Stochastic Optimization - III

Reservoir Inflow as Stochastic Process (Markov Process)
Inflow Transition Probabilities
Stochastic Dynamic Programming

Reservoir Inflow as Stochastic Process (Markov Process)

- In the development of the SDP recursive equations, the reservoir inflow is treated as a stochastic process.
- It is assumed that the reservoir inflows follow a first order Markov Chain.
- A stochastic process, $\{X_t\}$, is said to be a first order Markov Chain if the dependence of future values of the process on the past values is completely determined by its dependence on the current value alone.
- A first order Markov Chain has the property,

$$P[X_{t+1}/X_t, X_{t-1}, \ldots, X_0] = P[X_{t+1}/X_t]$$

- The assumption of a Markov Chain implies that the dependence of the inflow in the next period on the inflow during the current and all previous periods is completely described by its dependence on the inflow during the current period alone.

Transition Probabilities

- Transition probabilities are used to measure the dependence of the inflow during period t+1 on the inflow during the period t.
- Transition probability P_{ij}^t is defined as the probability that the inflow during the period t+1 will be in the class interval j, given that the inflow during the period t lies in the class interval i.

  $P_{ij}^t = P[Q_{t+1}=j \,/\, Q_t=i]$

  where $Q_t=i$, indicates that the inflow during the period t belongs to the discrete class interval i.
- In applications, the transition probabilities, P_{ij}^t are estimated from historical inflow data.
- A suitable inflow discretization scheme is arrived at first.
- Each inflow value in the historical data set is then assigned the class interval to which it belongs.
- The number of times the inflow in period t + 1 goes to class j, when the inflow in the preceding period t belongs to class i, divided by the number of times the inflow belongs to class i in period t is taken as the estimate of P_{ij}^t.
- Note that for this relative frequency approach of estimating the transition probabilities, inflow data must be available for a sufficiently long length of time.

Example

Sequence of inflows for 31 time periods t.

period t	flow Q_t	period t	flow Q_t	period t	flow Q_t
1	4.5	11	1.8	21	1.8
2	5.2	12	2.5	22	1.2
3	6.0	13	2.3	23	2.5
4	3.2	14	1.8	24	1.9
5	4.3	15	1.2	25	3.2
6	5.1	16	1.9	26	2.5
7	3.6	17	2.5	27	3.5
8	4.5	18	4.1	28	2.7
9	1.8	19	4.7	29	1.5
10	1.5	20	5.6	30	4.1
				31	4.8

Statistics of Observed Inflows

$$E[Q] = \sum_{1}^{31} q_t/31 = 3.155$$

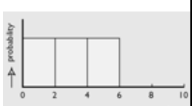
$$\mathrm{Var}[Q] = \sum_{1}^{31}(q_t - 3.155)^2/30 = 1.95$$

Lag-one correlation coefficient $= \rho$

$$= \left[\sum_{1}^{30}(q_{t+1} - 3.155)(q_t - 3.155)\right] \Big/ \sum_{1}^{31}(q_t - 3.155)^2 = 0.50$$

Transition Probabilities

- Probability distribution of the flows can be approximated by a histogram.
- Histograms can be created by subdividing the entire range of random variable values, such as flows, into discrete intervals.
- For example, let each interval be two units of flow.
- Counting the number of flows in each interval and then dividing those interval counts by the total number of counts results in the histogram shown in Figure.
- In this case, just to compare this with what will be calculated later, the first flow, q_1, is ignored.
- Figure shows a uniform unconditional probability distribution of the flow being in any of the possible discrete flow intervals.
- It does not show the possible dependency of the probabilities of the random variable from time t to t+1.



Histogram showing an equal 1/3 probability that the values of the random variable Q_t will be in any one of the three two-flow unit intervals.

