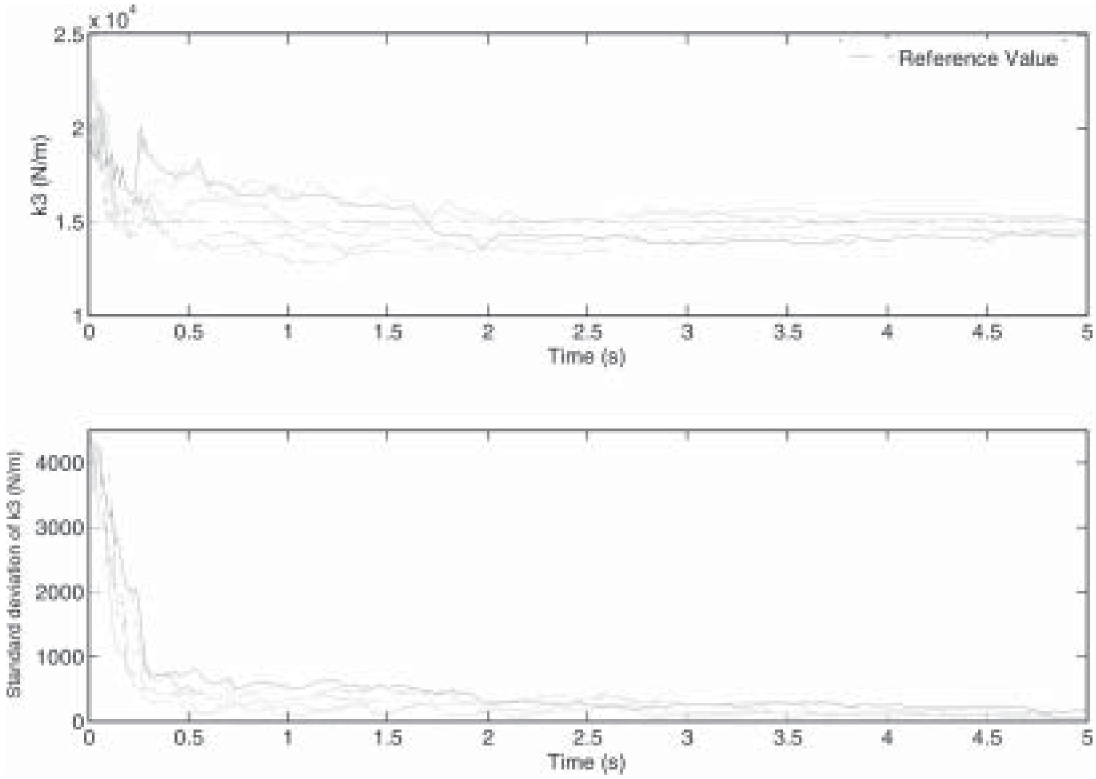

Figure 9: Parameter Estimations of the 3-DOF Linear Oscillator– Time Histories of the Mean of the Estimate of k_3 and the Associated Standard Deviation for Five Different Simulations

