Stochastic Optimization - II

Chance Constrained LP

Introduction

- In reservoir planning and operation problem, inflow, Q_t, is a random variable and is not known with certainty. Its probability distribution, however, may be estimated from the historical sequence of inflows.
- Being functions of inflow, Q_t, storage S_t and release R_t are also random variables.
- In a constraint containing two random variables, if the probability distribution of one is known, the probabilistic behavior of the second can be expressed as a measure of chance in terms of the probability of the first variable.
- If a constraint contains more than two random variables, we get into computational complications, and we need to understand the specific problem clearly to reformulate the problem, if necessary, and avoid those complications.

Chance Constraint

- The constraint, relating the release, R_t, (random) and demand, D_t, (deterministic), is expressed as a chance constraint, $P\,[R_t \geq D_t] \geq \alpha_1$;
- It means that the probability of release equaling or exceeding the known demand is at least equal to α_1, which is referred to as the reliability level. The interpretation of this chance constraint is simply that the reliability of meeting the demand in period *t* is at least α_1.
- Similarly, Chance constraints for the maximum release and the maximum and minimum storage can be written as

$$P[R_t \leq R_t^{max}] \geq \alpha_2$$
$$P[S_t \leq K] \geq \alpha_3$$
$$\text{and } P[S_t \geq S_{min}] \geq \alpha_4$$

- To use the above chance constraints in an optimization algorithm, we must first determine the probability distribution of R_t and S_t from the known probability distribution of Q_t.
- However, because S_t, Q_t and R_t are all interdependent through the continuity equation, it is, in general, not possible to derive the probability distributions of both S_t and R_t simultaneously.
- To overcome this difficulty and to enable the use of linear programming in the solution, a linear decision rule is appropriately defined.

Linear Decision Rule (LDR)

- The linear decision rule (LDR) relates the release, R_t, from the reservoir as a linear function of the water available in period *t*. The simplest form of such an LDR is

$$R_t = S_t + Q_t - b_t$$

where b_t is a deterministic parameter called the decision parameter.

- In this LDR, the entire amount, Q_t, is taken into account while making the release decision. Depending on the proportion of inflow, Q_t, used in the linear decision rule, a number of such LDRs may be formulated. A general form of this LDR may be written as

$$R_t = S_t + \beta_t Q_t - b_t \qquad 0 \leq \beta_t \leq 1$$

- $\beta_t = 0$ yields a relatively conservative release policy with release decisions related only to the storage, S_t; $\beta_t = 1$ yields an optimistic policy where the entire amount of water available $(S_t + Q_t)$, is used in the LDR.

LDR – Contd.

- Consider the LDR

$$R_t = S_t + Q_t - b_t$$

- Storage continuity equation is

$$S_{t+1} = S_t + Q_t - R_t$$

- Using above two equations

$$S_{t+1} = b_t$$

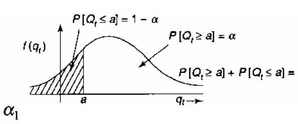
- Thus, the random variable, S_{t+1}, is set equal to a deterministic parameter b_t.
- Thus, the role of the LDR in this case is to treat S_t, deterministic in formulation.
- A main advantage of doing this is to do away with one of the random variables, S_t, so that the distribution of the other random variable, R_t, may be expressed in terms of the known distribution of Q_t.
- This implies that the variance of Q_t is entirely transferred to the variance of R_t.
- Including evaporation loss as a storage-dependent term in the storage continuity equation, the linear decision rule is written as,

$$R_t = Q_t - [A_o e_t] + [1 - (a\, e_t/2)]\, b_{t-1} - [1 + (a\, e_t/2)]\, b_t$$

Deterministic Equivalent of a Chance Constraint

- Knowing the probability distribution of inflow, Q_t , it is possible to obtain the deterministic equivalents of the chance constraints using the LDR, as follows:

$$P[R_t \geq D_t] \geq \alpha_1$$
$$P[S_t + Q_t - b_t \geq D_t] \geq \alpha_1$$
$$P[b_{t-1} + Q_t - b_t \geq D_t] \geq \alpha_1$$
$$P[Q_t \geq D_t + b_t - b_{t-1}] \geq \alpha_1$$
$$P[Q_t \leq D_t + b_t - b_{t-1}] \leq 1 - \alpha_1$$



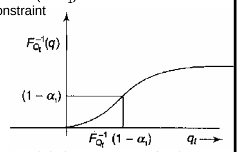
- The term $D_t + b_t - b_{t-1}$, is deterministic with b_t and b_{t-1} being decision variables and D_t being a known quantity for the period *t*.

$$F_{Qt}(D_t + b_t - b_{t-1}) \leq 1 - \alpha_1$$
$$(D_t + b_t - b_{t-1}) \leq F_{Qt}^{-1}(1 - \alpha_1)$$

Deterministic Equivalent of a Chance Constraint – Contd.