Transition Probabilities

- It is possible that the probability of being in a flow interval j in period t +1 depends on the actual observed flow interval i in period t.
- To see if the probability of being in any given interval of flows is dependent on the past flow interval, one can create a matrix.
- The rows of the matrix are the flow intervals i in period t. The columns are the flow intervals j in the following period t +1.
- Such a matrix is shown in Table.
- The numbers in the matrix are based on the inflows and indicate the number of times a flow in interval j followed a flow in interval i.
- Given an observed flow in an interval i in period t, the probabilities of being in one of the possible intervals j in the next period t +1 must sum to 1.

period t	flow Q_t	period t	flow Q_t	period t	flow Q_t
1	4.5	11	1.8	21	1.8
2	5.2	12	2.5	22	1.2
3	6.0	13	2.3	23	2.5
4	3.2	14	1.8	24	1.9
5	4.3	15	1.2	25	3.2
6	5.1	16	1.9	26	2.5
7	3.6	17	2.5	27	3.5
8	4.5	18	4.1	28	2.7
9	1.8	19	4.7	29	1.5
10	1.5	20	5.6	30	4.1
				31	4.8

flow interval in t: i	flow interval in t + 1: j = 1	2	3
1	5	4	1
2	3	4	3
3	2	2	6

Matrix showing the number of times a flow in interval i in period t was followed by a flow in interval j in period t+1.

Transition Probabilities

flow interval in t: *i*	flow interval in t + 1: *j* = 1	2	3
1	5	4	1
2	3	4	3
3	2	2	6

- Thus, each number in each row of the matrix in Table can be divided by the total number of flow transitions in that row (the sum of the number of flows in the row) to obtain the probabilities of being in each interval *j* in *t* +1 given a flow in interval *i* in period *t*.
- In this case there are ten flows that followed each flow interval *i*, hence by dividing each number in each row of the matrix by 10 defines the transition probabilities P_{ij}.
- These conditional or transition probabilities, shown in Table, correspond to the number of transitions shown in Table above.
- Transition probability Matrix.
- The sum of the probabilities in each row equals 1.
- Matrices of transition probabilities whose rows sum to 1 are also called stochastic matrices or first-order Markov chains.

flow interval in t: *i*	flow interval in t + 1: *j* 1	2	3
1	0.5	0.4	0.1
2	0.3	0.4	0.3
3	0.2	0.2	0.6

Matrix showing the probabilities P_{ij} of having a flow in interval *j* in period *t*+1 given an observed flow in interval *i* in period *t*.

Transition Probabilities

- Using the transition probability matrix, one can compute the probability of observing a flow in any interval at any period in the future given the present flow interval.
- This can be done one period at a time.
- For example assume the flow in the current time period *t* =1 is in interval *i* =3.
- The probabilities, $PQ_{j,2}$, of being in any of the three intervals in the following time period *t* =2 are the probabilities shown in the third row of the matrix in the Table.
- The probabilities of being in an interval *j* in the following time period *t*=3 is the sum over all intervals *i* of the joint probabilities of being in interval *i* in period *t* =2 and making a transition to interval *j* in period *t* =3.

$$PQ_{j,t+1} = \sum_i PQ_{it} P_{ij} \text{ for all intervals } j \text{ and periods } t$$

flow interval in t: *i*	flow interval in t + 1: *j* 1	2	3
1	0.5	0.4	0.1
2	0.3	0.4	0.3
3	0.2	0.2	0.6

Matrix showing the probabilities P_{ij} of having a flow in interval *j* in period *t*+1 given an observed flow in interval *i* in period *t*.