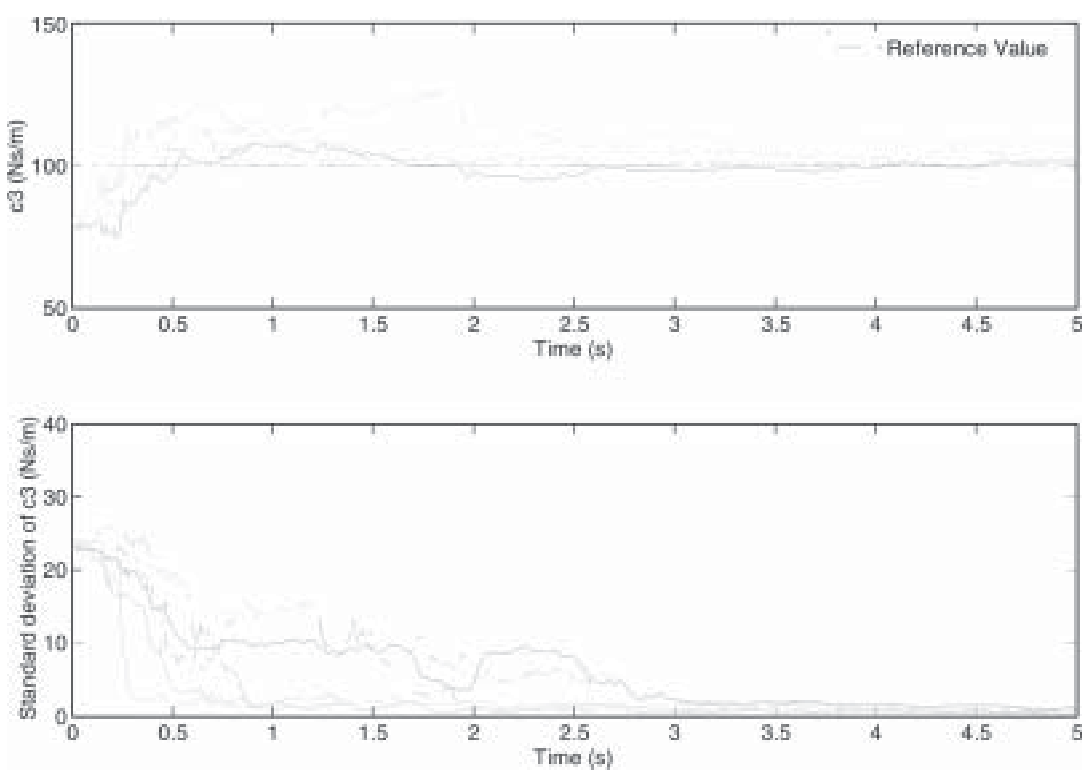

Figure 10: Parameter Estimations of the 3-DOF Linear Oscillator –Time Histories of the Mean of the Estimate of c_3 and the Associated Standard Deviation for Five Different Simulations