$(D_t + b_t - b_{t-1}) \le F_{Qt}^{-1}(1 - \alpha_1)$

- $F_{Qt}^{-1}(1 - \alpha_1)$ is the flow, q_t, at which the CDF value is $(1 - \alpha_1)$ as shown.
- Similarly, deterministic equivalent of Chance constraint
  $P(R_t \le R_t^{max}) \ge \alpha_2$ is
  $R_t^{max} + b_t - b_{t-1} \ge F_{Qt}^{-1}(\alpha_2)$



- Since the storage, S_t, is set equal to the deterministic parameter, b_t, the chance constraints containing only the storage random variable are written as deterministic constraints (without using the probability distribution of inflows).

Chance Constrained LP (CCLP)

- Complete deterministic equivalent of the CCLP is thus written as

$$\text{Min } K$$
subject to
$$D_t + b_t - b_{t-1} \le F_{Q_t}^{-1}(1 - \alpha_1) \quad \forall t$$
$$R_t^{\max} + b_t - b_{t-1} \ge F_{Q_t}^{-1}(\alpha_2) \quad \forall t$$
$$b_{t-1} \le K \quad \forall t$$
$$b_{t-1} \ge S_{\min} \quad \forall t$$
$$b_t \ge 0 \quad \forall t$$
$$K \ge 0$$

- While solving this model, for a problem with 12 periods (months) in a year, we also set $b_0 = b_{12}$ for a steady state solution.
- Further, depending on the nature of LDR used, the decision parameters, b_t, may be unrestricted in sign.
- For example, if we use the LDR, $R_t = S_t - b_t$, the decision parameter, b_t, may be allowed to take negative values.
- CCLP can also be applied for Reliability based reservoir sizing

Problem

- For the following chance constrained optimization problem, formulate the equivalent deterministic optimization problem using the LDR, $R_t = S_t - b_t$. Storage continuity should be maintained. Neglect losses. Following table gives the $F^{-1}(\,)$ values for the inflows and R_{max} and R_{min} values for different periods.

Minimize K
subject to
$P[S_{\min} \le S_t \le K] \ge 0.9 \ \forall t$
$P[R_t \le R^t_{\max}] \ge 0.95 \ \forall t$
$P[R_t \ge D_t] \ge 0.75 \ \forall t$

t	$F^{-1}(0.0)$	$F^{-1}(0.1)$	$F^{-1}(0.25)$	$F^{-1}(0.75)$	$F^{-1}(0.9)$	$F^{-1}(0.95)$	$R_{\max}$	$R_{\min}$	$S_{\min}$
1	0	12	33	60	90	93	90	24	2
2	0	3	20	48	60	80	84	20	2
3	0	6	21	36	72	85	84	20	2

t	$F^{-1}(0.0)$	$F^{-1}(0.1)$	$F^{-1}(0.25)$	$F^{-1}(0.75)$	$F^{-1}(0.9)$	$F^{-1}(0.95)$	$R_{\max}$	$R_{\min}$	$S_{\min}$
1	0	12	33	60	90	93	90	24	2
2	0	3	20	48	60	80	84	20	2
3	0	6	21	36	72	85	84	20	2

Solution

LDR is $R_t = S_t - b_t$
$$S_{t+1} = S_t + Q_t - R$$
$$= S_t + Q_t - S_t + b_t$$
$$= Q_t + b_t$$
$$S_t = Q_{t-1} + b_{t-1}$$

Deterministic equivalent of
$$P[S_{\min} \le S_t \le K] \ge 0.9$$
The constraint can be written as
$$P[S_{\min} \le S_t] \ge 0.9$$
$$P[S_t \le K] \ge 0.9$$