Transition Probabilities

flow interval in t: *i*	flow interval in t + 1: *j* 1	2	3
1	0.5	0.4	0.1
2	0.3	0.4	0.3
3	0.2	0.2	0.6

$$PQ_{j,t+1} = \sum_i PQ_{it} P_{ij} \text{ for all intervals } j \text{ and periods } t$$

- This operation can be continued to any future time period. Table illustrates the results of such calculations for six future periods, given a present period (*t* =1) flow in interval *i* = 3.
- Note that as the future time period *t* increases, the flow interval probabilities are converging to the unconditional probabilities – in this example 1/3, 1/3, 1/3 – as shown in Figure.
- The predicted probability of observing a future flow in any particular interval at some time in the future becomes less and less dependent on the current flow interval as the number of time periods increases between the current period and that future time period.

time period *t*	flow interval *i* 1	2	3
1	0	0	1
2	0.2	0.2	0.6
3	0.28	0.28	0.44
4	0.312	0.312	0.376
5	0.325	0.325	0.350
6	0.330	0.330	0.340
7	0.332	0.332	0.336
8	0.333	0.333	0.334
	probability PQ_{it}		

Probabilities of observing a flow in any flow interval *i* in a future time period t given a current flow in interval *i*=3. These probabilities are derived using the transition probabilities P_{ij}.

Steady State Probabilities

- When these unconditional probabilities are reached, PQ_{it} will equal $PQ_{i,t+1}$ for each flow interval *i*.
- To find these unconditional probabilities or **steady state probabilities** directly, we can solve $PQ_j = \sum_i PQ_i P_{ij}$ for all intervals *j*
- Along with $\sum_i PQ_i = 1$

(Use this equation and j-1 equations from above)

- Conditional or transition probabilities can be incorporated into stochastic optimization models of water resources systems.

time period *t*	flow interval *i* 1	2	3
1	0	0	1
2	0.2	0.2	0.6
3	0.28	0.28	0.44
4	0.312	0.312	0.376
5	0.325	0.325	0.350
6	0.330	0.330	0.340
7	0.332	0.332	0.336
8	0.333	0.333	0.334
	probability PQ_{it}		

Probabilities of observing a flow in any flow interval *i* in a future time period t given a current flow in interval *i*=3. These probabilities are derived using the transition probabilities P_{ij}.

Problem - 1

Compute the seasonal transitional probability matrices for the two seasons whose streamflow data is given below for 20 years.

Assume that the seasonal streamflow data follows first order Markov chain.

Divide the season 1 flows into three classes: 100-150, 151-200 and 201-250; and season 2 flows to four classes: 601-700, 701-800, 801-900 and 901-1000 units.

Year	Season 1	Season 2	Year	Season 1	Season 2
1	120	850	11	190	810
2	180	970	12	235	925
3	210	630	13	165	815
4	150	990	14	125	675
5	230	730	15	170	710
6	135	825	16	220	855
7	196	770	17	115	635
8	170	645	18	190	725
9	225	965	19	205	890
10	110	610	20	115	620

Problem - 2

Obtain steady state probabilities by solving the equations for the example problem of annual flows

Problem – 3

Obtain seasonal steady state probabilities For Problem 1 (Optional)

Stochastic Dynamic Programming for Reservoir Operation

- Stochastic Dynamic Programming (SDP) belongs to the Explicit Stochastic Optimization (ESO) class of optimization models.
- SDP application to the reservoir operation problem
  - Inflow to the reservoir is considered as a random variable.
- The reservoir storage at the beginning of period t and inflow during the period t are treated as state variables.
- All variables involved in the decision process, such as the reservoir storage, inflow, and release are discretized into a finite number of class intervals.
  - A class interval for a variable has a representative value, generally taken as its midpoint.

SDP for Reservoir Operation - Notation

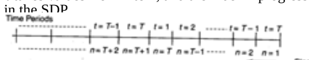
- Q denotes the inflow; i and j are the class intervals (also referred to as states) of inflow in period t and period $t + 1$, respectively;
- The representative values of inflow for the class i in period t and class j in period t+1 are denoted by Q_{it} *and* $Q_{j,t+1}$, respectively.
- S denotes the reservoir storage; and k and l are the storage class intervals in periods t and t +1, respectively.
- Similarly, the representative values for storage in the class intervals k and l are denoted by S_{kt}, and $S_{l,t+1}$, respectively.
- From the storage continuity, then, we may write

  $$R_{kilt} = S_{kt} + Q_{it} - E_{klt} - S_{l,t+1}$$

  where R_{kilt} is the reservoir release corresponding to the initial reservoir storage S_{kt}, the final reservoir storage $S_{l,t+1}$, and the evaporation loss E_{klt}.
- The loss E_{klt}, depends on the initial and final reservoir storages, S_{kt} and $S_{l,t+1}$.
- Since the inflow Q is a random variable, the reservoir storage and the release are also random variables.