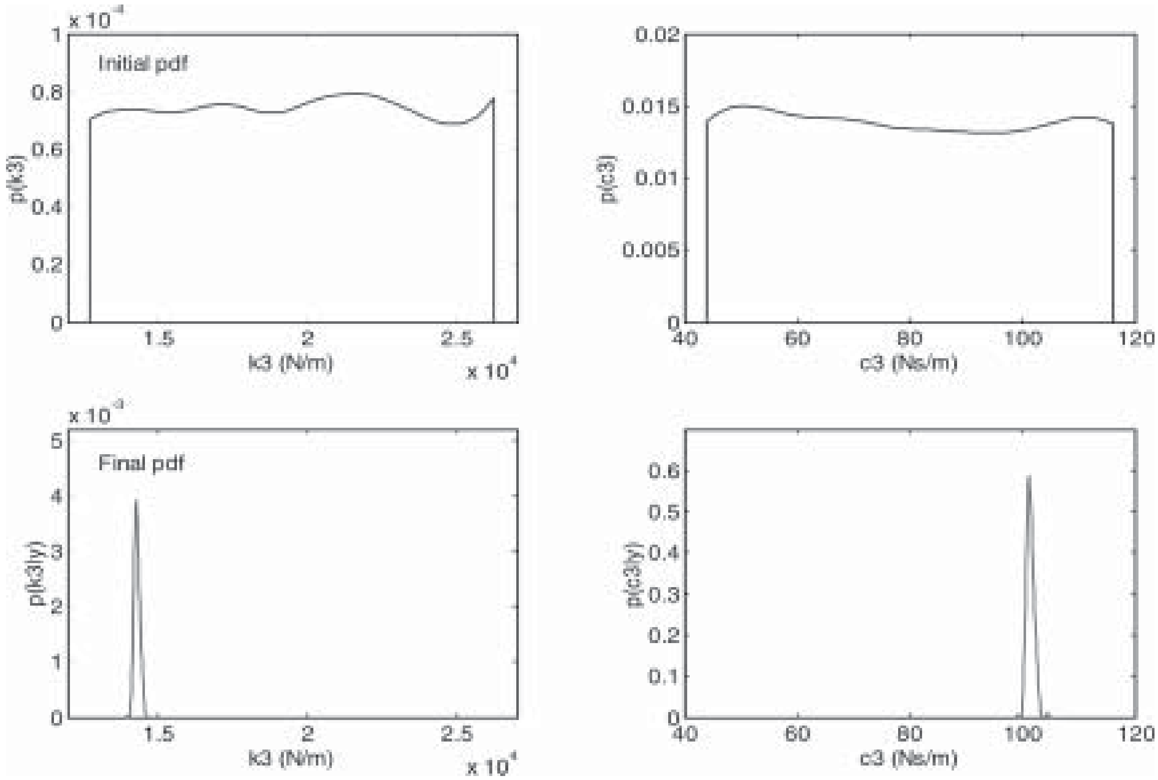

Figure 11: Parameter Estimations of the 3-DOF Linear Oscillator – initial (Assumed) and Final (Converged) pdf of the Estimated Quantities, in One of the Simulations

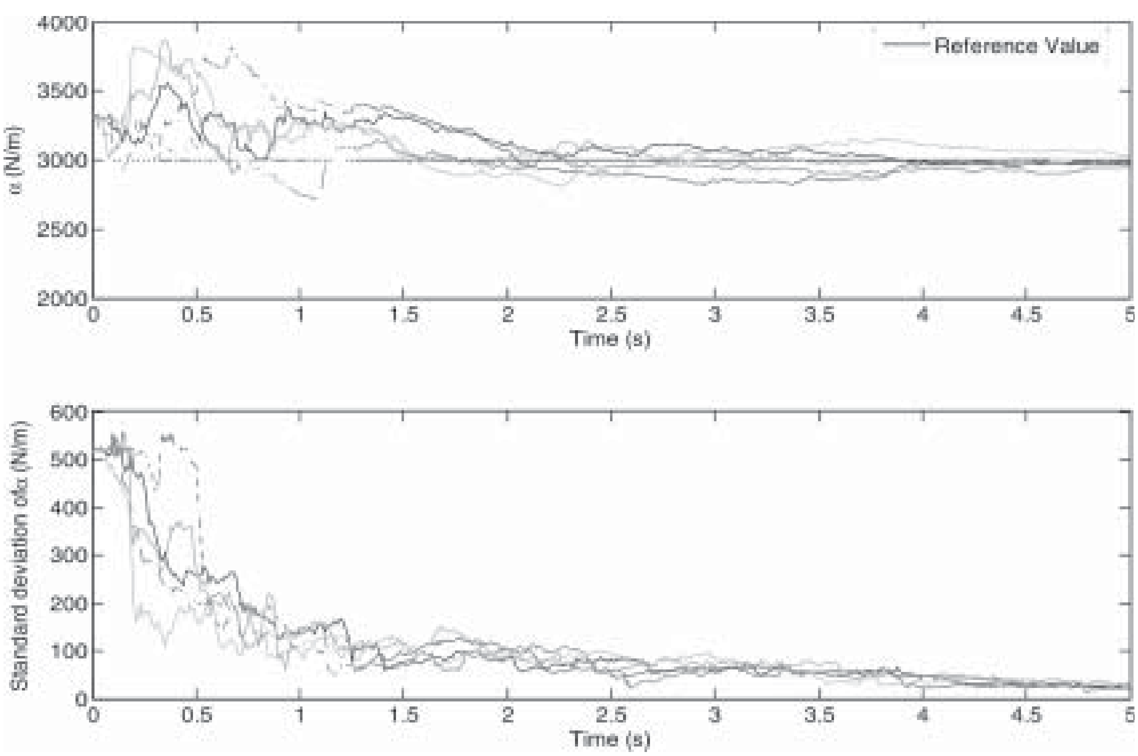

Figure 12: Parameter Estimations of the 3-DOF Nonlinear Oscillator – Time Histories of the Mean of the Estimate of α and the Associated Standard Deviation for Five Different Simulations