Deterministic equivalent of
$$P[S_{\min} \le S_t] \ge 0.9$$
$P[S_{\min} \le Q_{t-1} + b_{t-1}] \ge 0.9$ (as $S_{\min} = 2$)
$P[Q_{t-1} + b_{t-1} \ge S_{\min}] \ge 0.9$
$P[Q_{t-1} \ge S_{\min} - b_{t-1}] \ge 0.9$
$P[Q_{t-1} \le S_{\min} - b_{t-1}] \le (1 - 0.9)$
$P[Q_{t-1} \le 2 - b_{t-1}] \le 0.1$

$F_{Q_{t-1}}(2 - b_{t-1}) \le 0.1$
$2 - b_{t-1} \le F_{Q_{t-1}}^{-1}(0.1)$
$2 - b_3 \le F_{Q_3}^{-1}(0.1)$
$2 - b_1 \le F_{Q_1}^{-1}(0.1)$
$2 - b_2 \le F_{Q_2}^{-1}(0.1)$

$F^{-1}(.)$ values given,
$2 - b_3 \le 6 \quad t = 1$
$2 - b_1 \le 12 \quad t = 2$
$2 - b_2 \le 3 \quad t = 3$

t	$F^{-1}(0.0)$	$F^{-1}(0.1)$	$F^{-1}(0.25)$	$F^{-1}(0.75)$	$F^{-1}(0.9)$	$F^{-1}(0.95)$	$R_{\max}$	$R_{\min}$	$S_{\min}$
1	0	12	33	60	90	93	90	24	2
2	0	3	20	48	60	80	84	20	2
3	0	6	21	36	72	85	84	20	2

Solution – Contd.

Deterministic equivalent of
$$P[S_t \le K] \ge 0.9$$
$$P[Q_{t-1} + b_{t-1} \le K] \ge 0.9$$
$$P[Q_{t-1} \le K - b_{t-1}] \ge 0.9$$
$$K - b_{t-1} \ge F_{Q_{t-1}}^{-1}(0.9)$$
$K - b_3 \ge 72$
$K - b_1 \ge 90$
$K - b_2 \ge 60$

Deterministic equivalent of
$$P[R_t \le R^t_{\max}] \ge 0.95$$
$P[S_t - b_t \le R^t_{\max}] \ge 0.95$
$P[Q_{t-1} + b_{t-1} - b_t \le R^t_{\max}] \ge 0.95$
$P[Q_{t-1} \le R^t_{\max} + b_t - b_{t-1}] \ge 0.95$
$F_{Q_{t-1}}(R^t_{\max} + b_t - b_{t-1}) \ge 0.95$
$R^t_{\max} + b_t - b_{t-1} \ge F_{Q_{t-1}}^{-1}(0.95)$
$90 + b_1 - b_3 \ge F_{Q_3}^{-1}(0.95)$
$84 + b_2 - b_1 \ge F_{Q_1}^{-1}(0.95)$
$84 + b_3 - b_2 \ge F_{Q_2}^{-1}(0.95)$
$90 + b_1 - b_3 \ge 85 \quad t = 1$
$84 + b_2 - b_1 \ge 93 \quad t = 2$
$84 + b_3 - b_2 \ge 80 \quad t = 3$

Solution – Contd.

Deterministic equivalent of
$$P[R_t \ge D_t] \ge 0.75$$
$P[Q_{t-1} + b_{t-1} - b_t \ge D_t] \ge 0.75$
$P[Q_{t-1} \ge D_t + b_t - b_{t-1}] \ge 0.75$
$P[Q_{t-1} \le D_t + b_t - b_{t-1}] \le (1 - 0.75)$
$F_{Q_{t-1}}(D_t + b_t - b_{t-1}) \le 0.25$
$D_t + b_t - b_{t-1} \le F_{Q_{t-1}}^{-1}(0.25)$

$24 + b_1 - b_3 \le F_{Q_3}^{-1}(0.25)$
$20 + b_2 - b_1 \le F_{Q_1}^{-1}(0.25)$
$20 + b_3 - b_2 \le F_{Q_2}^{-1}(0.25)$

$24 + b_1 - b_3 \le 21 \quad t = 1$
$20 + b_2 - b_1 \le 33 \quad t = 2$
$20 + b_3 - b_2 \le 20 \quad t = 3$

The solution of this model results in
$K = 90;\ b_1 = 0;\ b_2 = 9;\ b_3 = 5.$

Thank You