SDP – System Performance Measure

- The system performance measure depends on the state of the system defined by the storage class intervals k and l, and the inflow class interval i for the period t.
- We denote the system performance measure for a period t as B_{kilt} which corresponds to an initial storage state k, inflow state i, and final storage state l in period t.
- The system performance measure may be, for example, the amount of hydropower generated when a release of R_{kilt} is made from the reservoir, and the reservoir storages (which determine the head available for power generation) at the beginning and end of the period are respectively S_{kt} and $S_{l,t+1}$.
- Following backward recursion, the computations are assumed to start at the last period T of a distant year in the future and proceed backwards.
- Each time period denotes a stage in the dynamic programming.
  That is, n = 1 when t=T; n=2 when t = T - 1, etc.
- The index t takes values from T to 1, and the index n progressively increases with the stages in the SDP



SDP – Recursive Relationship

- Let $f_n^t(k, i)$ denote the maximum expected value of the system performance measure up to the end of the last period T (i.e. for periods t, t + 1, ..., T), when n stages are remaining, and the time period corresponds to t.
- With only one stage remaining (i.e. n = 1 and t = T), we write,

  $$f_1^T\ (k,\ i) = \underset{\{\text{feasible } l\}}{\text{Max}}\ [B_{kilT}] \quad \forall\ k,\ i$$

- Note that for a given k and i, only those values of l are feasible that result in a non-negative value of release, R_{kilt}.
- Since this is the last period in computation, the performance measure B_{kilt} is determined with certainty for the known values of k, i and l.

SDP – Recursive Relationship

- When we move to the next stage, (n = 2, t = T - 1), the maximum value of the expected performance of the system is written as

  $$f_2^{T-1}\ (k,\ i) = \underset{\{\text{feasible } l\}}{\text{Max}}\ [B_{kil\ T-1} + \sum_j P_{ij}^{T-1} f_1^T\ (l,\ j)\ ]\ \forall\ k,\ i$$

- When the computations are carried out for stage 2, period T - 1, the inflow during the period is known.
- However, since we are interested in obtaining the maximum expected system performance up to the end of the last period T, we must know the inflow during the succeeding period T also.
- Since this is not known with certainty, the expected value of the system performance is got by using the inflow transition probabilities P_{ij}^{T-1} for the period T - 1.
- It must be noted that the term within the summation denotes the maximized expected value of the system performance up to the end of the last period T, when the inflow state during the period T - 1 is i.
- The search for the optimum value of the performance is made over the end-of-the period storage l.
- Since $f_1^T(k, i)$ is already determined in stage 1, for all values of k and i, $f_2^{T-1}(k, i)$ given by above equation may be determined.
- The term {feasible l}, indicates that the search is made only over those end-of-the-period storages which result in a non-negative release R_{kilt} or satisfy any other constraints.

SDP – Recursive Relationship

- The relationship may be generalized for any stage n and period t as

  $$f_n^t\ (k,\ i) = \underset{\{\text{feasible } l\}}{\text{Max}}\ [B_{kilt} + \sum_j P_{ij}^t f_{n-1}^{t+1}(l,\ j)\ ]\ \forall\ k,\ i$$

- Solving the equation recursively will yield a steady state policy within a few annual cycles, if the inflow transition probabilities P_{ij}^t are assumed to remain the same every year, which implies that the reservoir inflows constitute a stationary stochastic process.
- In general, the steady state is reached when the expected annual system performance, $[f^t_{n+T}(k, i) - f^t_n(k, i)]$ remains constant for all values of k, i, and t.
- When the steady state is reached, the optimal end-of-the-period storage class intervals, l, are defined for given k and i for every period t in the year.
- This defines the optimal steady state policy and is denoted by $l^*(k, i, t)$.