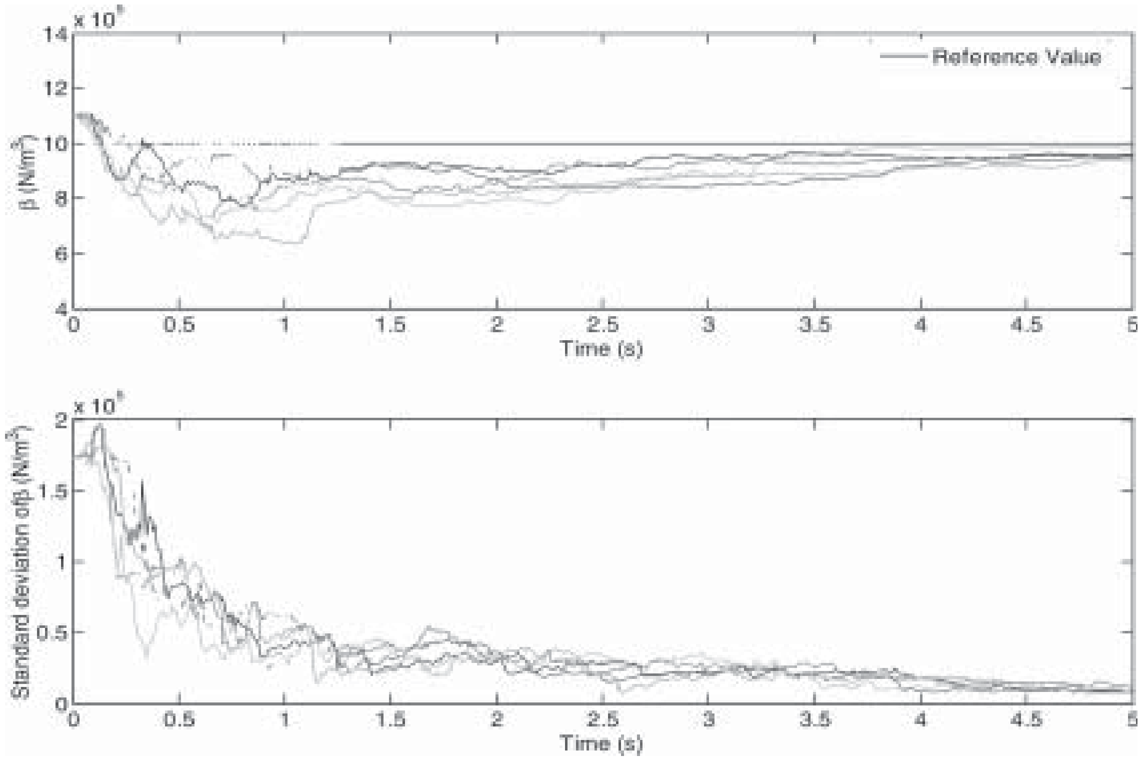

Figure 13: Parameter Estimations of the 3-DOF Nonlinear Oscillator – Time Histories of the Mean of the Estimate of β and the Associated Standard Deviation for Five Different Simulations

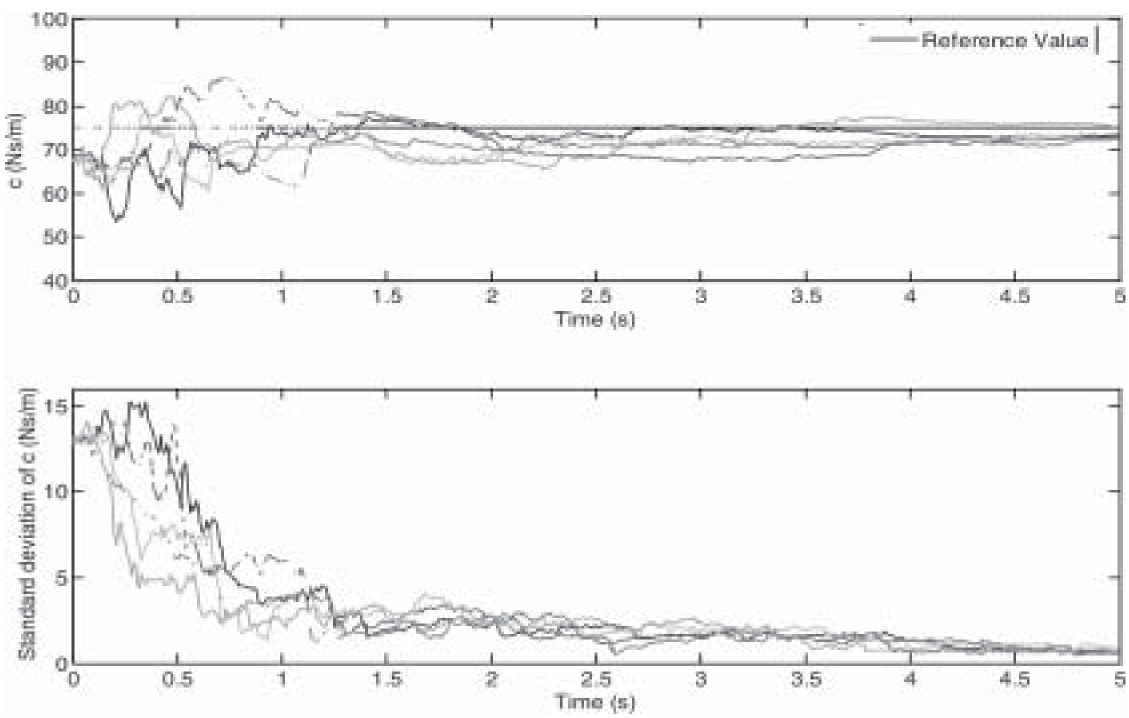

Figure 14: Parameter Estimations of the 3-DOF Nonlinear Oscillator – Time Histories of the Mean of the Estimate of *c* and the Associated Standard Deviation for Five Different Simulations