SDP – Example

- Obtain steady state policy for the following data, when the objective is to minimize the expected value of the sum of the square of deviations of release and storage from their respective targets, over a year with two periods.
- Neglect evaporation loss.
- If the release is greater than the release target, the deviation is set to zero.
- Target Storage, T_S=30; Target Release, T_r=30; $B_{kilt} = (R_{kilt} - T_r)^2 + (S_k^t - T_s)^2$

For period 1

i	Q_{it}	k	S_{kt}
1	15	1	30
2	25	2	40

For period 2

i	Q_{it}	k	S_{kt}
1	35	1	20
2	45	2	30

Inflow transition probabilities

	$t = 2$				$t = 1$		
		j				j	
$t = 1$	i,	1	2	$t = 2$	i,	1	2
	1	0.5	0.5		1	0.4	0.6
	2	0.3	0.7		2	0.8	0.2

SDP – Example – Solution

Find B_{kilt} values for all k, i, l, and t

For period 1

k	S_k^t	i	Q_i^t	l	S_l^{t+1}	E_{kilt}	R_{kilt}	$(S_k^t - T_s)^2$	$(R_{kilt} - T_r)^2$	B_{kilt}
1	30	1	15	1	20	0	25	0	25	25
1	30	1	15	2	30	0	15	0	225	225
1	30	2	25	1	20	0	35	0	0	0
1	30	2	25	2	30	0	25	0	25	25
2	40	1	15	1	20	0	35	100	0	100
2	40	1	15	2	30	0	25	100	25	125
2	40	2	25	1	20	0	45	100	0	100
2	40	2	25	2	30	0	35	100	0	100

For period 2

k	S_k^t	i	Q_i^t	l	S_l^{t+1}	E_{kilt}	R_{kilt}	$(S_k^t - T_s)^2$	$(R_{kilt} - T_r)^2$	B_{kilt}
1	20	1	35	1	30	0	25	100	25	125
1	20	1	35	2	40	0	15	100	225	325
1	20	2	45	1	30	0	35	100	0	100
1	20	2	45	2	40	0	25	100	25	125
2	30	1	35	1	30	0	35	0	0	0
2	30	1	35	2	40	0	25	0	25	25
2	30	2	45	1	30	0	45	0	0	0
2	30	2	45	2	40	0	35	0	0	0

SDP – Example – Solution – Contd.

n=1, t=2

$$f_1^2 (k, i) = \underset{\{\text{feasible } l\}}{\text{Min}} [B_{kil2}] \quad \forall . k, i$$

k	i	B_{kilt} $l = 1$	B_{kilt} $l = 2$	$f_1^2(k,i)$	l^*
1	1	125.00	325.00	125.00	1
1	2	100.00	125.00	100.00	1
2	1	0.00	25.00	0.00	1
2	2	0.00	0.00	0.00	1, 2

SDP – Example – Solution – Contd.

n=2, t=1

$$f_2^1(k,i) = \underset{\{\text{feasible } l\}}{\text{Min}} [B_{kil1} + \sum_j P_{ij}^1 f_1^2(l, j)] \quad \forall \; k, i$$

$k = 1;\ i = 1\ l = 1;\ B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j) = 25.0 + 0.50*125.0 + 0.50*100.0$
$= 137.5$

$k = 1,\ i = 1,\ l = 2;\ B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j) = 225.0 + 0.50*0.0 + 0.50*0.0$
$= 225.0$

$k = 1,\ i = 2,\ l = 1;\ B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j) = 0.00 + 0.30*125.0 + 0.70*100.0$
$= 107.5$