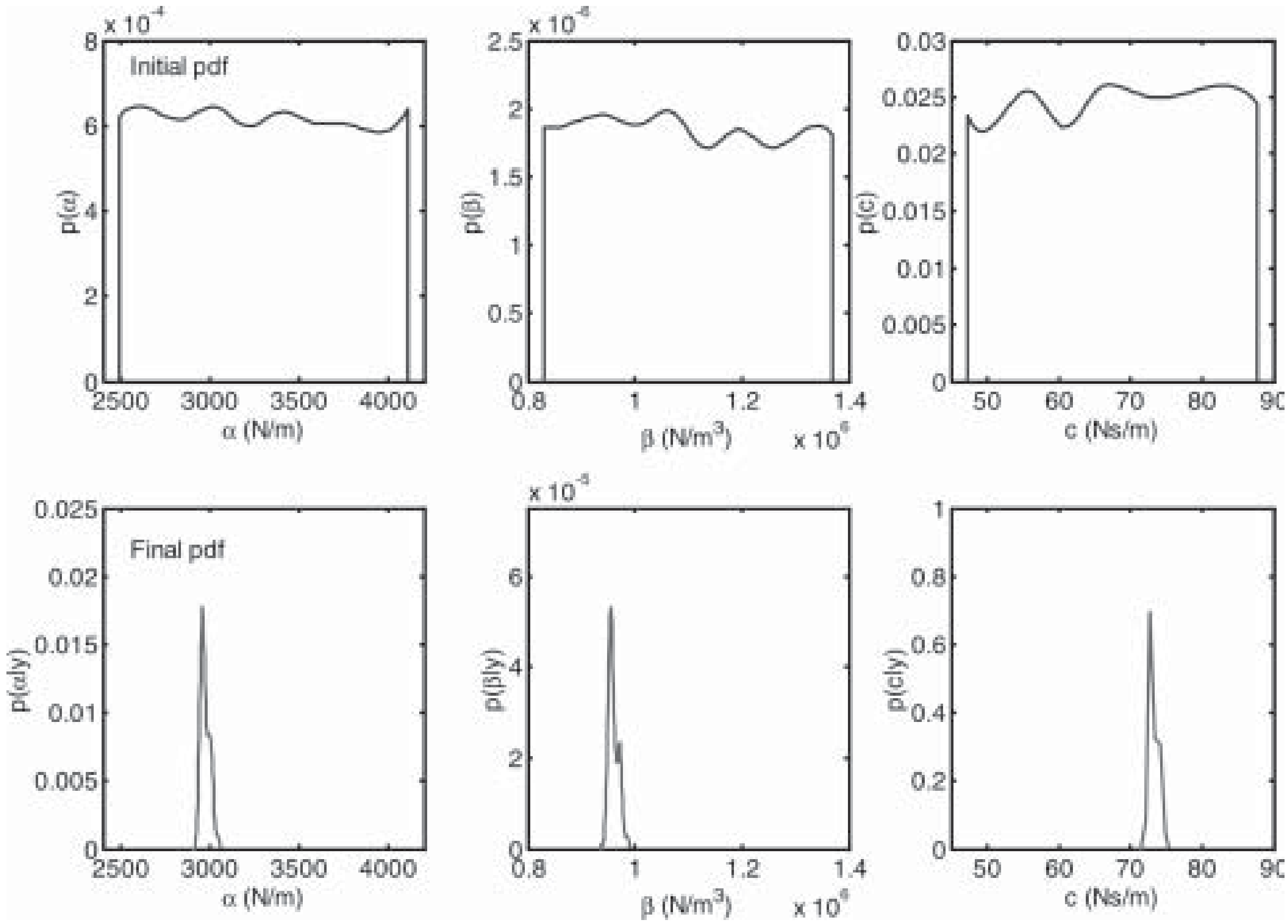

Figure 15: Parameter Estimations of the 3-DOF Nonlinear Oscillator – Initial (Assumed) and Final (Converged) pdf of the Estimated Quantities, in One of the Simulations

precision arithmetic and approximations due to truncated Ito-Taylor expansions. It is important to note that the standard Rao-Blackwellization is not directly applicable to substructures of a mechanical system as they do not usually correspond to each other via a cascading effect. To the authors' knowledge, the present work constitutes the first attempt at exploiting Rao-Blackwellization in the context of state/parameter estimations of a mechanical oscillator with a rational framework for handling the coupling of the associated substructures. The limited numerical illustrations on a few low-dimensional oscillators emphasize the robustness and numerical superiority of the proposed RBPF over the standard particle filter and provide an adequate pointer to the applicability of the method to higher dimensional systems. The new RBPF has the potential to be applied for state and parameter estimations of a majority of engineering structures, since nonlinearity generally appears in a localized form. We have presently used the bootstrap filter for particle filtering, and, with more efficient particle filters, the proposed RBPF may as well offer a robust tool for online parameter estimations.

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References

Kalman, R. E. (1960), "A New Approach to Linear Filtering and Prediction Problems", Transactions of the *ASME–Journal of Basic Engineering*. **82**(Series D), 35-45.

Brown, R. G. and Hwang, P. Y. C. (1992), *Introduction to Random Signals and Applied Kalman Filtering*, John Wiley and Sons, Inc., New York.

Gordon, N. J., Salmond, D. J., and Smith, A. F. M. (1993), "Novel Approach to Nonlinear/non-Gaussian Bayesian State Estimation", *IEE Proceedings-F*, **140**, pp. 107-113.

Tanizaki, H. (1996), *Nonlinear Filters: Estimation and Applications*. Springer Verlag: Berlin.

Tanizaki, H. and Mariano, R. S. (1998), "Nonlinear and Non-Gaussian State-space Modeling with Monte Carlo Simulations", *Journal of Econometrics*, **83**, 263-290.

Doucet, A. (1998), *On Sequential Simulation-based Methods for Bayesian Filtering*. Technical Report CUED/F-INFENG/TR.310(1998), Dept. Electrical Engineering, University of Cambridge, UK.

Doucet, A., Godsill, S., and Andrieu C. (2000), "On Sequential Monte Carlo Sampling Methods for Bayesian Filtering", Statistics and Computing, **10**, pp. 197-208.

Doucet, A., de Freitas, N., and Gordon, N. (2001), *Sequential Monte Carlo Methods in Practice*, Springer, New York.

Ristic, B., Arulampalam, S., and Gordon, N. (2004), *Beyond the Kalman Filter–Particle Filters for Tracking Applications*, Artech House, Boston, London.

Robert, C. P. and Casella, G. (2004), *Monte Carlo Statistical Methods*, 2nd Edition, Springer, New York.

Liu, J. S. (2001), *Monte Carlo Strategies in Scientific Computing*, Springer, New York.

Murphy, K. and Russell, S. (2001), "Rao-Blackwellised Particle Filtering for Dynamic Bayesian Networks", In Doucet, A., de Freitas, N., and Gordon, N., editors, *Sequential Monte Carlo Methods in Practice*, Springer-Verlag, New York.

Nordlund, P. and Gustafsson, F. (2001), "Sequential Monte Carlo Filtering Techniques Applied to Integrated Navigation Systems", *Proceedings of the American Control Conference*, pp. 4375-4380.

Manohar, C. S., and Roy, D. (2006), "Monte Carlo Filters for Identification of Nonlinear Structural Dynamical Systems", Sadhana, Academy Proceedings in Engineering Sciences, Indian Academy of Sciences, **31(4),** 399-427.

Ching, J., Beck, J., L., and Porter, K., A. (2006), "Bayesian State and Parameter Estimation of Uncertain Dynamical Systems", Probabilistic Engineering Mechanics, **21**, pp. 81-96.

Ching, J., Beck, J., L., Porter, K., A., and Shaikhutdinov, R. (2006), "Bayesian State Estimation Method for Nonlinear Systems and its Application to Recorded Seismic Response", *ASCE Journal of Engineering Mechanics*, **132(4)**, 396-410.

Li, S. J., Suzuki, Y., and Noori, M. (2004), "Improvement of Parameter Estimation for Non-linear Hysteretic Systems with Slip by a Fast Bayesian Bootstrap Filter", *International Journal of Nonlinear Mechanics*, **39**, 1435–1445.

Gustafsson, F., Gunnarsson, F., Bergman, N., Forssell, U., Jansson, J., Karlsson, R., and Nordlund, P., 2002,"Particle Filters for Positioning, Navigation, and Tracking", *IEEE Transactions on Signal Processing*, 50(2), 425-437.

de Freitas, N. (2002), "Rao-Blackwellised Particle Filtering for Fault Diagnosis", *Proceedings of the IEEE Aerospace Conference*, **4**, 1767-1772.

Li, M., Hong, B., Cai, Z. and Luo, R. (2006), "Novel Rao-Blackwellised Particle Filter for Mobile Robot SLAM Using Monocular Vision", *International Journal of Intelligent Technology*, 1(1), 63-69.

Kloeden, P. E., and Platen, E. (1992), *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin.

Milstein, G. N. (1995), *Numerical Integration of Stochastic Differential Equations*, Kluwer Academic Publishers: Dordrecht.

Oksendal, B. (1992), *Stochastic Differential Equations: An Introduction with Applications*, Springer-Verlag, Berlin.