$k = 1,\ i = 2,\ l = 2;\ B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j) = 25.00 + 0.30*0.0 + 0.70*0.0$
$= 25.0$

$k = 2,\ i = 1,\ l = 1;\ B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j)$
$= 100.00 + 0.50*125.00 + 0.50*100.00$
$= 212.5$

K	i	$B_{kil1} + \sum P_{ij}^1 f_1^2 (l, j)$ $l = 1$	$l = 2$	$f_2^1\ (k, i)$	l^*
1	1	137.50	225.00	137.50	1
1	2	107.50	25.00	25.00	2
2	1	212.50	125.00	125.00	2
2	2	207.50	100.00	100.00	2

SDP – Example – Solution – Contd.

n=3, t=2

$$f_3^2\ (k, i) = \underset{\{\text{feasible } l\}}{\text{Min}} [B_{kil2} + \sum_j P_{ij}^1 f_2^1\ \ (l, j)] \quad \forall\ k, i$$

k	i	$B_{kil2} + \sum P_{ij}^1 f_2^1\ (i,j)$ $l = 1$	$l = 2$	$f_3^2\ (k, i)$	l^*
1	1	195.00	435.00	195.00	1
1	2	215.00	245.00	215.00	1
2	1	70.00	135.00	70.00	1
2	2	115.00	120.00	115.00	1

SDP – Example – Solution – Contd.

n=4, t=1

$$f_4^1\ (k,i) = \underset{\{\text{feasible } l\}}{\text{Min}} [B_{kil1} + \sum_j P_{ij}^2 f_3^2\ \ (l,j)] \quad \forall k,i$$

k	i	$B_{kil1} + \sum P_{ij}^2 f_3^2\ (l, j)$ $l = 1$	$l = 2$	$f_4^1\ (k,l)$	l^*
1	1	230.00	317.50	230.00	1
1	2	209.00	126.50	126.50	2
2	1	305.00	217.50	217.50	2
2	2	309.00	201.50	201.50	2

SDP – Example – Solution – Contd.

n=5, t=2

k	i	l:1	l:2	f_5^2 (k, i)	l*
1	1	292.90	532.90	292.90	1
1	2	309.30	339.30	309.30	1
2	1	167.90	232.90	167.90	1
2	2	209.30	214.30	209.30	1

n=6, t=1

k	i	l:1	l:2	f_6^1 (k, i)	l*
1	1	326.10	413.60	326.10	1
1	2	304.38	221.88	221.88	2
2	1	401.10	313.60	313.60	2
2	2	404.38	296.88	296.88	2

SDP – Example – Solution – Contd.

n=7, t=2

k	i	l:1	l:2	f_7^2 (k, i)	l*
1	1	388.57	628.57	388.57	1
1	2	405.26	435.26	405.26	1
2	1	263.57	328.57	263.57	1
2	2	305.26	310.26	305.26	1

n=8, t=1

k	i	l:1	l:2	f_8^1 (k,i)	l*
1	1	421.91	509.41	421.91	1
1	2	400.25	317.75	317.75	2
2	1	496.91	409.41	409.41	2
2	2	500.25	392.75	392.75	2

SDP – Example – Solution – Contd.

The computations are terminated after this stage because it is verified that the annual system performance measure remains constant, (being nearly 96).

$$f_8^1\ (1, 1) - f_6^1\ (1, 1) = 421.91 - 326.10 = 95.81)$$

Steady state policy for period 1

k	i	l*
1	1	1
1	2	2
2	1	2
2	2	2

Steady state policy for period 2

k	i	l*
1	1	1
1	2	1
2	1	1
2	2	1

SDP – Exercise Problems

1. Solve the previous problem, if the storage is greater than the storage target, the deviation is set to zero whereas squared deviations from the release targets on either side should be minimized.
2. Solve the previous problem, if only the absolute sum of deficits from the storage and release targets are to be minimized.

Thank You